MODULE-III

Syllabus:

Transient behavior and initial conditions: Behavior of circuit elements under switching condition and their Representation, evaluation of initial and final conditions in RL, RC and RLC circuits for AC and DC excitations.
Laplace Transformation & Applications: Solution of networks, step, ramp and impulse responses, waveform synthesis.

Text Books:
1. M.E. Van Valkenberg (2000), “Network analysis”, Prentice Hall of India, 3rd edition, 2000, ISBN: 9780136110958.
2. Roy Choudhury, “Networks and systems”, 2nd edition, New Age International Publications, 2006, ISBN: 9788122427677.

Reference Books:
1. Hayt, Kemmerly and Durbin “Engineering Circuit Analysis”, TMH 7th Edition, 2010.
2. J. David Irwin /R. Mark Nelms, “Basic Engineering Circuit Analysis”, John Wiley, 8th ed, 2006.
3. Charles K Alexander and Mathew N O Sadiku, “Fundamentals of Electric Circuits”, Tata McGraw-Hill, 3rd Ed, 2009.

Transient behavior and initial conditions

Objectives:

- To know why initial conditions are important.
- To understand behavior of circuit elements under switching condition and their representation.
- To evaluate initial and final conditions in RL, RC and RLC circuits.

An electric switch is turned on or off in some circuit (for example in a circuit consisting of resistance and inductance), transient currents or voltages will occur for a short period after these switching actions. After the transient has ended, the current or voltage in question returns to its steady state situation. Duration of transient phenomena are over after only a few micro or milliseconds, or few seconds or more depending on the values of circuit parameters (like *R, L and C*).The situation relating to the sudden application of dc voltage to circuits possessing resistance (*R*), inductance (*L*), and capacitance (*C*) will now be investigated in this chapter. We will continue our discussion on transients occurring in a dc circuit. It is needless to mention that transients also occur in ac circuit but they are not included in this chapter.

There are many reasons for studying initial conditions:

- The most important reason is that initial conditions must be known to evaluate the arbitrary constants that appear in the general solution of the differential equations.

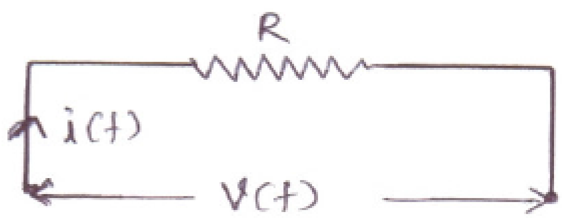
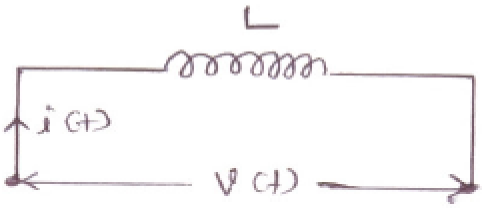
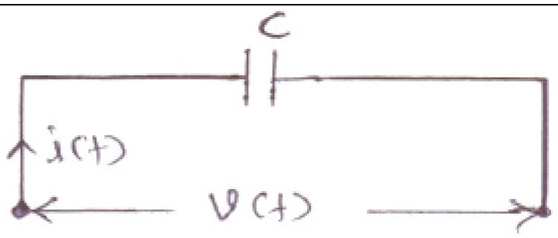
- The initial conditions give knowledge of the behavior of the elements at the instant of switching.
- Knowledge of the initial values of derivatives of a response is helpful in anticipating the nature of response.
- It gives a better understanding of non linear switching circuits.

What are Initial Conditions:

- Finding the value of selected variables when one or more circuit switches are moved from one position to another.
- t=0- represents time just before switch changing the position.
- t=0 represents the time when switch changes its position.
- t=0+ represents time just after switch changing the position.
- Initial conditions focuses on currents and voltages of energy storing elements (Inductor and capacitor) as these determine the behavior of the circuit at t>0.
- Past history of the circuit is shown up using capacitor voltage and inductor current.

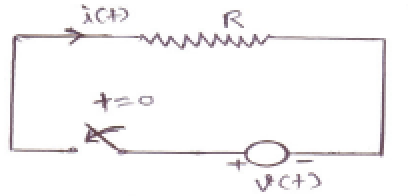
The evaluation of all voltages and currents and their derivatives at t =0+, constitutes the **evaluation of initial conditions.**
Sometimes we use conditions at t = ∞; these are known as **final conditions.**

V-I Relationships of Network Elements:

Element	Voltage	Current
	**v(t)=R.i(t)**	**i(t) = v(t)/R**
	**v(t)= L di(t)/dt**	**i(t) = ∫v(t) dt**
	**v(t) = (1/C) ∫i(t) dt**	**i(t) = C dv(t)/dt**

Initial Conditions in Network Elements:

1. Resistor

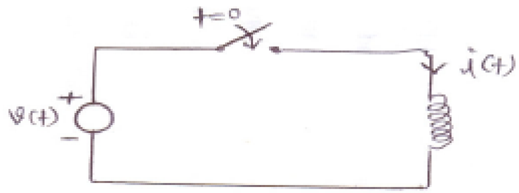


The current voltage relation for a resistor is given by:- v(t)=i(t)*R.

From the above relation it can be said the instantaneous value of current depends on instantaneous value of voltage and vice versa.

Past value of current or voltage does not affect present value of current or voltage i.e. the behavior of resistance remains same irrespective of past value of current and voltage.

2. Inductor



When switch is closed at t = 0, the current through an inductor cannot change instantaneously. As a result, closing of a switch to connect an inductor to a source of energy will not cause current to flow at that instant and inductor will act as **an open circuit.**

The current through inductor is given by

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)dt$$

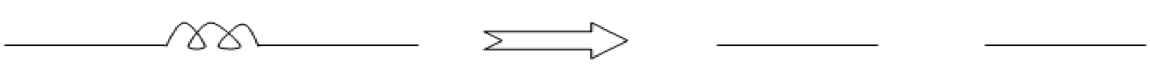
$$i(t) = \frac{1}{L}\int_{-\infty}^{0-} v(t)dt + \frac{1}{L}\int_{0-}^{t} v(t)dt$$

$$i(t) = i(0^-) + \frac{1}{L}\int_{-0}^{t} v(t)dt$$

$$i(0^+) = i(0^-) + \frac{1}{L}\int_{0-}^{0+} v(t)dt$$

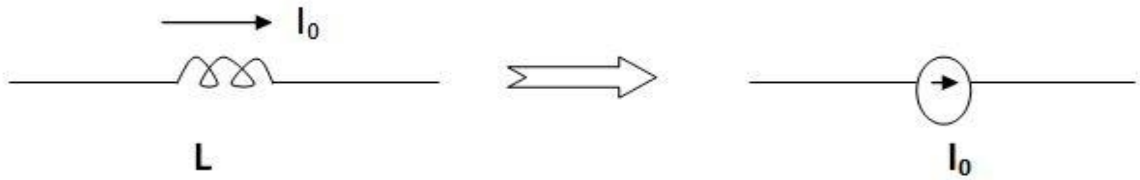
$$i(0^+) = i(0^-)$$

Hence the current at 0+ is equal to the current at 0-. So if i(0-)=0, then i(0+)=0 also. So it acts as open circuit as shown.

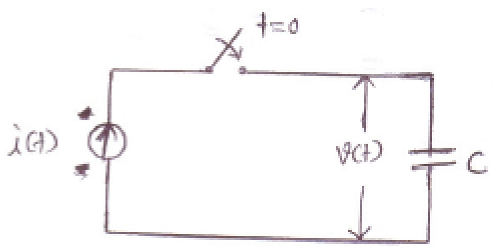


L OC

If a current of I_0 amps flows in the inductor at the instant of switching takes place, that current will continue to flow & for the initial instant the ($t=0+$) inductor can be considered as a current source of I_0 amps



3. Capacitor



Voltage across capacitor is given by

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{0-} i(t)dt + \frac{1}{C}\int_{0-}^{t} i(t)dt$$

$$v(t) = v(0^-) + \frac{1}{C}\int_{-0}^{t} i(t)dt$$

$$v(0^+) = v(0^-) + \frac{1}{L}\int_{0-}^{0+} i(t)dt$$

$$v(0^+) = v(0^-)$$

If v(0-)=0, then v(0+)=0 also, so it acts as short circuit at t=0+

If v(0-)=V_0 volts, then capacitor acts as a constant voltage source at t=0+ as shown below

Behavior of the elements at t=0+ and at t=∞

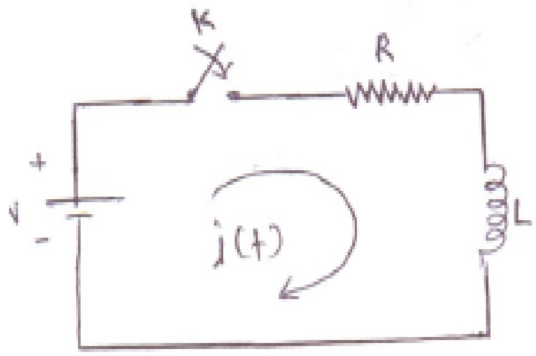
Element	Equivalent circuit at t=0+	At t=∞
Resistor	Resistor	Resistor
Inductor	Open circuit	Short circuit
Capacitor	Short circuit	Open circuit
Inductor with initial current	Current source Io	Finally short
Capacitor with initial charge	Voltage source + -	Finally open circuit

Procedure for Evaluating Initial Conditions

- Find the current through inductor and voltage across the capacitor before switching i.e. at t=0- find i(0-) and v(0-) [i.e. History of the network]
- Draw the circuit at t=0+ using equivalent for each circuit element.
- Determine i(0+) and v(0+)
- Draw the general circuit after switching.
- Write the integro-differential equation for the circuit.
- Obtain an expression for di/dt or dv/dt
- Using initial conditions like i(0+) find di(0+)/dt or using v(0+) find dv(0+)/dt
- Obtain an expression for $d^2i(0+)/dt^2$ or $d^2v(0+)/dt^2$
- Using initial conditions like i(0+) , di(0+)/dt, find out $d^2i(0+)/dt^2$ or Using v(0+) , dv(0+)/dt, find out $d^2v(0+)/dt^2$.

Numerical Examples:

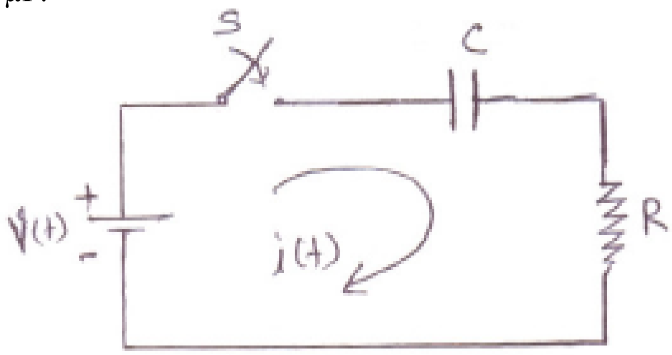
1. In the network shown below, If at t=0, switch 'K' is closed, Find the values of i, di/dt and d^2i/dt^2 at t =0+. Assume V= 100V, R= 1000Ω and L= 1H.



Solution:

- Before the switch is closed i**(0-) =0**
- Draw the circuit immediately after switching, Inductor acts as open , as inductor won't allow current to change instantaneously Hence **i(0+) = 0**
- Draw general network after switching and writing KVL
  V= Ri + L di/dt
- Obtain an expression for the first derivative
  di/dt = (V – R(i))/L, substituting the values we get
  di/dt = 100 A/s
- Obtain an expression for the second derivative:
  $d^2i(0+)/dt^2$ = - R/L di/dt , substituting the known values we get
  $d^2i(0+)/dt^2$ = -100000A/s^2

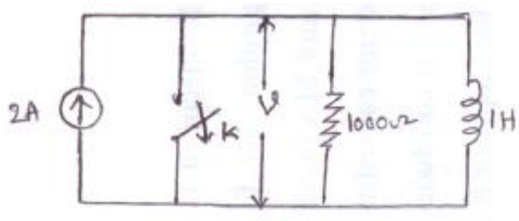
2. In the network shown below, If at t=0, switch 'S' is closed with no initial charge on the capacitor, Find the values of i, di/dt and d^2i/dt^2 at t =0+. Assume V= 100V, R= 1000Ω and C= 1μF.



Solution:

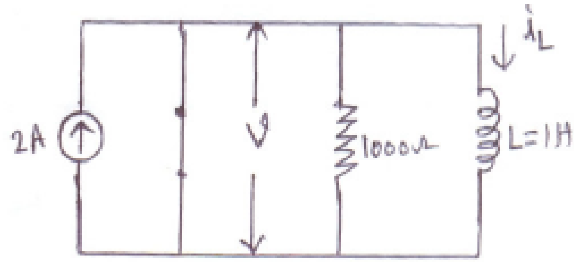
- Before switch is closed Vc(0-) = 0 , i(0-)=0
- Capacitor acts as a short circuit after switching therefore i(0+)=V/R = **0.1 A**
- Draw the general network after switch is closed
- V= Ri + 1/c ∫i dt
- Differentiate : 0 = R di/dt + i/c
  di(0+)/dt= - i(0+)/RC, substituting the values we get
  di(0+)/dt= – 100 A/s
- Differentiate again to obtain second derivative
  $d^2i(0+)/dt^2$ = - (1/RC) di(0+)/dt , substituting we get
  $d^2i(0+)/dt^2$ = + 100000 A/s^2

3. In the circuit shown, switch 'k' is opened at t=0. Find the values of v, dv/dt,d^2v/dt^2 at t = 0+.



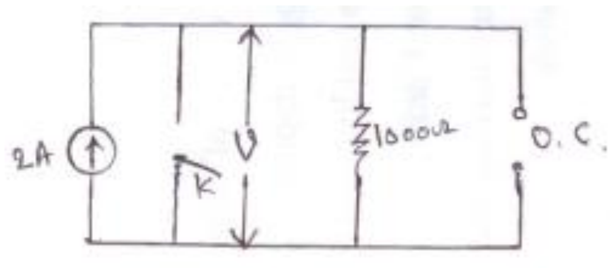
Solution:

At t=0-



v(0-) = 0, iL(0-) = 0 and iL(0+) = 0 since current through the inductor cannot change suddenly.

At t=0+



v(0+) = i x R =2x1000=2000V

- Apply KCL to the given circuit at t = 0+, we get
  v/R+(1/L)∫vdt = 2 ……….…(1)
- Differentiate equation (1) with respect to t, we get
  (1/R)dv/dt +(1/L)v = 0….. (2)
  At t = 0+, equation (2) becomes
  (1/R) dv(0+)/dt +(1/L)v(0+) = 0

dv(0+)/dt = - (R/L)v(0+)

= (-1000/1) x 2000

= -2x10^6 V/S

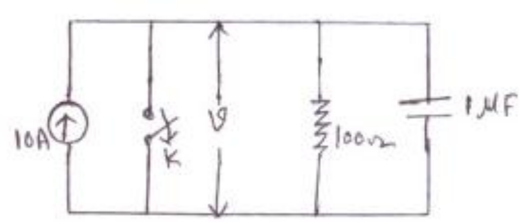
- Differentiate equation (2) with respect to t, we get
  (1/R)d2v/dt2 +(1/L)dv/dt = 0 ....(3)
  At t = 0+, equation (2) becomes
  (1/R)d2v(0+)/dt2 +(1/L)dv(0+)/dt = 0

d^{2}v(0+)/dt^2 = -(R/L)dv(0+)/dt

= - (1000/1)x(-2x106)

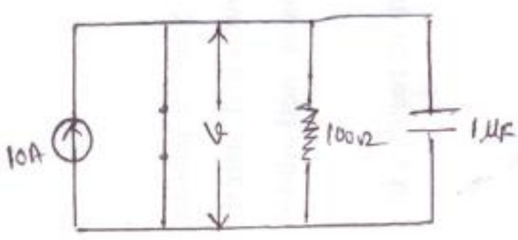
= 2x10^9 V/S^2

4. In the circuit shown, switch 'k' is opened at t=0. Find the values of v, dv/dt, d^{2}v/dt^2 at t = 0+.



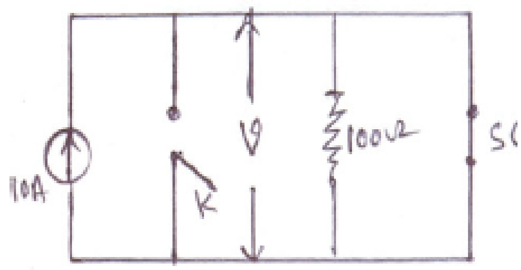
Solution:

At t=0-



v(0-)=0

At t=0+



v (0+) = 0 since voltage across the capacitor cannot change instantaneously.

- Apply KCL to the given circuit at t = 0+, we get

V/R+ C dV/dt = 10 .... (1)

At t = 0+, equation (1) becomes

dV(0+)/dt = (10/C) – v(0+)/RC

= 10/(1x10-6) – 0

=10^7 Volts/sec

- Differentiate equation (1) with respect to t, we get

(1/R) dV/dt + cd2v /dt2 = 0 ....(2)

At t = 0+, equation (2) becomes

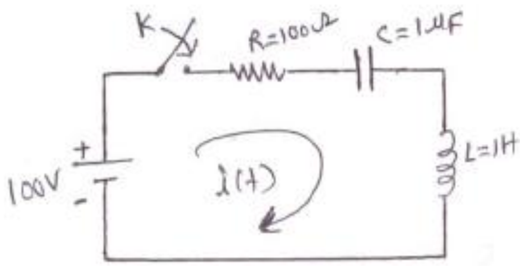
cd2V (0+)/dt2 = - (1/R) dV(0+)/dt

$d^2V(0+)/dt^2$ = -(1/RC)dv(0+)/dt

= - [1/(100x1x10-6)]x107)

= -10^{11} V/s^2

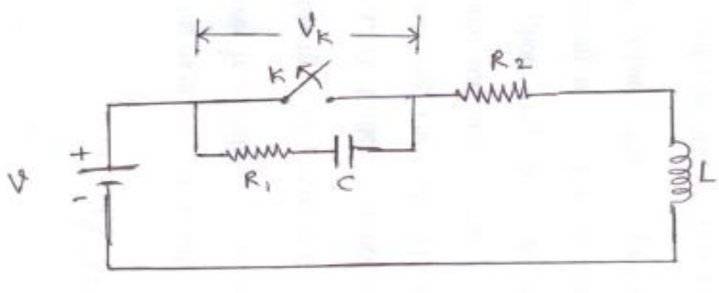
5. In the given circuit, switch 'K' is closed at t=0 with capacitor uncharged and zero current in the inductor. Find di(t)/dt and $d^2i(t)/dt^2$ at t = 0+



Solution:

- At t=0- switch is open so i(0-)=0.
- At t=0+, the inductor will act as an open circuit and capacitor will act as an short circuit, so i(0+)=0.
- Applying KVL, V (t)= R i(t) + L di(t)/dt +1/C ∫ i(t)dt ……..(1)
  V (0+)= R i(0+) + L di(0+)/dt +1/C ∫ i(0+)dt
  100 = 100x0 +1x di(0+)/dt +(1/1x10-6) 0 dt
  Therefore **di(0+)/dt=100 A/S**
- Differentiate equation (1) with respect to t, we get
  0 = Rdi(t)/dt + L d2i(t)/dt2 +i(t)/C ………..(2)
  0 = Rdi(0+)/dt + L d2i(0+)/dt2 + 0
  Therefore $\mathbf{d^2i(0+)/dt^2}$ **= - (R/L)di(t)/dt = (100/1)x100**
  = -10000A/S^2

6. In the network shown below, the switch 'K' is opened at t=0 after the network has attained a steady state with the switch closed.
   Find (a) the expression for the voltage across the switch at t =0+
   (b) If the parameters are adjusted such that i(0+)=1 and di(t)/dt = -1, what is the value of the derivative of the voltage across the switch $dV_k(0+)/dt$ =?



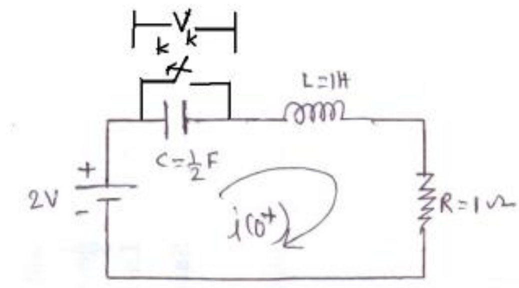
Solution:

- Initially switch is closed indicates that the voltage across R1 and C is zero, inductor acts as short circuit as steady state having been reached, Therefore
  i(0-) = V/R_2 , voltage across cap =0,
- Switch is opened at t=0

Inductor acts as a current source of value V/R_2, capacitor acts as short

- Hence: i(0+)= V/R_2 only as inductor does not allow any sudden change in current.
- Now general network after switching:
  V= R_1i+ 1/c∫idt + R_2i + L di/dt
  But the voltage across the switch Vk= R_1. i + (1/C) ∫ idt
- At t=0+ , (1/C) ∫ idt=0
  Therefore Vk= R_1x i(0+)= R_1 xV/R_2= **Vx(R_1/R_2)**
  dV_k/dt = R_1 di/dt + i/c = **1/C - R_1**

7. In the following circuit with switch 'k' is closed, steady state has been reached. At t=0, the switch is open. Find di(t)/dt, $d^2i(t)/dt^2$ and $d^3i(t)/dt^3$ at t = 0+ also Find Vk(0+) and $d^2Vk(0+)/dt^2$.



Solution:

Just before opening the switch i.e. at t=0-, the capacitor is redundant and inductor is shorted so

i(0-) = V/R= 2/1= **2A = i(0+)**

Apply KVL to the circuit, we get

V= (1/C)∫idt +Ldi/dt +Rxi ....(1)

At t=0+ , 2 = 0 + 1 di(0+)/dt +1x i(0+)

di(0+)/dt=2- 1x i(0+)=2-(1x 2)

= 0 A/S

Differentiate equation (1) w. r. t. t, we get

0 = i/C + Ld2i/dt2+ R di/dt

At t=0+, **$d^2i(0+)/dt^2$=-(R/L) di(0+)/dt –i(0+)/LC**

= 1/1 di(0+)/dt – i(0+)/[1x(1/2)]

= 0 –2x 2

= -4 A/S^2

Differentiate equation (1) twice w. r. t. t, we get

0= (1/C) di/dt + L d3i/dt3+ R d2i/dt2

At t=0+,

$d^3i(0+)/dt^3$=-2di(0+)/dt –d2i(0+)/dt2

= -2x0-(-4)

= 4 A/S^3

From (1)

Vk+Ldi/dt +Rx i=2 (since (1/C) ∫idt= Vk

Differentiate above equation twice w. r. t. t, we get

d^2 Vk /dt^2 + Ld^3i/dt^3+Rd^2i/dt^2=0

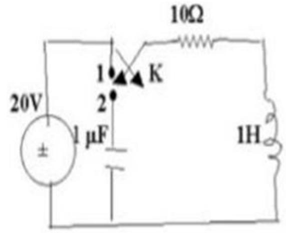
At t=0+,

$d^2Vk(0+)/dt^2$ = - $Ld^3i(0+)/dt^3$-$Rd^2i(0+)//dt^2$

= -1x 4-1x(-4)

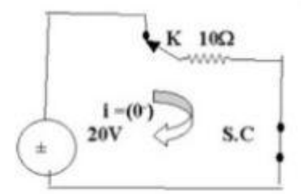
=0 V/S^2

8. Position of the switch is changed from 1 to 2 at t=0. Steady state was achieved at position 1 determine i(t), di/dt and d^2i/dt^2 at t=0+.

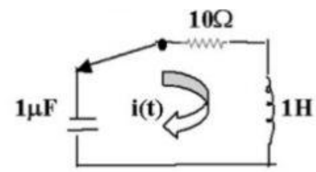


Solution:

- At t=0-, the circuit is as shown
- Inductor in steady state assumed as short circuit.
- So the current through it is
  $I_L(0^-)=20/10=$**2A.**
  $V_c(0^-)=0V$



- At t=0+, the inductor behaves as current source of 2A and circuit so, $I_L(0^+)=$**2A** and $V_c(0^+)=0V$
- For t>=0+, the circuit is as shown.
- By applying KVL in the circuit above we get.



$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int i(t)dt$$

$$Ri(t) + L\frac{di}{dt} + V_c(t) = 0$$

At $t=0^+$, $Ri(0^+) + L\frac{di}{dt}(0^+) + V_c(0^+) = 0$

$$2R + L\frac{di}{dt}(0^+) + 0 = 0$$

$$\frac{di}{dt}(0^+) = -\frac{2R}{L} = -\mathbf{20A/sec}$$

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i(t)}{C} = 0$$

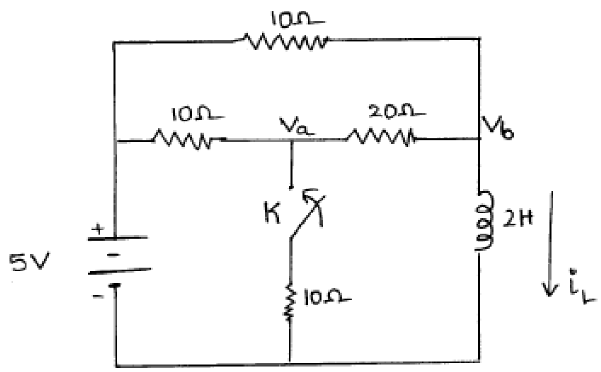
At $t=0^+$,

$$R\frac{di}{dt}(0^+) + L\frac{d^2i}{dt^2}(0^+) + \frac{i(0^+)}{C} = 0$$

$$-20R + L\frac{d^2i}{dt^2}(0^+) + \frac{2}{C} = 0$$

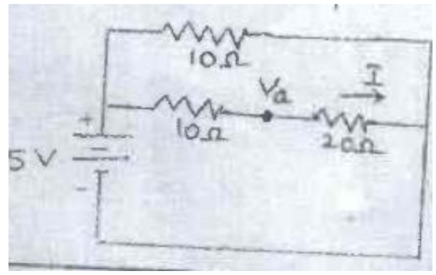
$$\frac{d^2i}{dt^2}(0^+) = -2x10^6A/sec^2$$

9. In the network shown below, a steady state is reached with the switch K open. At t = 0, the switch is closed. For the element values given, determine the values of Va(0-) and Va(0+).



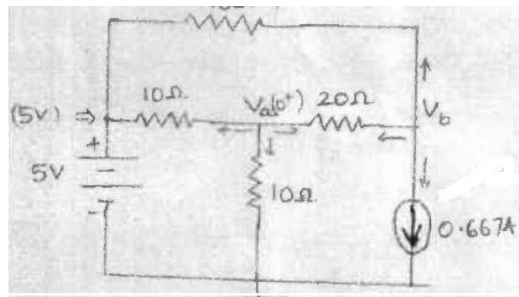
Solution:

- **At t=0-**



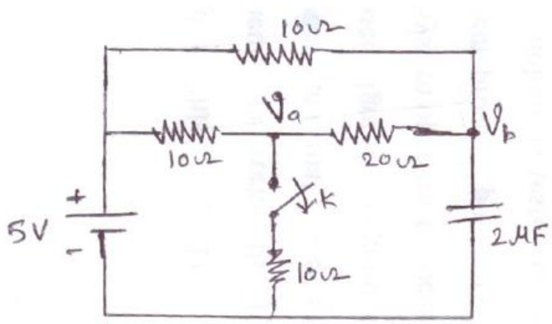
Req=10||(10+20)= 7.5Ω
I_T=5/Req = 0.667A
I=I_T *10/(10+30) = 0.167A
Va(0-) = 20*I= 3.34V
Since i_L(0-) = I_T= **0.667A** = i_L(0+)

- **At t=0+**



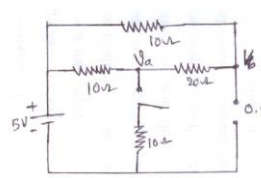
Applying KCL at Node a
(Va-5)/10+ Va/10+(Va-Vb)/20 = 0
0.25Va-0.05Vb=0.5………(1)
Applying KCL at Node b
(Vb-5)/10+(Vb-Va)/20+0.667=0
-0.05Va+0.15Vb-0.167=0……(2)
Solving equations 1 and 2 we get
Va=1.904V & Vb=-0.479V
Therefore
Va(0+) = 1.904V

10. In the circuit shown below, the steady state is reached with switch 'k' is open. At t=0, switch 'k' is closed. For the element values given, determine the value of Va(0-) and Va(0+).



Solution:

- **At t=0-**

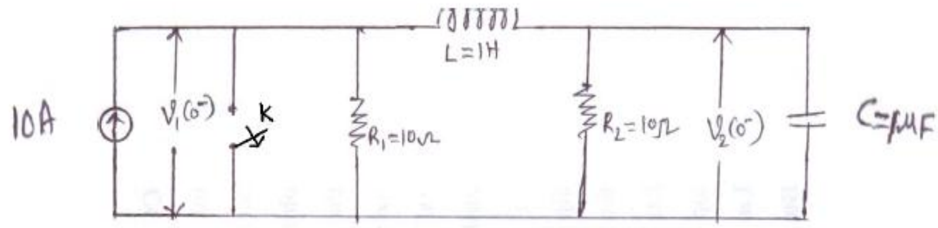


Va(0-)=Vb(0-)=5 V
Vb(0-)=5V= Vb(0+)

- **At t = 0+,**

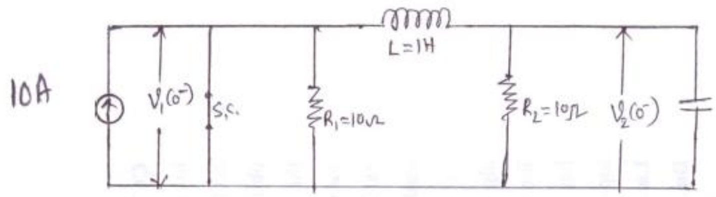
Apply KCL to the given circuit, we get
(Va-5)/10 + (Va/10) + (Va -Vb)/20 = 0 .....(1)
(Vb-5)/10 + (Vb-Va)/20 + CdVb/dt = 0 …..(2)
From (1), At t = 0+,
[Va(0+) -5]/10 + [Va(0+)] /10 + [Va(0+) -Vb(0+)]/20 = 0
[Va(0+) -5]/10 + [Va(0+)] /10 + [Va(0+) -5]/20 = 0
[2Va(0+) -10 + 2Va(0+) + Va(0+) -5]/20 = 0
[5Va(0+) -15]/20 = 0
Va(0+) = 15/5=3V0lts

11. In the given circuit, switch 'k' is opened at t=0. Find the values of v_1, v_2, dv_1/dt and dv_2/dt at t = 0+.



Solution:

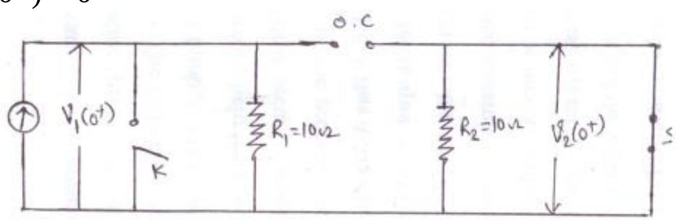
At t= 0- $v_1(0-) = 0$ and $v_2(0-) = 0$



At t= 0+

$v_1(0+) = ixR_1 = 10x10=100V$

$v_2(0+) = 0$ and $i_2(0+) = 0$



Apply KCL to the given circuit at t = 0+, we get

$v1/R1+ (1/L)\int(v1- v2)dt = 0$....... (1)

$v2/R2 +(1/L)\int(v2- v1)dt + c\, dv2/dt = 0$.... (2)

Differentiate equation (1) with respect to t, we get

$(1/R1)\, dv1/dt +(1/L)(v1- v2) = 0$

At t = 0+,

$(1/R1)\, dv1(0+)/dt + (1/L)v1(0+) - (1/L)v2(0+) = 0$

$dv_1(0+)/dt = -R_1/L\ v1\ (0+) +(R/L)v_2(0+)$

$= -(10/1)x\ 100 + (10/1)\ x\ 0$

= -1000 volts/sec

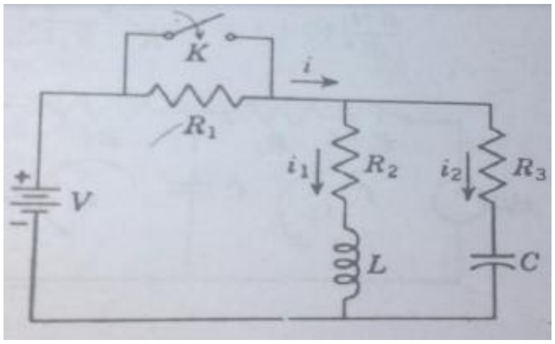
From equation (2)

$v_2(0+)/R_2 + i_L(0+)+ Cdv_2(0+)/dt = 0$

$0 + 0 + cdv_2(0+)/dt = 0$

$Cdv_2(0+)/dt = 0$ volts/sec

12. Steady state reached with switch k open, Switch is closed at t=0. V=100V, R_1=10 Ω, $R_2=R_3$=20Ω, L=1H, C=1μF. Find voltage across C, initial values of i_1, i_2, di_1/dt, di_2/dt at t=0+ and also find di/dt at t=∞.



Solution:

Switch k is opened : Steady State being established

Inductor acts as short circuit: capacitor acts as open:

$I_1(0\text{-})= V/R_1+R_2 = 3.33$ A, $I_2(0\text{-})=0$;

$Vc(0\text{-})= i_1 * R_2 = 66.67V$

At t=0 switch is closed: hence R_1 becomes redundant.

Inductor acts as a current source of value 3.33 A, Vc(0+)= 66.67 V

$I_2(0+) = (100 – 66.67)/20 =$ **1.67A**

$I_1(0+)=$ **3.33 A**

General network after switching :

$V=i_1R_2 + L\, di_1/dt$-------------------(a)

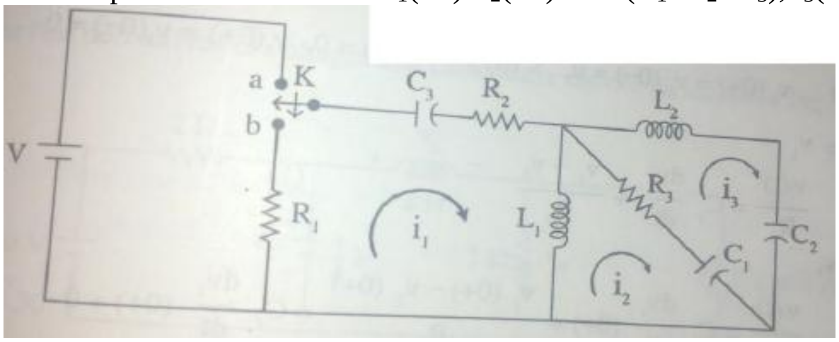
$V=i_2 R_3 + 1/c \int i_2dt$ ------------------(b)

$di_1(0+)/dt = (V-i_1R_2) /L =$ **33.3A/s**

diff eqn (b) we get

$di_2(0+)/dt=- i_2(0+)/R_3C =$ **-83500 A/s**

13. In the network shown , switch K is changed from position a to b at t=0, steady state being established at position a. Show that $i_1(0+)=i_2(0+)= -V/(R_1+R_2+R_3)$, $i_3(0+)=0$



Solution:

Steady state being established at position a, Means inductors act as short circuit, capacitors act as open circuit

Switch being at position a, we can see that

i1(0-)=0, i2(0-)=0 i3(0-)=0

and

Vc1(0-)=0, Vc2(0-)=0 and Vc3(0-)=V.

When switch is moved to position b, **Only C3 will act as a voltage source,** No current i1, i2 and i3 hence inductor L1 and L2 acts as open and Vc3 = V

We get i3(0+)=0 And i1=i2= -V/ $R_1+R_2+R_3$

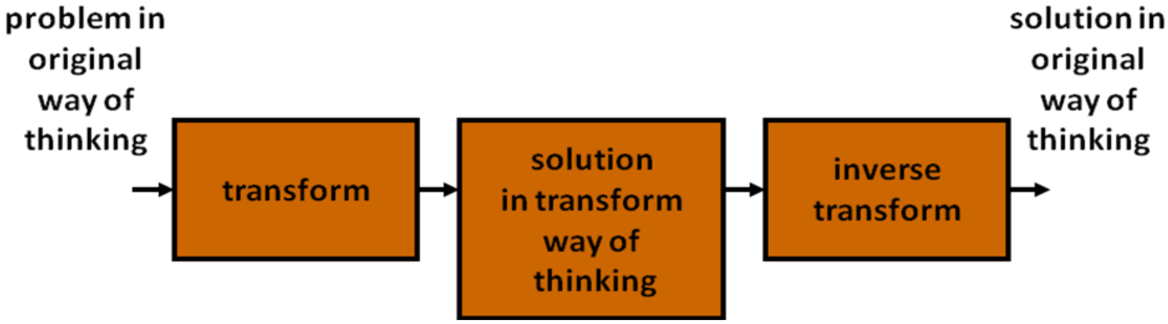
Laplace Transformation and Applications

Objectives:

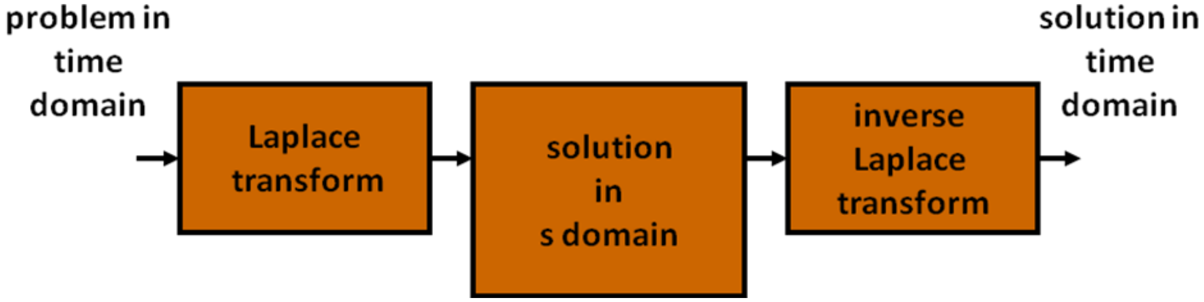
- To Know What is Laplace Transform
- What are advantages of Laplace Transform
- Laplace Transforms of Standard Functions
- Initial and Final Value Theorems
- Applications of Laplace Transform
- Waveform Synthesis

What is Transform?

A mathematical conversion from one way of thinking to another to make a problem easier to solve is called as transform.



Laplace Transform:



It's a transformation method used for solving differential equation, which transforms signal from time domain t to complex frequency domain S.
It is defined as

$$\mathbf{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Where S is the complex frequency

Inverse Laplace Transform is defined as

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{ts}\,ds$$

Advantages of Laplace Transform

- The solution of differential equation using LT, progresses systematically. We use Laplace transform to convert equations having complex differential equations to relatively simple equations having polynomials which are easier to solve.
- With Laplace transform nth degree differential equation can be transformed into an nth degree polynomial. One can easily solve the polynomial to get the result and then change it into a differential equation using inverse Laplace transform.
- Initial conditions are automatically specified in transformed equation.
- The method gives complete solution in one operation. (Both complementary function and particular Integral in one operation)

Standard Functions

The three important singularity functions employed in circuit analysis are:

- Unit Step Function [u(t)]
- Delta function [δ(t)]
- Ramp Function [r(t)]

They are called singularity functions because they are either not finite or they do not possess finite derivatives everywhere.

Unit Step Function [u(t)]

- The *unit step* function, $u(t)$
  - Mathematical definition

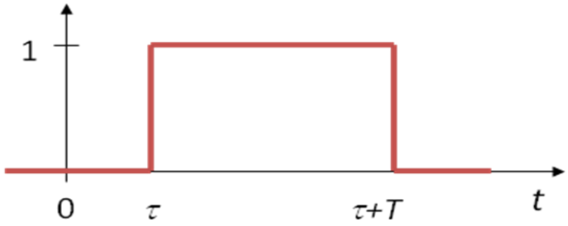
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

- A more general *unit step* function is *u*(*t*-*a*)

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

- The gate function can be constructed from *u*(*t*)
  - a rectangular pulse that starts at $t=\tau$ and ends at $t=\tau+T$
  - like an on/off switch

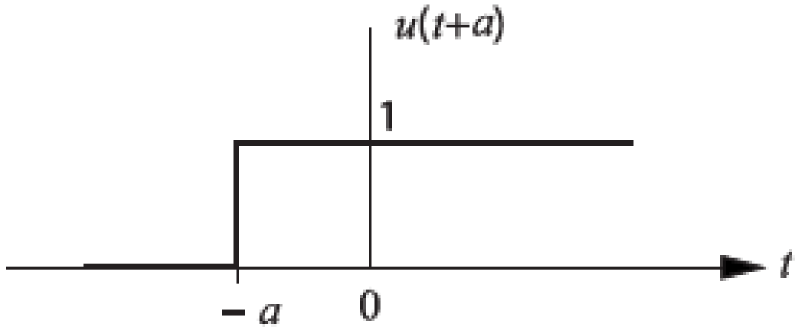
$u(t-\tau) - u(t-\tau-T)$



Similarly, the unit step function that occurs at $t = -a$ is expressed as $u(t+a)$.

Thus,

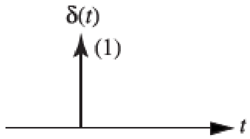
$$u(t+a) = \begin{cases} 0, & t+a < 0 \text{ or } t < -a \\ 1, & t+a > 0 \text{ or } t > -a \end{cases}$$



Delta function [δ(t)]

The derivative of the unit step function u(t) is the unit impulse function δ(t).

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$



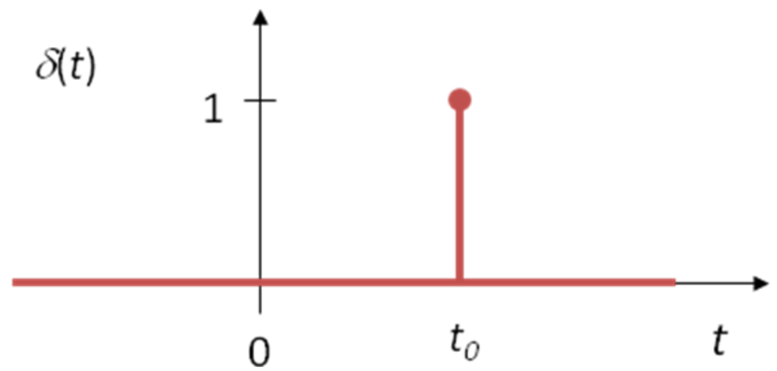
The unit impulse may be visualized as very short duration pulse of unit area.

$$\int_{0^-}^{0^+} \delta(t)dt = 1$$

- The *delta* or *unit impulse* function, δ(*t*)
  - Mathematical definition (non-pure version)

$$\delta(t - t_0) = \begin{cases} 0 & t \neq t_0 \\ 1 & t = t_0 \end{cases}$$

  - Graphical illustration



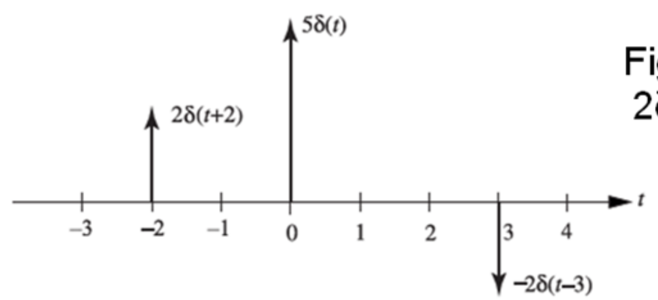


Figure shows impulse functions, 2δ(t+ 2), 5δ(t) and -2δ(t-3)

An important property of the unit impulse function is what is often called the sifting property; which is exhibited by the following integral:

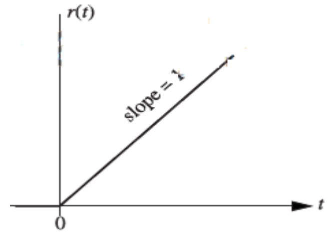
$$\int_{t_1}^{t_2} f(t)\delta(t - t_0)dt = \begin{cases} f(t_0), & t_1 < t_0 < t_2 \\ 0, & t_1 > t_0 > t_2 \end{cases}$$

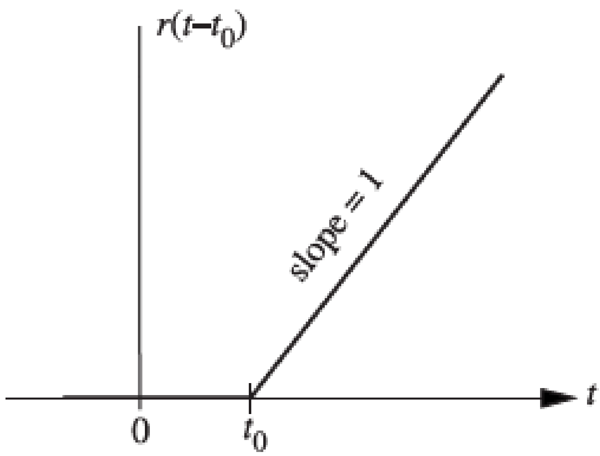
Ramp function [r(t)]

Integrating the unit step function u(t) results in the unit ramp function r(t).

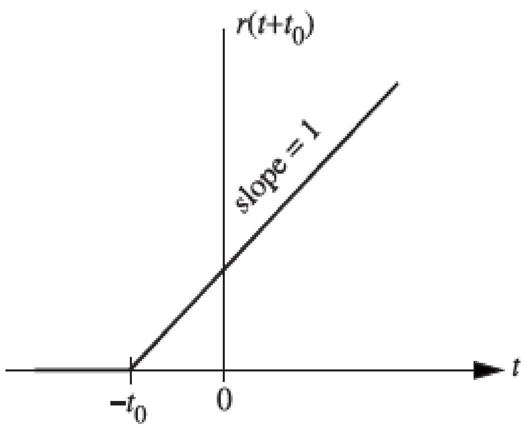
$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau = tu(t)$$

or $$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$





$$r(t-t_0) = \begin{cases} 0, & t \le t_0 \\ t - t_0, & t \ge t_0 \end{cases}$$



$$r(t+t_0) = \begin{cases} 0, & t \le -t_0 \\ t + t_0, & t \ge -t_0 \end{cases}$$

• Relation Between Three Singularity Functions:

It is very important to note that the three sigularity functions are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt}$$

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt, \quad r(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

Functional Laplace Transforms

1. Unit Step Function [u(t)]

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$L[u(t)] = F(s) = \int_0^{\infty} 1 e^{-st} dt = \frac{1}{s}$$

2. Unit Impulse Function [δ(t)]

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ 1 & t = 0 \end{cases}$$

$$L[\delta(t)] = F(s) = \int_0^{\infty} \delta(t) e^{-st} dt = 1$$

3. Unit Ramp Function [r(t)]

$$r(t) = \begin{cases} 0 & t \leq 0 \\ t & t \geq 0 \end{cases}$$

$$\mathrm{L}\{t\,\mathrm{u}(t)\} = \int_{0^-}^{\infty} t e^{-st} dt$$

$$= t\left(-\frac{1}{s}e^{-st}\right)\Bigg|_{0^-}^{\infty} - \int_{0^-}^{\infty} 1\left(-\frac{1}{s}e^{-st}\right) dt$$

$$= 0 + \frac{1}{s}\int_{0^-}^{\infty} e^{-st} dt$$

$$= \frac{1}{s}\left(-\frac{1}{s}e^{-st}\right)\Bigg|_{0^-}^{\infty} = \frac{1}{s^2}$$

Laplace Transform Properties

Theorem	Property	$f(t)$	$\mathbf{F}(s)$
1	Scaling	$A\, f(t)$	$A\, \mathbf{F}(s)$
2	Linearity	$f_1(t) \pm f_2(t)$	$\mathbf{F}_1(s) \pm \mathbf{F}_2(s)$
3	Time Scaling	$f(a \cdot t)$	$\frac{1}{a}\mathbf{F}\left(\frac{s}{a}\right) \quad a > 0$
4	Time Shifting	$f(t-t_0)\, u(t-t_0)$	$e^{-s \cdot t_0}\, \mathbf{F}(s) \quad t_0 \geq 0$
6	Frequency Shifting	$e^{-a \cdot t}\, f(t)$	$\mathbf{F}(s+a)$
9	Time Domain Differentiation	$\frac{d\, f(t)}{dt}$	$s\, \mathbf{F}(s) - f(0)$
7	Frequency Domain Differentiation	$t\, f(t)$	$\frac{-d\, \mathbf{F}(s)}{ds}$
10	Time Domain Integration	$\int_0^t f(\tau)\, d\tau$	$\frac{1}{s}\mathbf{F}(s)$
11	Convolution	$\int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau$	$\mathbf{F}_1(s)\, \mathbf{F}_2(s)$

Laplace Transforms of some important functions

1. Exponential Function:

$f(t) = e^{-at}u(t)$, where $a > 0$ and $u(t)$ is the unit step function.

$$\mathscr{L}\{e^{-at}u(t)\} = F(s) = \int_0^\infty f(t)dt$$

$$= \int_0^\infty e^{-at}e^{-st}dt$$

$$= \left.\frac{-e^{-(s+a)t}}{(s+a)}\right|_{t=0}^{\infty}$$

$$= \frac{1}{s+a}$$

2. **Sinusoidal Function:**

$$L[\sin(wt)] = \int_0^\infty \frac{(e^{jwt} - e^{-jwt})}{2j} e^{-st} dt$$

$$= \frac{1}{2j}\left[\frac{1}{s - jw} - \frac{1}{s + jw}\right]$$

$$= \frac{w}{s^2 + w^2}$$

3. **Cosine Function:**

$$L[\cos(wt)] = \int_0^\infty \frac{(e^{jwt} + e^{-jwt})}{2} e^{-st} dt$$

$$= \frac{1}{2}\left[\frac{1}{s - jw} - \frac{1}{s + jw}\right]$$

$$= \frac{s}{s^2 + w^2}$$

4. **Find L[e^{-at} cos(wt)]:**

Let f(t) = cos(wt)
Then

$$F(s) = \frac{s}{s^2 + w^2}$$

$$and \;\; F(s+a) = \frac{(s+a)}{(s+a)^2 + w^2}$$

5. Laplace Transform of cosh(at) and sinh(at)

$$\mathscr{L}\{\cosh(at)\} = \frac{1}{2}\mathscr{L}\{e^{at}\} + \frac{1}{2}\mathscr{L}\{e^{-at}\}$$
$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$
$$= \frac{s}{s^2 - a^2}$$

(b) $\sinh at = \frac{1}{2}[e^{at} - e^{-at}]$

Applying linearity property,

$$\mathscr{L}\{\sinh(at)\} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$
$$= \frac{a}{s^2 - a^2}$$

Laplace Transform of Periodic Functions

- If f(t) is periodic with period T then

$$\mathcal{L}\{f(t)\} = \frac{\int_{0^-}^{T} f(t) e^{-st} dt}{1 - e^{-sT}}$$

Initial and Final Values

The initial-value and final-value properties allow us to find the initial value $f(0)$ and $f(\infty)$ of $f(t)$ directly from its Laplace transform $F(s)$.

Initial-value theorem

$$f(0) = \lim_{s \to \infty} sF(s)$$

The initial-value theorem allows us to find the initial value $x(0)$ directly from its Laplace transform $X(s)$.

If $x(t)$ is a causal signal,

then, $$x(0) = \lim_{s \to \infty} sX(s)$$

***Proof*:**

To prove this theorem, we use the time differentiation property.

$$\mathscr{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0) = \int_0^\infty \frac{dx}{dt} e^{-st}\, dt$$

If we let $s \to \infty$, then the integral on the right side of equation (5.10) vanishes due to damping factor, e^{-st}.

Thus, $$\lim_{s \to \infty} [sX(s) - x(0)] = 0$$

$$\Rightarrow \qquad x(0) = \lim_{s \to \infty} sX(s)$$

Example:

Given;

$$F(s) = \frac{(s+2)}{(s+1)^2 + 5^2}$$

Find f(0)

$$f(0) = \lim sF(s) = \lim s \frac{(s+2)}{(s+1)^2 + 5^2} = \lim \left[\frac{s^2 + 2s}{s^2 + 2s + 1 + 25}\right]$$

$$= \lim \frac{s^2/s^2 + 2s/s^2}{s^2/s^2 + 2s/s^2 + (26/s^2)} = 1$$

Final-value theorem

$$f(\infty) = \lim_{s \to 0} sF(s)$$

The final-value theorem allows us to find the final value $x(\infty)$ directly from its Laplace transform $X(s)$.

If $x(t)$ is a causal signal,

then
$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

***Proof*:**

The Laplace transform of $\frac{dx(t)}{dt}$ is given by

$$sX(s) - x(0) = \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt$$

Taking the limit $s \to 0$ on both the sides, we get

$$\begin{aligned}
\lim_{s \to 0}[sX(s) - x(0)] &= \lim_{s \to 0} \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt \\
&= \int_0^\infty \frac{dx(t)}{dt} \left[\lim_{s \to 0} e^{-st}\right] dt \\
&= \int_0^\infty \frac{dx(t)}{dt} dt \\
&= x(t)|_0^\infty \\
&= x(\infty) - x(0)
\end{aligned}$$

$$x(\infty) = \lim_{s \to 0} sX(s)$$

Example:

Given:

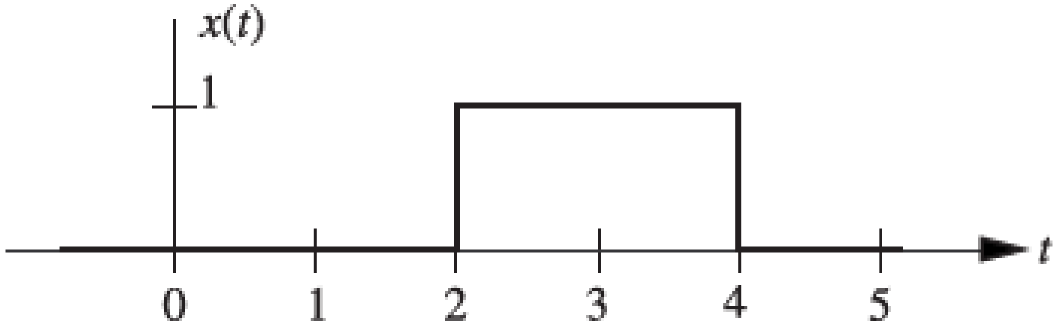
$$F(s) = \frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} \qquad \textit{note } F^{-1}(s) = te^{-2t}\cos 3t$$

Find $f(\infty)$

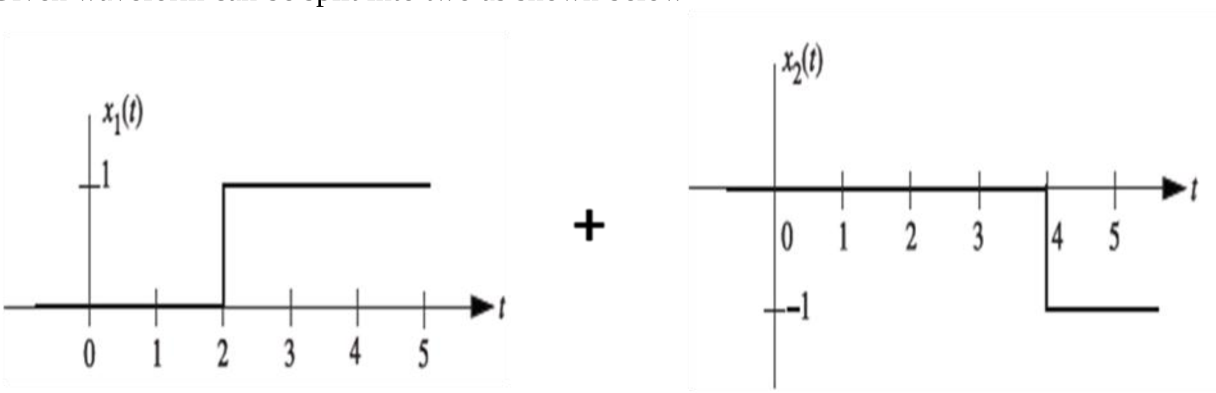
$$f(\infty) = \lim_{s \to 0} sF(s) = \lim s \frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} = 0$$

Waveform Synthesis:

1.Find the Laplace transform of x(t), shown in Figure below



Given waveform can be split into two as shown below

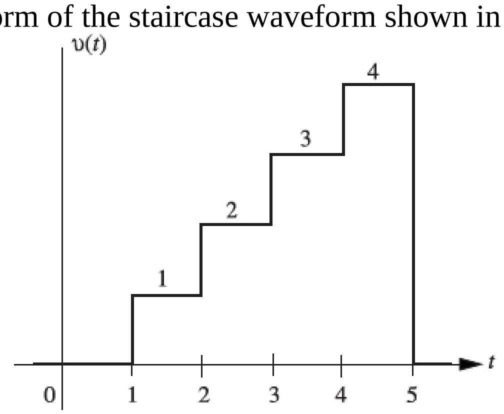


$$x(t) = x_1(t) + x_2(t) = u(t-2) - u(t-4)$$

$$\mathscr{L}\{x(t)\} = X(s) = \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-4s}$$

$$X(s) = \frac{1}{s}(e^{-2s} - e^{-4s})$$

2.Find the Laplace transform of the staircase waveform shown in Figure below



We can express mathematically, the voltage waveform shown in Fig.

$$v(t) = \begin{cases} 1, & 1 < t < 2 \\ 2, & 2 < t < 3 \\ 3, & 3 < t < 4 \\ 4, & 4 < t < 5 \\ 0, & \text{elsewhere} \end{cases}$$

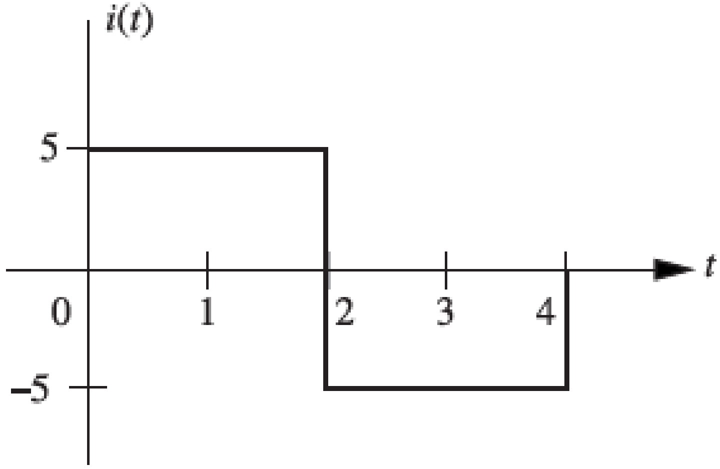
or

$$\begin{aligned} v(t) &= [u(t-1) - u(t-2)] + 2[u(t-2) - u(t-3)] \\ &\quad + 3[u(t-3) - u(t-4)] + 4[u(t-4) - u(t-5)] \\ &= u(t-1) + u(t-2) + u(t-3) + u(t-4) - 4u(t-5) \end{aligned}$$

Taking the Laplace transform, we get

$$V(s) = \frac{1}{s}\left[e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} - 4e^{-5s}\right]$$

3.Express the current pulse in Figure below in terms of the unit step.
Find a) L{i(t)} b) L{∫i(t)dt}



The given waveform can be split into three parts

i(t) = i1(t) + i2(t) + i3(t)

i(t)= 5u(t) – 10u(t-2) + 5u(t- 4)

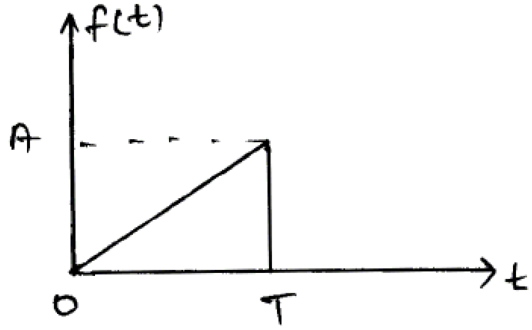
Taking Laplace Transform we get

$$\begin{aligned} I(s) &= \frac{5}{s} - \frac{10}{s}e^{-2s} + \frac{5}{s}e^{-4s} \\ &= \frac{5}{s}\left[1 - 2e^{-2s} + e^{-4s}\right] \\ &= \frac{5}{s}\left[1 - e^{-2s}\right]^2 \end{aligned}$$

Let f(t) = ∫i(t)dt
then f(t) = ∫[5u(t)-10u(t-2) + 5u(t-4)]dt
= 5r(t)- 10r(t- 2) + 5r(t- 4)

$$\begin{aligned}\mathcal{L}\{f(t)\} &= F(s)\\ &= \mathcal{L}\{5r(t) - 10r(t-2) + 5r(t-4)\}\\ &= \frac{5}{s^2} - \frac{10}{s^2}e^{-2s} + \frac{5}{s^2}e^{-4s}\\ &= \frac{5}{s^2}\left[1 - 2e^{-2s} + e^{-4s}\right]\end{aligned}$$

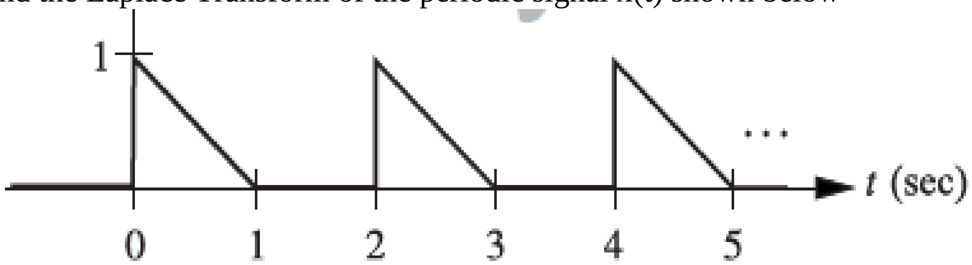
4.Obtain the Laplace Transform of saw tooth waveform shown in Figure below



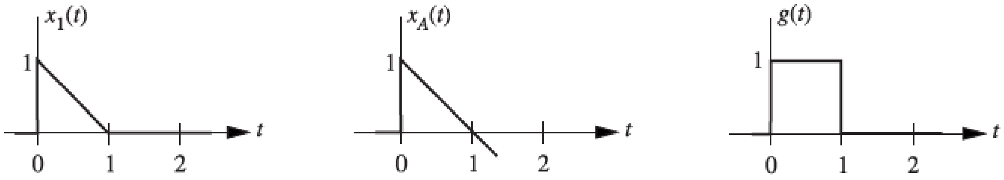
The above saw tooth signal can be split into three parts i.e. initial ramp of slope A/T, ramp of slope –A/T and a step from A to zero.

f(t)= A/T tu(t) - A/T(t-T)u(t-T) – Au(t-T)
Taking LT we get
F(s) = A/TS^2 -A/TS^{2}e^{-ST} -A/Se^{-TS}
= A/TS^2[**1 -e^{-ST} - STe^{-TS}**]

5.Find the Laplace Transform of the periodic signal x(t) shown below



- The given signal is a periodic signal with period T= 2 sec
- Let x1(t) be the signal x(t) over one period which may be viewed as multiplication of xA(t) and g(t) as shown below

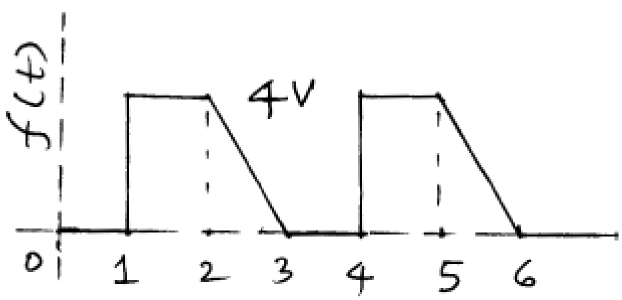


$$x_1(t) = x_A(t)g(t)$$
$$= [-t+1][u(t)-u(t-1)]$$
$$x_1(t) = -tu(t) + tu(t-1) + u(t) - u(t-1)$$
$$= -tu(t) + (t-1+1)u(t-1) + u(t) - u(t-1)$$
$$= -tu(t) + (t-1)u(t-1) + u(t-1) + u(t) - u(t-1)$$
$$= u(t) - tu(t) + (t-1)u(t-1)$$
$$= u(t) - r(t) + r(t-1)$$

$$X_1(s) = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-s}$$
$$= \frac{s-1+e^{-s}}{s^2}$$

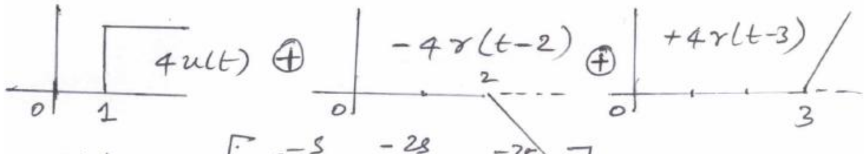
$$\mathbf{X(s) = X1(s)/(1-e^{-2S})}$$

6.Using standard waveforms express the waveform given (periodic) in Figure below and obtain its Laplace transform.



- The one cycle of above waveform can be split into three parts a step of 4V at t=1, ramp of slope -4 at t=2 and ramp of slope +4 at t=3.

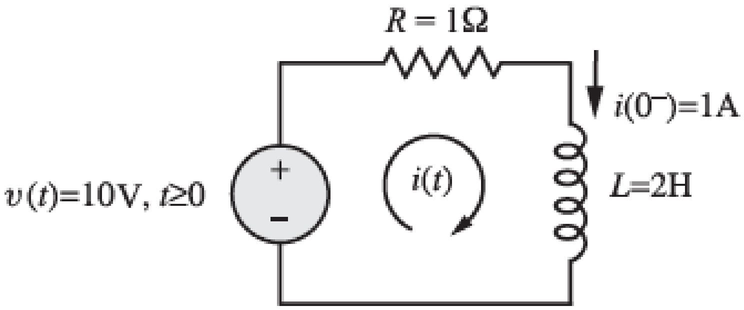
$f(t) = f_1(t) + f_2(t) + f_3(t)$ for 1 cycle



$$F(s) = 4\left[\frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}\right]$$

For periodic $F(s) = \dfrac{4}{1-e^{-3s}}\left[\dfrac{e^{-s}}{s} - \dfrac{e^{-4s}}{s^2} + \dfrac{e^{-3s}}{s^2}\right]$

7.Refer the circuit shown below and Find i(0) and i(∞) using initial and final value theorems.



Applying KVL we get

iR + Ldi/dt = v(t)

i + 2di/dt = 10

Taking LT we get

I(s) + 2[sI(s) – i(0-)] = 10/s

I(s)[1 + 2s] = 10/s + 2

I(s) = (10 +2s)/s(1+2s)

= (5+s)/s(s+1/2)

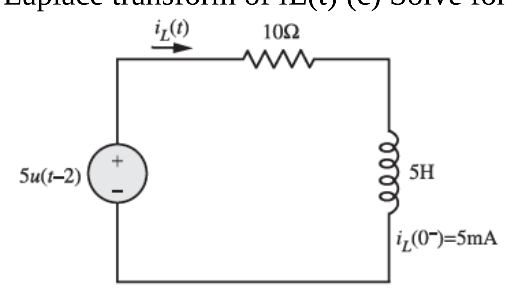
According to initial value theorem

$$i(0) = \lim_{s\to\infty} sI(s) = \lim_{s\to\infty} s\frac{(s+5)}{s\left(s+\frac{1}{2}\right)} = \lim_{s\to\infty}\frac{1+\frac{5}{s}}{1+\frac{1}{2s}} = 1$$

According to initial value theorem

$$i(\infty) = \lim_{s\to 0} sI(s) = \lim_{s\to 0}\frac{s(s+5)}{s\left(s+\frac{1}{2}\right)} = \frac{5}{1/2} = 10\ \mathbf{A}$$

8.For the circuit shown below (a) write a differential equation for the inductor current iL(t). (b) Find IL(s) the Laplace transform of iL(t) (c) Solve for iL(t).



Applying KVL we get

$i_L(t)R + L di_L(t)/dt = v(t)$

$10i_L(t) + 5di_L(t)/dt = 5u(t - 2)$

Taking LT we get

$10I_L(s) + 5[sI_L(s) - i_L(0-)]$
$= 5/s e^{-2s}$

$$I_L(s) = \frac{\frac{5}{s}e^{-2s} + 5i_L(0^-)}{5s+10}$$

$$= \frac{e^{-2s} + 5\times 10^{-3}s}{s(s+2)}$$

$$= e^{-2s}\left[\frac{K_1}{s} + \frac{K_2}{s+2}\right] + \frac{5\times 10^{-3}s}{s(s+2)}$$

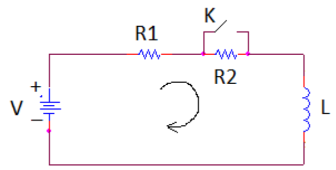
$$I_L(s) = \frac{1}{2}e^{-2s}\left[\frac{1}{s} - \frac{1}{s+2}\right] + \frac{5\times 10^{-3}}{(s+2)}$$

Taking Inverse Laplace transform, we get

$$i_L(t) = \frac{1}{2}\left[u(t) - e^{-2t}u(t)\right]_{t\to t-2} + 5\times 10^{-3}e^{-2t}u(t)$$

$$= \frac{1}{2}\left[u(t-2) - e^{-2t}u(t-2)\right] + 5\times 10^{-3}e^{-2t}u(t)$$

9.In the circuit shown in figure, steady state is reached with switch K open. Obtain the expression for current when switch K is closed at t=0. Assume R1=1Ω, R2=1 Ω, L=1H V=10V



Applying KVL, with the switch is closed

$$Vu(t) = R1i(t) + L\frac{di(t)}{dt}$$

Taking Laplace transform of the above equation yields

$$\frac{V}{s} = R1I(s) + L[sI(s) - i(0)]$$

$$i(0) = \frac{V}{R1+R2} = \frac{10}{3} = 3.333Amp$$

Substituting the values of R1, L and i(0-)

$$I(s) = \frac{10 + 3.333s}{s(s+1)}$$

$$i(t) = [10 - 6.667e^{-t}]u(t) \ \ Amp$$