Module 1

Stress & Strain

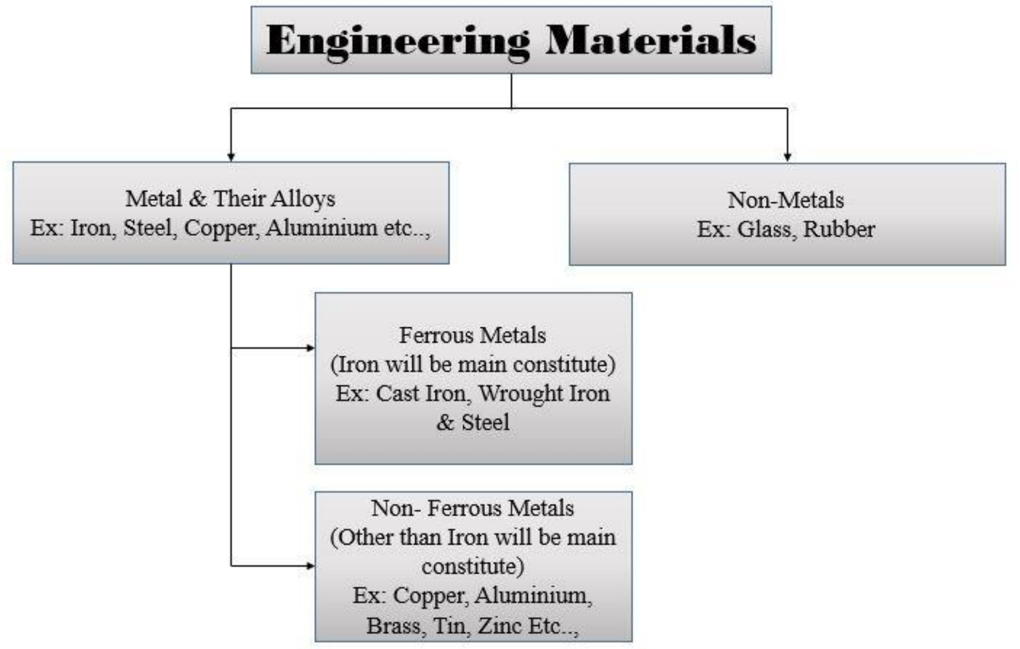
Objectives:

Classify the stresses into various categories and define elastic properties of materials and compute stress and strain intensities caused by applied loads in simple and compound sections and temperature changes.

Learning Structure

- Classification Of Engineering Materials
- Choice Of Selection Of Engineering Materials
- Physical Properties Of Materials
- Mechanical Properties
- Stress, Strain And Hook's Law
- Stress – Strain Relation Or Diagram For Ductile Material
- Stress – Strain Relation Or Diagram For Brittle Material
- Problems
- Elongation Of Tapering Bars Of Circular Cross Section
- Elongation Of Tapering Bars Of Rectangular Cross Section
- Elongation In Bar Due To Self-Weight
- Compound Or Composite Bars
- Temperature Stresses In A Single Bar
- Temperature Stresses In A Composite Bar
- Simple Shear Stress And Shear Strain
- Complementary Shear Stresses
- Volumetric Strain
- Bulk Modulus
- Relation Between Elastic Constants
- Exercise Problems
- Outcomes

1.1 Classification of Engineering Materials



1.2 Choice of Selection of Engineering Materials

- Availability of materials.
- Sustainability of materials for the working conditions in service.
- Cost of materials.

1.3 Physical Properties of Materials

- Lustre
- Colour
- Size
- Density
- Shape

1.4 Mechanical Properties

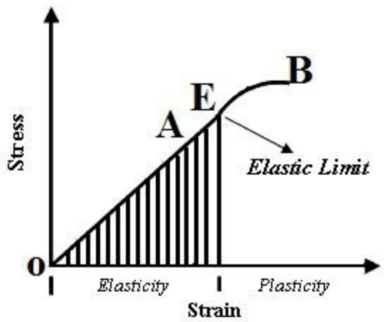
- **Load (F or P)**

  It is defined as any external force acting on a body

- **Elasticity**

  It is the property by virtue of which a material deformed under the load is enable to return to its original dimension when load is removed.

  If the body regains completely its original shape, it is said to be perfectly elastic.



In the above figure, the specimen is loaded upto point A, well within the elastic limit E. When load corresponding to point A is gradually removed the curve follows the same path AO and Strain completely disappears. Such a behaviour is known as Elastic behaviour. Steel is more elastic than rubber.

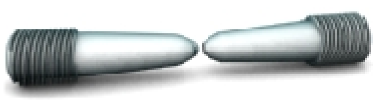
- **Plasticity**

  It is the converse of Elasticity. It is the property of a material which retains the deformation produced under the load permanently.

- **Ductility**

  It is the property of a material which exhibits large deformations in longitudinal direction under the application of tensile force before failure.

  A ductile material must be strong and plastic. The ductility is measured in terms of % elongation or % reduction in cross-sectional area of test specimen.



Ex: Mild steel, Brass, Aluminium, Nickel, Zinc, Tin, Lead etc..,

- **Brittleness**

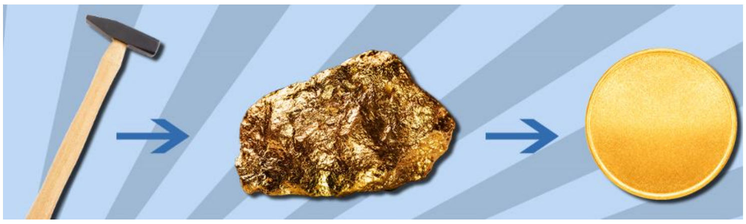
  It is the property of a material which exhibits little or no yielding before failure. Generally brittle materials are have higher strength in compression than in tension.

  Ex: Cast Iron, High carbon steel, Concrete, Stone, Glass, Ceramic materials etc..,

- **Malleability**

  It is the property of a material which permits the material to be extended in all directions without rupture.

  A malleable material possesses a high degree of plasticity but not necessarily great strength



  Ex: Gold, Lead, Soft steel, wrought iron, Copper, Aluminium, etc..,

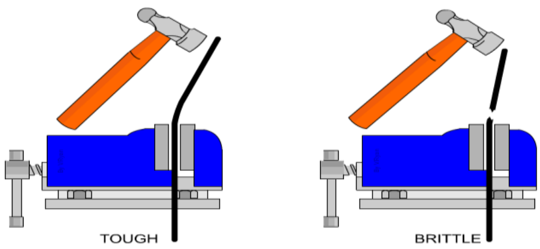
- **Strength**

  It is the ability of a material to resist the externally applied forces without breaking or yielding.

  The load required to cause fracture divided by the area of the test specimen is termed as ultimate strength of the material.

- **Toughness**

  It is the property of a material which enables it to absorb energy without fracture.

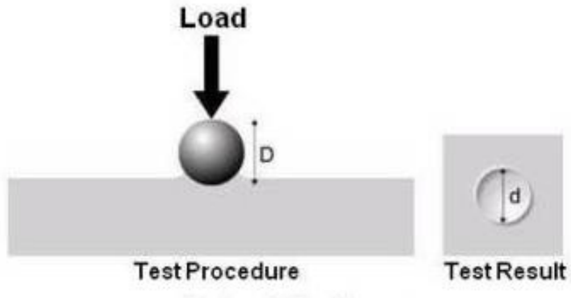


This property is desirable in parts subjected to impact and shock loads. Toughness is measured in terms of energy required per unit volume of the material to cause rupture under the action of gradually increasing tensile load.

- **Hardness**

  It is the ability of the material to resist indentation or surface abrasion.

  It embraces many different properties such as resistance to wear, scratching, deformation, machinability etc..,



- **Stiffness**

  It is the ability of a material to resist deformation under stress.

  The stiffness is measured by the modulus of elasticity in case of axially loaded members

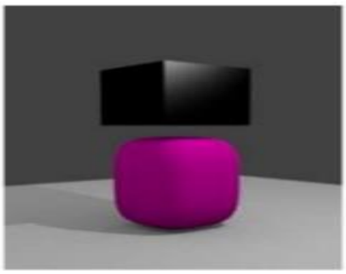
- **Creep**

  Whenever a member or part of a machine subjected to a constant stress at high temperature for a longer period of time, it will undergo a slow and permanent deformation called creep.

- **Resilience**

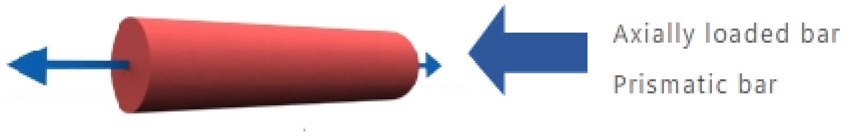
  It is the property of the material to absorb energy and to resist shock and Impact loads.

  It is measured by the amount of energy absorbed per unit volume within elastic limit,



1.5 Stress, Strain and Hook's law

The most fundamental concepts in mechanics of materials are **stress** and **strain**. These concepts can be illustrated in their most elementary form by considering a prismatic bar subjected to axial forces. A **prismatic bar** is a straight structural member having the same cross section throughout its length, and an **axial force** is a load directed along the axis of the member, resulting in either tension or compression in the bar.



1.5.1 Stress

When a body is acted upon by external force F, or Load P, internal resisting force is setup in the body such a body is said to be in state of stress, hence the resistance offered by the body against deformation due to the application of load is called as stress.

Or

The Internal resisting force per unit area at any section of the body is known as Stress

It is denoted by σ (Sigma),

$$\text{Stress } \sigma = \frac{\text{Applied Load or Force}}{\text{Cross}-\text{sectional Area}} = \frac{\text{F or P}}{\text{A}} \frac{\text{N}}{\text{mm}^2}$$

In general, the stresses s acting on a plane surface may be uniform throughout the area or may vary in intensity from one point to another.

1.5.1.1 Types of Stresses

1) Normal Stress
    a) Tensile Stress
    b) Compressive Stress
2) Shear Stress
3) Bearing Stress

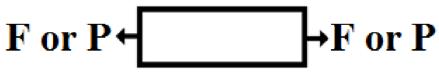
1. Normal Stress

A normal stress is a stress that occurs when a member is loaded by an axial force. (Axial force is the force acting along the axis of the specimen).

Normal stress can be either tensile or compressive in nature.

a) Tensile stress

When a load is acting in such a way that it tends to extend the material in the direction of application of load is called tensile load and the corresponding stress is called tensile stress.



$$\text{Tensile stress, } \sigma = \frac{\text{P or F}}{\text{A}} \; \frac{\text{N}}{\text{mm}^2}$$

b) Compressive stress

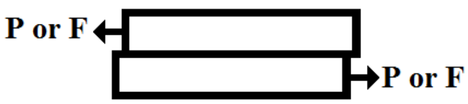
When a load is acting in such a way that it tends to shorten the material in the direction of application of load is called compressive load and the corresponding stress is called compressive stress.

$$\text{Compressive stress, } \sigma = \frac{\text{P or F}}{\text{A}} \; \frac{\text{N}}{\text{mm}^2}$$

When a **sign convention** for normal stresses is required, it is customary to define tensile stresses as positive and compressive stresses as negative.

2. Shear Stress

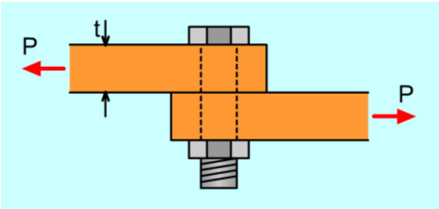
Shearing stress is a force that causes two contacting parts or layers to slide upon each other in opposite directions. The stress developed at the contacting surfaces is known as shear stress.



$$\text{Shear Stress, } \tau = \frac{\text{Shearing Force}}{\text{Shearing Area}} = \frac{\text{P or F}}{\text{A}} \; \frac{\text{N}}{\text{mm}^2}$$

3. Bearing Stress

A Localised compressive stress at the surface of contact between two members of a machine part that are relatively at rest is known as Bearing stress or crushing stress.



$$\text{Bearing Stress} = \frac{P}{A} = \frac{P}{td} \frac{N}{mm^2}$$

Where,

t = Thickness of Plate

d = Diameter of the bolt

1.5.2 Strain

When a body is subjected to some external force there is some change in dimensions of the body.

The ratio of change in dimensions of the body to the original dimensions is known as Strain (ε)

$$\text{Strain } \varepsilon = \frac{\text{Change in Dimension}}{\text{Original Dimension}}$$

Strain is dimensionless

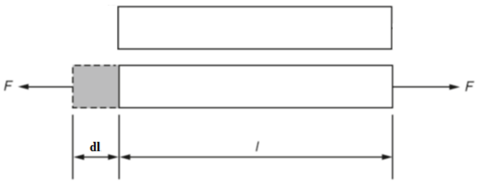
1.5.2.1 Types of Strain

1) Linear Strain
   a) Tensile Strain
   b) Compressive Strain
2) Lateral Strain
3) Shear Strain
4) Volumetric Strain

1. Linear Strain

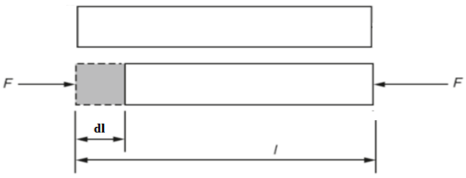
A straight bar will change in length when loaded axially, becoming longer when in tension and shorter when in compression. This change in dimensions in axial direction is known as Linear Strain.

Tensile Strain,



$$\text{Tensile Strain } \varepsilon = \frac{\text{Change in length (Extension)}}{\text{Original length}} = \frac{dl}{l} = \frac{\delta l}{l}$$

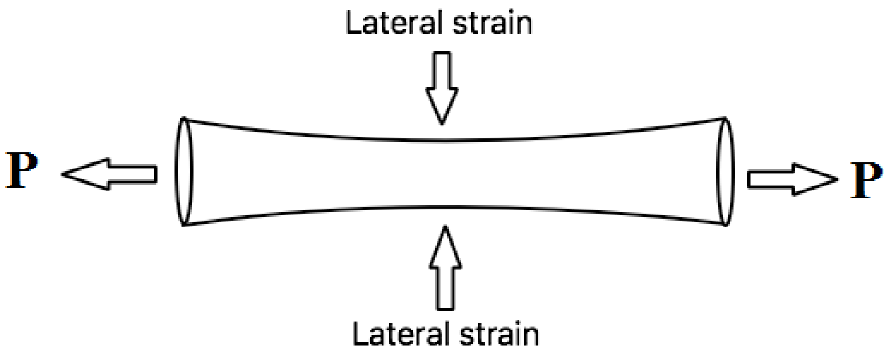
Compressive Strain,



$$\text{Compressive Strain } \varepsilon = \frac{\text{Change in Length (Reduction)}}{\text{Original Length}} = \frac{dl}{l} = \frac{\delta l}{l}$$

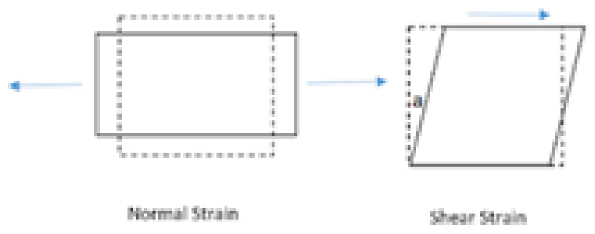
2. Lateral Strain

Lateral strain, also known as transverse strain, which takes place at right angles to the direction of applied load is known as lateral strain.



3. Shear Strain

Shear strain is the ratio of deformation to original dimensions. In the case of shear strain, it is the amount of deformation perpendicular to a given line rather than parallel to it.



4. Volumetric Strain

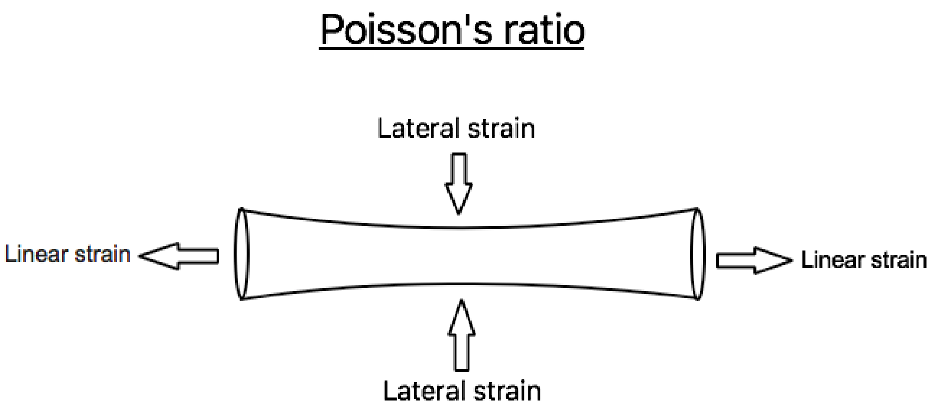
It is the ratio of change in volume to its original volume

$$\text{Volumetric Strain, } \varepsilon_v = \frac{\delta v}{v}$$

1.5.3 Poison's ratio

It is the ratio of lateral strain to linear strain

$$\text{Poison's ratio } \mu = \frac{\text{Lateral Strain}}{\text{Linear Strain}}$$



1.5.4 Hook's Law

It states that **"When a material is loaded within its elastic limit, stress is directly proportional to the strain"**

$$\text{Stress} \; \alpha \; \text{Strain}$$

$$\text{i.e } \frac{\text{Stress}}{\text{Strain}} = \text{Constant}$$

$$\text{i.e } \frac{\sigma}{\varepsilon} = E$$

Where,

E = A constant of proportionality known as Modulus of Elasticity E

σ = Stress & ε = Strain

Hook's law holds good for tension as well as compression.

1.5.5 Modulus of Elasticity or Young's Modulus (E)

Modulus of Elasticity or Young's Modulus (E) is the constant of proportionality and is defined as the ratio of linear stress to linear strain within elastic limit.

$$\text{Modulus of Elasticity, } E = \frac{\text{Linear stress (Tensile or Compressive)}}{\text{Linear Strain (Tensile or Compressive)}} = \frac{\sigma}{\varepsilon}$$

$$\therefore E = \frac{\sigma}{\varepsilon} \text{ MPa or GPa}$$

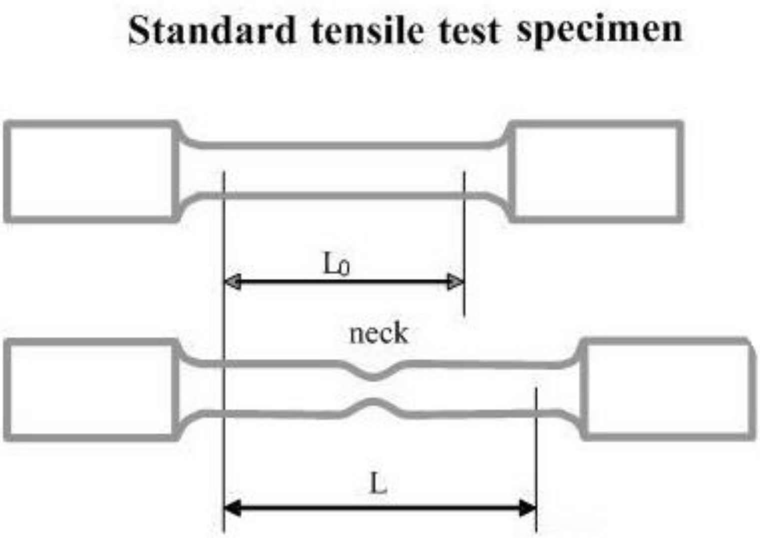
1.5.6 Factor of Safety (FOS)

It is defined as the ratio of ultimate stress or yield stress to the working or allowable or design stress

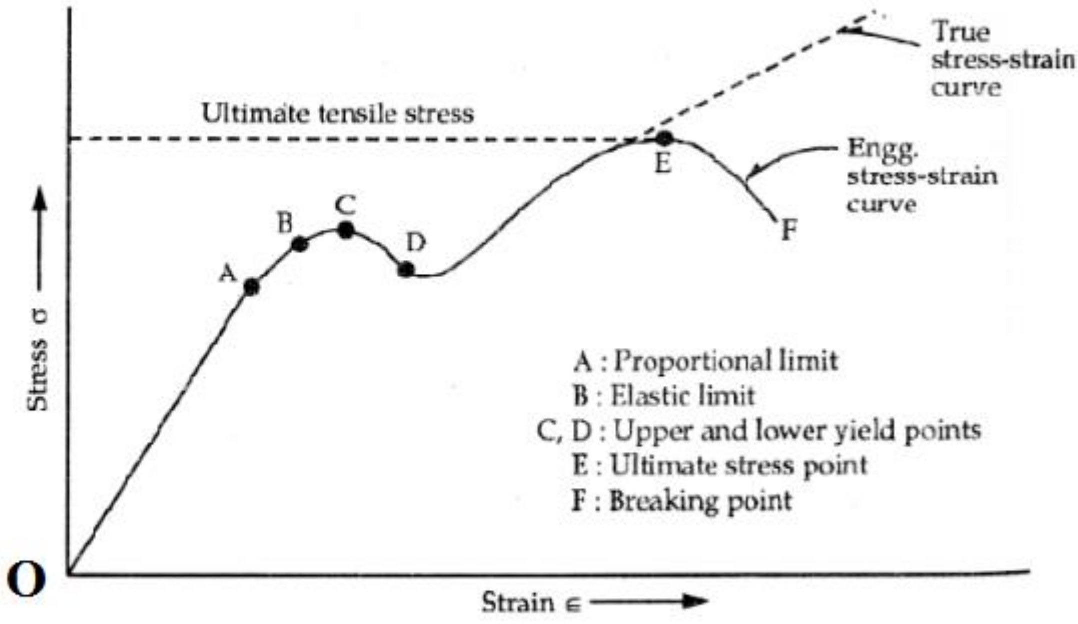
$$\text{FOS} = \frac{\text{Ultimate or Yield Stress}}{\text{Working or Allowable or Design Stress}}$$

1.6 Stress – Strain Relation or Diagram for Ductile Material (Mild Steel or Low carbon steel)

A stress-strain diagram for a typical structural steel as a specimen in tension is shown in Figure. Strains are plotted on the horizontal axis and stresses on the vertical axis.



The load on the test specimen is increased gradually from zero in suitable increments till the specimen fails and the corresponding graph will be computed as shown in the figure below.



Proportional Limit (A)

From O to A the curve is straight and linear and hence proportional limit is the limiting value of stress upto which stress is directly proportional to strain and hence Hooke's law holds good upto point A.

$$\text{Stress} \ \alpha \ \text{Strain}$$

Elastic Limit (B)

The point B is slightly beyond point A and is known as Elastic limit. Upto point B, the material will regain its original size and shape when load is removed. This indicates that the material has elastic properties upto point B.

Upper Yield Point (C)

If the material is stressed beyond point B, plastic deformation starts and the material does not regain its original size and shape upon unload and this phenomenon is called as **Yielding**.

A point at which Maximum load or stress required to initiate the plastic deformation or yielding of the material is called as **Upper yield point "C"**. At this point the dislocations or slip in the crystalline structure starts moving.

Lower Yield Point (D)

As the dislocations or slip is taking place in the material, it offers less resistance to the material and hence curve falls slightly.

A point at which minimum load or stress required to maintain the plastic deformation or yielding of the material is called as **Lower yield point "D"** and this point depicts the end of plastic deformation of the material.

Dislocations or slip become too much in number and they restrict each other's movement.

Ultimate Stress point (E)

After Lower Yield point D, Strain Hardening in the materials takes place. **Strain hardening**, also known as **work hardening**, is the strengthening of a metal occurs because of dislocation movements within the crystal structure of the material and hence there is a positive rise in curve from D to E. In this region as stress increases strain also increases

At point E the specimen takes maximum load, and the corresponding stress at point E is called the **ultimate stress point "E"**.

Breaking Point (F)

Beyond the ultimate stress point is reached **Necking** takes place and the cross sectional area considerably decreases, the load carrying capacity of the specimen reduces and hence in the portion E to F the strain increases with decrease in stress. At point F the specimen breaks. The stress at this point is called breaking stress or fracture stress.

1.7 True Stress - Strain and Engineering Stress - Strain

Let P be the load, A_o be the original area of Cross-section, A be the area of cross-section at any instant.

Engineering stress is the applied load divided by the original cross-sectional area of a material. Also known as nominal stress.

$$\text{Engineering Stress } \sigma = \frac{\text{Load}}{\text{Original Area of Cross-section}} = \frac{P}{A_o} \frac{N}{mm^2}$$

True stress is the applied load divided by the actual cross-sectional area (the changing area with respect to time) of the specimen at that load

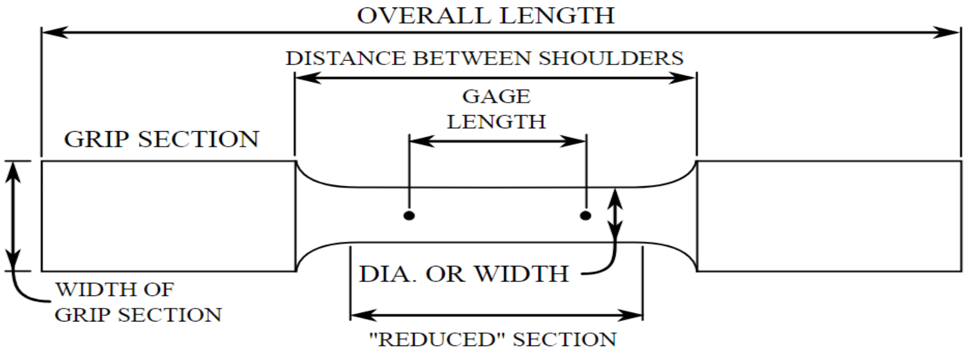
$$\text{True Stress } \sigma = \frac{\text{Load}}{\text{Actual Area of Cross-section at any instant}} = \frac{P}{A} \frac{N}{mm^2}$$

Engineering strain is the change in length to its original length in a tensile test. Also known as nominal strain.

$$\text{Engineering Strain } \varepsilon = \frac{\delta l}{l}$$

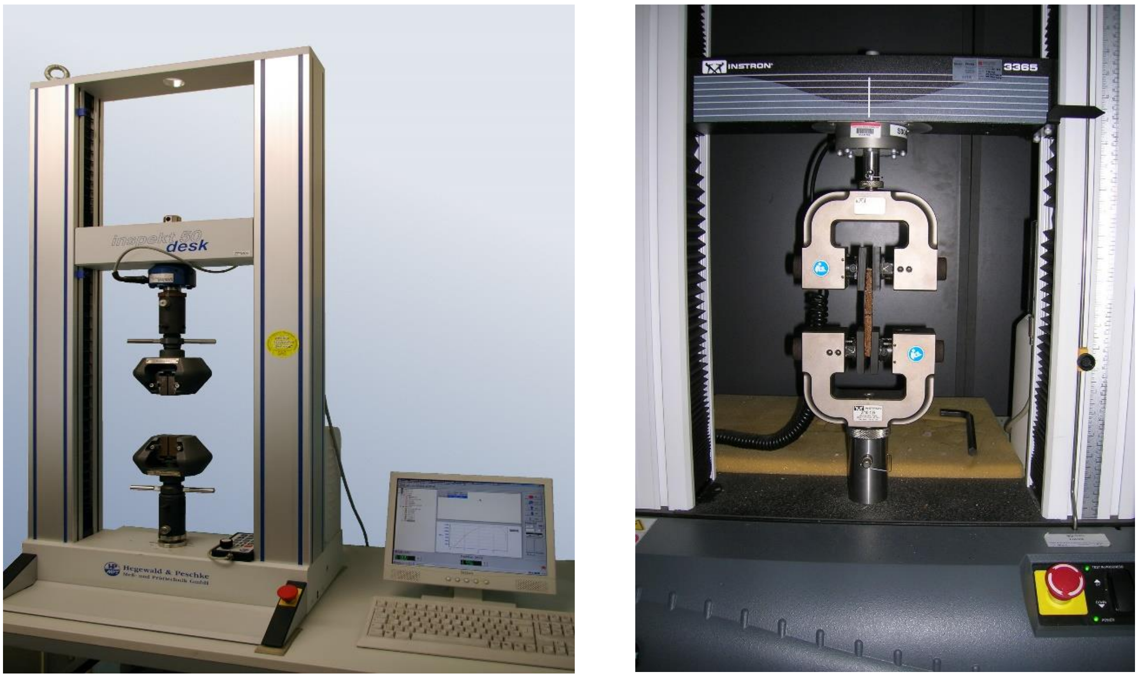
True strain is the sum of all the strains over the original length. True Strain $\varepsilon = \sum \frac{\delta l}{l}$

Additional Information – Photo Gallery

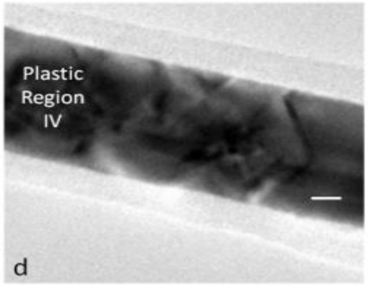


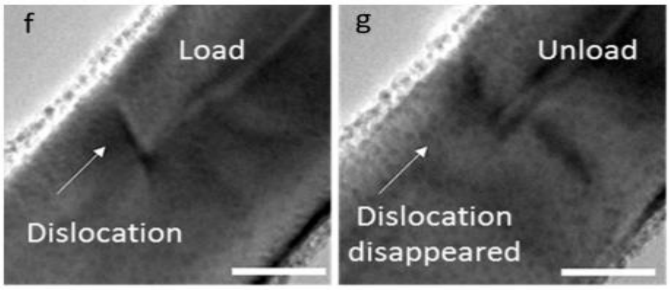
Detailed information for specimen preparation

Various Specimens



UTM for testing and Computer to plot Stress-Strain curve.

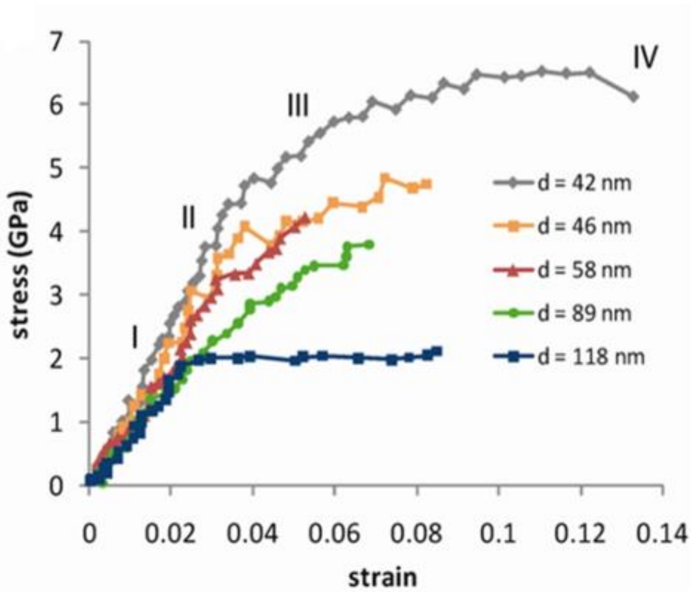




Location of plastic region

Location of Dislocation upon Loading and Unloading

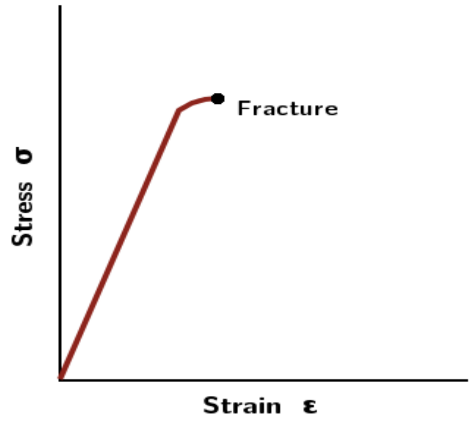
Necking in Tensile specimen



Specimens undergoing test

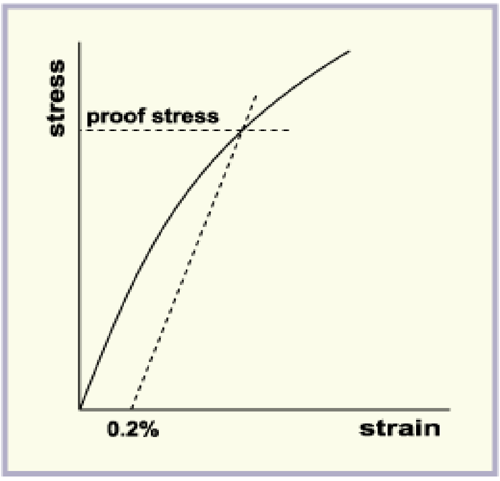
Actual Stress-Strain curve

1.7 Stress – Strain Relation or Diagram for Brittle Material



Brittle materials, which includes cast iron, glass, and stone do not have a yield point, and do not strain-harden. Therefore, the ultimate strength and breaking strength are the same. A typical stress–strain curve is shown in the figure.

Proof Stress:



For materials which do not have clearly defined yield point, an arbitrary yield point is defined by drawing a line which is offset by a certain strain value and is parallel to the original stress-strain line (within proportional limit). The strain by which line is offset can be 0.1% or 0.2% and the corresponding stress is the Proof Stress at 0.1% or 0.2% strain respectively.

Additional formulas to remember:

Percentage Elongation in Length,

$$\text{\% Elongation} = \frac{\text{Final Length} - \text{Initial Length}}{\text{Initial Length}} \text{ x } 100$$

$$\text{\% Elongation} = \frac{L_f - L_i}{L_i} \text{ x } 100$$

Percentage Reduction in Area,

$$\text{\% Reduction} = \frac{\text{Initial Area} - \text{Final Area}}{\text{Initial Area}} \text{ x } 100$$

$$\text{\% Reduction} = \frac{A_i - A_f}{A_i} \text{ x } 100$$

1.8 Problems

1. The following data refer to a mild steel specimen tested in a laboratory

- **Diameter of the specimen - 25mm**
- **Length of the specimen – 300 mm**
- **Extension under a load of 15kN – 0.045 mm**
- **Load at yield point – 127.65kN**
- **Maximum load – 208.60kN**
- **Length of the specimen after failure – 375mm**
- **Neck diameter – 17.75mm**

Determine Young's modulus, Yield stress, Ultimate stress, % Elongation, %Reduction in area, safe or permissible stress adopting a factor of safety 2

Given Data:

d_o = 25mm, d_f = 17.75mm, L_o = 300mm, L_f = 375mm, δl = 0.045mm, F_{max} = 208.60kN

Area of the specimen, $A = \frac{\pi {d_o}^2}{4} = \frac{\pi x 25^2}{4} = \mathbf{490.87\ mm^2}$

$\sigma = \frac{F}{A} = \frac{15x10^3}{490.87} = \mathbf{30.55\ \frac{N}{mm^2}}$

$\varepsilon = \frac{\delta l}{l} = \frac{0.045}{300} = \mathbf{1.5x10^{-4}}$

→Young's Modulus $E = \frac{\sigma}{\varepsilon} = \frac{30.55}{1.5x10^{-4}} = \mathbf{203.66x10^3\ \frac{N}{mm^2}}$

→ Yield stress $\sigma_y = \frac{F}{A} = \frac{127.65x10^3}{490.87} = \mathbf{260.04\ \frac{N}{mm^2}}$

→ Ultimate stress $\sigma_u = \frac{F}{A} = \frac{208.60x10^3}{490.87} = \mathbf{424.95\ \frac{N}{mm^2}}$

→ % Elongation = $\frac{L_f - L_i}{L_i}$ x 100 = $\frac{375-300}{300}$ x100 = **25%**

→% Reduction = $\frac{A_i - A_f}{A_i}$ x 100

$$A_f = \frac{\pi {d_f}^2}{4} = \frac{\pi x 17.75^2}{4} = \mathbf{247.44\ mm^2}$$

$\therefore \quad \frac{490.87-247.44}{490.87}$ x100 = **49.59 %**

$\rightarrow$ FOS = $\frac{\text{Ultimate or Yield Stress}}{\text{Working or Allowable or Design Stress}}$

Note:

In question they have asked for safe permissible stress hence take yield stress for calculation

For Maximum permissible stress take ultimate stress for calculation

$$\therefore \text{ FOS} = \frac{\text{Yield Stress}}{\text{Working or Allowable or Design Stress}}$$

Wkt, FOS = 2

$$2 = \frac{260.04}{\text{Working or Allowable or Design Stress}}$$

$$\text{Working stress or Design stress} = \frac{260.04}{2} = \mathbf{130.02\,\frac{N}{mm^2}}$$

2. A rod 150 cm long and a diameter 2 cm is subjected to an axial pull of 20kN. If the modulus of elasticity of material is 200 GPa. Determine stress, strain and Elongation of rod.

Given data:

l = 150 cm = 1500 mm

d = 2 cm = 20 mm

P = 20kN = $20\text{x}10^3$ N

E = 200 GPa = $200\text{x}10^3 \, \frac{\text{N}}{\text{mm}^2}$

$$\text{Area} = \frac{\pi d^2}{4} = \frac{\pi 20^2}{4} = \mathbf{314.15\ mm^2}$$

$$\rightarrow \sigma = \frac{\text{F}}{\text{A}} = \frac{20x10^3}{314.15} = \mathbf{63.66\,\frac{N}{mm^2}}$$

$$\rightarrow \text{E} = \frac{\sigma}{\varepsilon} \longrightarrow \quad \varepsilon = \frac{\sigma}{\text{E}} = \frac{63.66}{200\text{x}10^3} = \mathbf{3.183x10^{-4}}$$

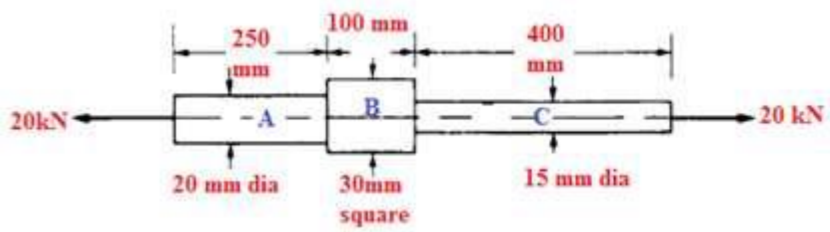
$$\rightarrow \varepsilon = \frac{\delta l}{l} \longrightarrow \quad \delta l = \varepsilon \text{ x } l = 3.183\text{x}10^{-4} \text{ x } 1500$$

$$\mathbf{\delta l = 0.477\ mm}$$

3. A bar of a rectangular section of 20 mm × 30 mm and a length of 500 mm is subjected to an axial compressive load of 60 kN. If E = 102 kN/mm2 and v = 0.34, determine the changes in the length and the sides of the bar.

- Since the bar is subjected to compression, there will be decrease in length, increase in breadth and depth. These are computed as shown below
- L = 500 mm, b = 20 mm, d = 30 mm, P = 60 x1000 = 60000 N, E = 102000 N/mm^2
- Cross-sectional area A = 20 × 30 = 600 mm^2
- Compressive stress σ = P/A = 60000/600 = 100 N/mm^2
- Longitudinal strain $\varepsilon_L = \sigma/E$ = 100/102000 = 0.00098
- Lateral strain $\varepsilon_{lat} = \nu\ \varepsilon_L$ = 0.34 x 0.00098 = 0.00033
- Decrease in length $\delta L = \varepsilon_L L$ = 0.00098 x 500 = 0.49 mm
- Increase in breadth $\delta b = \varepsilon_{lat}\, b$ = 0.00033 x 20 = 0.0066 mm
- Increase in depth $\delta d = \varepsilon_{lat}\, d$ = 0.00033 x 30 = 0.0099 mm

4. Determine the stress in each section of the bar shown in the following figure when subjected to an axial tensile load of 20 kN. The central section is of square cross-section; the other portions are of circular section. What will be the total extension of the bar? For the bar material E = 210000MPa.



The bar consists of three sections with change in diameter. Loads are applied only at the ends. The stress and deformation in each section of the bar are computed separately. The total extension of the bar is then obtained as the sum of extensions of all the three sections. These are illustrated in the following steps.

The bar is in equilibrium under the action of applied forces

Stress in each section of bar = P/A and P = 20000N

i. Area of Bar A = π x 202/4 = 314.16 mm2

ii. Stress in Bar A : σA = 20000/ 314.16 = 63.66MPa

iii. Area of Bar B = 30 x30 = 900 mm2

iv. Stress in Bar B : σ_B = 20000/ 900 = 22.22MPa

v. Area of Bar C = π x 15$_2$/4 = 176.715 mm$_2$

vi. Stress in Bar C : σ_C = 20000/ 176.715 = 113.18MPa

Extension of each section of bar = σL/E and E = 210000 MPa

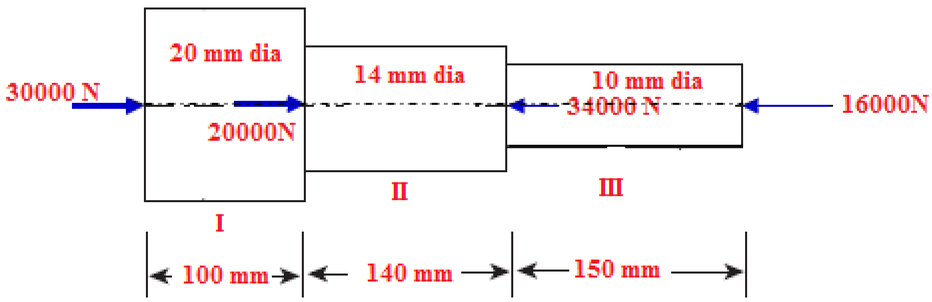
i. Extension of Bar A = 63.66 x 250 / 210000= 0.0758 mm

ii. Extension of Bar B = 22.22 x 100 / 210000= 0.0106 mm

iii. Extension of Bar C = 113.18 x 400 / 210000= 0.2155 mm

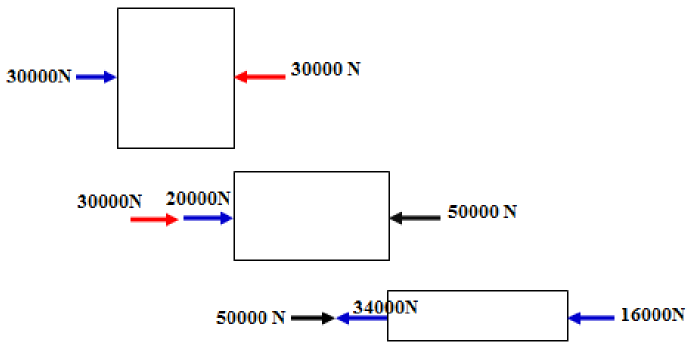
Total extension of the bar = **0. 302mm**

5. Determine the overall change in length of the bar shown in the figure below with following data: E = 100000 N/mm$_2$



The bar is with varying cross-sections and subjected to forces at ends as well as at other interior locations. It is necessary to study the equilibrium of each portion separately and compute the change in length in each portion. The total change in length of the bar is then obtained as the sum of extensions of all the three sections as shown below.

Forces acting on each portion of the bar for equilibrium



Sectional Areas

$A_I = \frac{\pi \times 20^2}{4} = 314.16\ mm^2$; $A_{II} = \frac{\pi \times 14^2}{4} = 153.94\ mm^2$; $A_{III} = \frac{\pi \times 10^2}{4} = 78.54\ mm^2$

Change in length in Portion I

Portion I of the bar is subjected to an axial compression of 30000N. This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_I L_I}{A_I E} = \frac{30000 \times 100}{314.16 \times 100000} = 0.096\ mm$$

Change in length in Portion II

Portion II of the bar is subjected to an axial compression of 50000N (30000 + 20000). This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_{II} L_{II}}{A_{II} E} = \frac{50000 \times 140}{153.94 \times 100000} = 0.455mm$$

Change in length in Portion III

Portion III of the bar is subjected to an axial compression of (50000 – 34000) = 16000N. This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_{III} L_{III}}{A_{III} E} = \frac{16000 \times 150}{78.54 \times 100000} = 0.306mm$$

Since each portion of the bar results in decrease in length, they can be added without any algebraic signs.

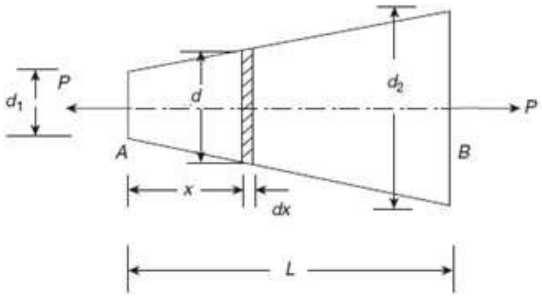
Hence Total decrease in length = 0.096 + 0.455 + 0.306 = **0.857mm**

Note:

For equilibrium, if some portion of the bar may be subjected to tension and some other portion to compression resulting in increase or decrease in length in different portions of the bar. In such cases, the total change in length is computed as the sum of change in length of each portion of the bar with proper algebraic signs. Generally positive sign (+) is used for increase in length and negative sign (-) for decrease in length.

1.9 Elongation of tapering bars of circular cross section

Consider a circular bar uniformly tapered from diameter **d1** at one end and gradually increasing to diameter **d2** at the other end over an axial length **L** as shown in the figure below.



Since the diameter of the bar is continuously changing, the elongation is first computed over an elementary length and then integrated over the entire length. Consider an elementary strip of diameter ***d*** and length ***dx*** at a distance of ***x*** from end ***A***.

Using the principle of similar triangles the following equation for d can be obtained

$$d = d_1 + \frac{d_2 - d_1}{L} x = d_1 + kx \text{ , } where\ k = \frac{d_2 - d_1}{L}$$

Cross–sectional area of the bar at x : $A_x = \dfrac{\pi\,(d_1+kx)^2}{4}$

Axial stress at x:$\sigma_x = \dfrac{P}{A_x} = \dfrac{4P}{\pi\,(d_1+kx)^2}$

Change in length over dx :$\delta dx = \dfrac{\sigma_x\, dx}{E} = \dfrac{4P\, dx}{\pi E\,(d_1+kx)^2}$

Total change in length: $\delta L = \int_0^L \dfrac{4P\, dx}{\pi E\,(d_1+kx)^2} = \dfrac{4P}{\pi E}\left[\dfrac{(d_1+kx)^{-1}}{-k}\right]_0^L$

$After\ rearranging\ the\ terms: \delta L = -\dfrac{4P}{\pi E k}\left[\dfrac{1}{(d_1+kx)}\right]_0^L$

$Upon\ substituting\ the\ limits:\ \delta L = -\dfrac{4P}{\pi E k}\left[\dfrac{1}{(d_1+kL)} - \dfrac{1}{d_1}\right]$

$After\ rearranging\ the\ terms: \delta L = \dfrac{4P}{\pi E k}\left[\dfrac{1}{d_1} - \dfrac{1}{(d_1+kL)}\right]$

$But\ (d_1 + kL) = d_1 + \dfrac{d_2 - d_1}{L} L = d_2$

$With\ the\ above\ substitution: \delta L = \dfrac{4P}{\pi E k}\left[\dfrac{1}{d_1} - \dfrac{1}{d_2}\right] = \dfrac{4P}{\pi E k}\left[\dfrac{d_2 - d_1}{d_1 d_2}\right]$

Substituting for $k = \frac{d_2 - d_1}{L}$ in the above expression, following equation for elongation of tapering bar of circular section can be obtained

Total change in length: $\delta L = \dfrac{4P\,L}{\pi E\, d_1 d_2}$

Problem:

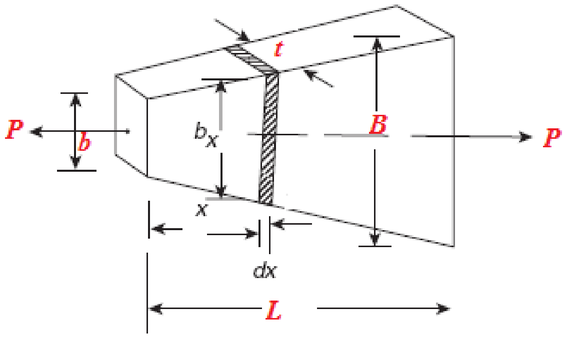
A bar uniformly tapers from diameter 20 mm at one end to diameter 10 mm at the other end over an axial length 300 mm. This is subjected to an axial compressive load of 7.5 kN. If *E* = 100 kN/mm2, determine the maximum and minimum axial stresses in bar and the total change in length of the bar.

$P = 7500$ N, $E = 100000$ N/mm^2, $d_1 = 10$mm, $d_2 = 20$mm, $L = 300$mm

- Minimum compressive stress occurs at $d_2 = 20$mm as the sectional area is maximum.
- Area at $d_2 = \frac{\pi \times 20^2}{4} = 314.16mm^2$
- $\sigma_{min} = \frac{7500}{314.16} = 23.87MPa$
- Maximum compressive stress occurs at $d_1 = 10$mm as the sectional area is minimum.
- Area at $d_1 = \frac{\pi \times 10^2}{4} = 78.54mm^2$
- $\sigma_{min} = \frac{7500}{78.54} = 95.5MPa$
- Total decrease in length: $\delta L = \frac{4PL}{\pi E\, d_1 d_2} = \frac{4 \times 7500 \times 300}{\pi \times 100000 \times 10 \times 20} = 0.143\text{mm}$

1.10 Elongation of tapering bars of rectangular cross section

Consider a bar of same thickness **t** throughout its length, tapering uniformly from a breadth **B** at one end to a breadth **b** at the other end over an axial length **L**. The flat is subjected to an axial force **P** as shown in the figure below.



Since the breadth of the bar is continuously changing, the elongation is first computed over an elementary length and then integrated over the entire length. Consider an elementary strip of breadth ***bx*** and length ***dx*** at a distance of ***x*** from left end.

Using the principle of similar triangles the following equation for *bx* can be obtained

$$b_x = b + \frac{B-b}{L}x = b + kx \text{ , where } k = \frac{B-b}{L}$$

Cross–sectional area of the bar at x : $A_x = b_x t = (b + kx)t$

Axial stress at x:$\sigma_x = \frac{P}{A_x} = \frac{P}{(b+kx)t}$

Change in length over dx :$\delta dx = \frac{\sigma_x \, dx}{E} = \frac{P\,dx}{Et(b+kx)}$

Total change in length: $\delta L = \int_0^L \frac{P\,dx}{Et(b+kx)} = \frac{P}{Etk}[ln(b + kx)]_0^L$

Upon substituting the limits : $\delta L = \frac{P}{Etk}[ln(b + kL) - \ln(b)]$

$But\ (b + kL) = b + \frac{B-b}{L} L = B$

With the above substitution: $\delta L = \frac{P}{Etk}[ln(B) - \ln(b)] = \frac{P}{Etk}\ln(B/b)$

Substituting for $k = \frac{B-b}{L}$ in the above expression, following equation for elongation of tapering bar of rectangular section can be obtained

$$\delta L = \frac{P\,L}{Et(B-b)}\ln(B/b)$$

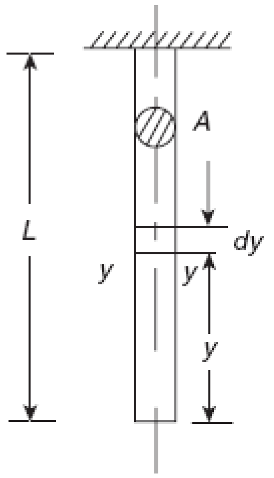
Problem

An aluminium flat of a thickness of 8 mm and an axial length of 500 mm has a width of 15 mm tapering to 25 mm over the total length. It is subjected to an axial compressive force *P*, so that the total change in the length of flat does not exceed 0.25 mm. What is the magnitude of *P*, if *E* = 67,000 N/mm2 for aluminium?

t = 8mm, B = 25mm,b = 15mm, L = 500 mm, δL = 0.25 mm, E = 67000MPa, P =?

$$P = \frac{Et(B-b)\delta L}{ln(B/b)\,L} = \frac{67000 \times 8 \times (25-15) \times 0.25}{ln(25/15) \times 500} = 5.246kN$$

1.11 Elongation in Bar Due to Self-Weight



Consider a bar of a cross-sectional area of **A** and a length **L** is suspended vertically with its upper end rigidly fixed as shown in the adjoining figure. Let the weight density of the bar is ρ. Consider a section y- y at a distance y from the lower end.

Weight of the portion of the bar below y-y = ρ A y

Stress at y-y : σy = ρ A y /A = ρ y

Strain at y-y : εy = ρ y / E

Change in length over dy: δdy = ρ y dy / E

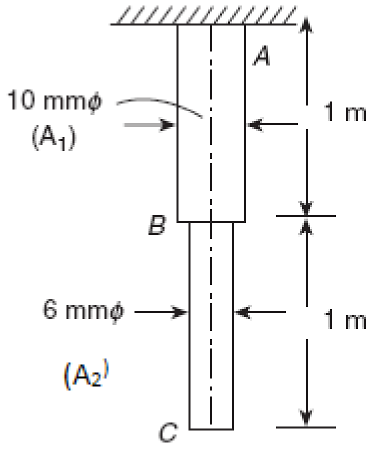
Total change in length : $\delta L = \int_0^L \frac{\rho\, y\, dy}{E} = [\frac{\rho y^2}{2E}]_0^L = \frac{\rho L^2}{2E}$

This can also be written as : $\delta L = \frac{(\rho AL)L}{2AE} = \frac{WL}{2AE}$

W = ρ A L represents the total weight of the bar

Problem:

A stepped steel bar is suspended vertically. The diameter in the upper half portion is 10 mm, while the diameter in the lower half portion is 6 mm. What are the stresses due to self-weight in sections B and A as shown in the figure. E = 200 kN/mm^2. Weight density, ρ = 0.7644x10^{-3} N/mm^3. What is the change in its length if E = 200000 MPa?



Stress at B will be due to weight of portion of the bar BC

Sectional area of BC: $A_2 = \pi \times 6^2/4 = 28.27$ mm^2

Weight of portion BC: $W_2 = \rho\ A_2\ L_2 = 0.7644\times 10^{-3} \times 28.27 \times 1000 = 21.61$N

Stress at B: $\sigma_B = W_2/A_2 = 21.61/28.27 =$ **0.764 MPa**

Stress at A will be due to weight of portion of the bar BC + AB

Sectional area of AB: $A_1 = \pi \times 10^2/4 = 78.54$ mm^2

Weight of portion AB: $W_1 = \rho\ A_1\ L_1 = 0.7644\times 10^{-3} \times 78.54 \times 1000 = 60.04$N

Stress at A: $\sigma_c = (W_1+W_2)/A_1 = (60.04+ 21.61) / 78.54 =$ **1.04 MPa**

Change in Length in portion BC

This is caused due to weight of BC and is computed as:

$$\delta L_{BC} = \frac{W_2 L_2}{2A_2 E} = \frac{21.61\times1000}{2\times28.27\times200000} = 0.00191\text{mm}$$

Change in Length in portion AB

This is caused due to weight of AB and due to weight of BC acting as a concentrated load at B and is computed as:

$$\delta L_{AB} = \frac{W_1 L_1}{2A_1 E} + \frac{W_2 L_1}{E\ A_1} = \frac{60.04\times1000}{2\times78.54\times200000} + \frac{21.61\times1000}{200000\times78.54} = 0.0033\text{mm}$$

Total change in length = 0.00191+ 0.0033 = **0.00521mm**

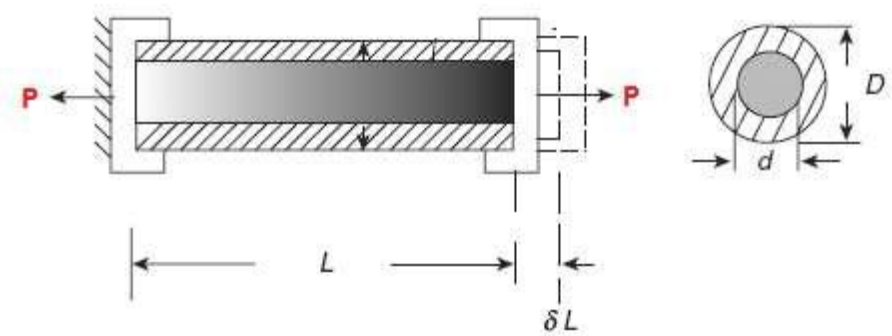
1.12 Compound or composite bars

A composite bar can be made of two bars of different materials rigidly fixed together so that both bars strain together under external load. As the strains in the two bars are same, the stresses in the two bars will be different and depend on their respective modulus of elasticity. A stiffer bar will share major part of external load.

In a composite system the two bars of different materials may act as suspenders to a third rigid bar subjected to loading. As the change in length of both bars is the same, different stresses are produced in two bars.

1.10.1 Stresses in a Composite Bar

Let us consider a composite bar consisting of a solid bar, of diameter ***d*** completely encased in a hollow tube of outer diameter ***D*** and inner diameter ***d*,** subjected to a tensile force **P** as shown in the following figure.



Let the extension of composite bar of length L be δL. Let E_S and E_H be the modulus of elasticity of solid bar and hollow tube respectively. Let σ_S and σ_H be the stresses developed in the solid bar and hollow tube respectively.

Since change in length of solid bar is equal to the change in length of hollow tube, we can establish the relation between the stresses in solid bar and hollow tube as shown below :

$$\frac{\sigma_S L}{E_S} = \frac{\sigma_H L}{E_H} \text{ or } \sigma_S = \sigma_H \frac{E_S}{E_H}$$

$$\frac{\sigma_S L}{E_S} = \frac{\sigma_H L}{E_H} \text{ or } \sigma_S = \sigma_H \frac{E_S}{E_H}$$

Area of cross section of the hollow tube : $A_H = \frac{\pi(D^2 - d^2)}{4}$

Area of cross section of the solid bar : $A_S = \frac{\pi d^2}{4}$

Load carried by the hollow tube : $P_H = \sigma_H A_H$ and Load carried by the solid bar : $P_S = \sigma_S A_S$

But $P = P_S + P_H = \sigma_S A_S + \sigma_H A_H$

With $\sigma_S = \sigma_H \frac{E_S}{E_H}$, the following equation can be written

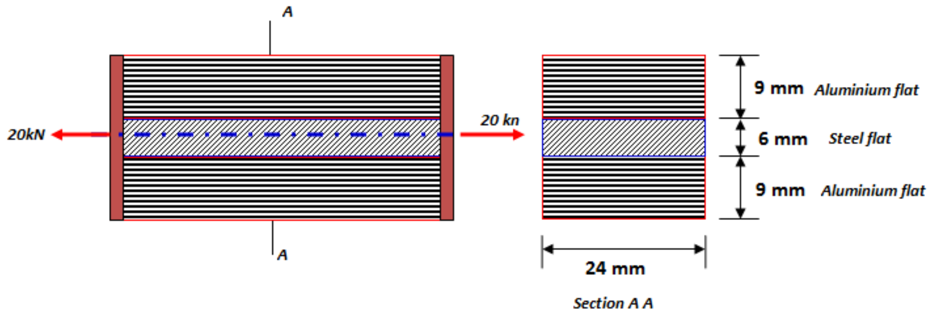
$$P = \sigma_H \frac{E_S}{E_H} A_s + \sigma_H A_H = \sigma_H \left(A_H + \frac{E_S}{E_H} A_s\right)$$

E_S/E_H is called *modular ratio*. Using the above equation stress in the hollow tube can be calculated. Next, the stress in the solid bar can be calculated using the equation

$$P = \sigma_S A_S + \sigma_H A_H$$

Problems.

A flat bar of steel of 24 mm wide and 6 mm thick is placed between two aluminium alloy flats 24 mm × 9 mm each. The three flats are fastened together at their ends. An axial tensile load of 20 kN is applied to the composite bar. What are the stresses developed in steel and aluminium alloy? Assume E_S = 210000 MPa and E_A = 70000MPa.



Area of Steel flat: $A_S = 24 \times 6 = 144\ mm^2$

Area of Aluminium alloy flats: $A_A = 2 \times 24 \times 9 = 432\ mm^2$

Since all the flats elongate by the same extent, we have the condition that $\frac{\sigma_S L}{E_S} = \frac{\sigma_A L}{E_A}$.

The relationship between the stresses in steel and aluminum flats can be established as:

$$\sigma_S = \sigma_A \frac{E_S}{E_A} = 3\,\sigma_A$$

Since $P = P_S + P_A = \sigma_S A_S + \sigma_A A_A$. This can be written as

$$P = 3\sigma_A A_s + \sigma_A A_A = \sigma_A (3A_s + A_A)$$

From which stress in aluminium alloy flat can be computed as:

$$\sigma_A = \frac{P}{(3A_s + A_A)} = \frac{20 \times 1000}{(3 \times 144 + 432)} = \mathbf{23.15 MPa}$$

Stress in steel flat can be computed as:

$$\sigma_S = 3 \times 23.15 = \mathbf{69.45 MPa}$$

2. A short post is made by welding steel plates into a square section and then filling inside with concrete. The side of square is 200 mm and the thickness t = 10 mm as shown in the figure. The steel has an allowable stress of 140 N/mm2 and the concrete has an allowable stress of 12 N/mm2. Determine the allowable safe compressive load on the post. E_C = 20 GPa, E_s = 200 GPa.

Since the composite post is subjected to compressive load, both concrete and steel tube will shorten by the same extent. Using this condition following relation between stresses in concrete and steel can be established.

$$\frac{\sigma_C L}{E_C} = \frac{\sigma_S L}{E_S} \text{ or } \sigma_S = \sigma_C \frac{E_S}{E_C} = 10\,\sigma_C$$

Assume that load is such that σ_s = 140 N/mm^2. Using the above relationship, the stress in concrete corresponding to this load can be calculated as follows:

$$140 = 10\,\sigma_C \quad or \quad \sigma_C = 14\ N/mm^2 > 12\ \text{N/mm}^2$$

Hence the assumed load is not a safe load.

Instead assume that load is such that σ_c = 12 N/mm^2. The stress in steel corresponding to this load can be calculated as follows:

$$\sigma_s = 12 \times 10 \quad or \quad \sigma_s = 120\ N/mm^2 < 140\ \text{N/mm}^2$$

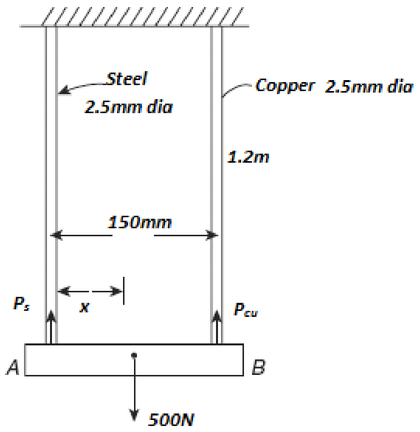
Hence the assumed load is a safe load which is calculated as shown below.

Area of concrete section Ac = 180 x180 = 32400mm^2.

Area of steel tube As = 200 x 200 – 32400 = 7600 mm^2.

$$P = \sigma_C A_C + \sigma_s A_s = 12 \times 32400 + 120 \times 7600 = \mathbf{1300.8}\boldsymbol{kN}$$

3. A rigid bar is suspended from two wires, one of steel and other of copper, length of the wire is 1.2 m and diameter of each is 2.5 mm. A load of 500 N is suspended on the rigid bar such that the rigid bar remains horizontal. If the distance between the wires is 150 mm, determine the location of line of application of load. What are the stresses in each wire and by how much distance the rigid bar comes down? Given $E_s = 3E_{cu} = 201000$ N/mm^2.



i. Area of copper wire (Acu) = Area of steel wire(As) = $\pi \times 2.5^2/4 = 4.91$ mm^2

ii. For the rigid bar to be horizontal, elongation of both the wires must be same. This condition leads to the following relationship between stresses in steel and copper wires as:

$$\sigma_s = \frac{E_s}{E_{cu}}\sigma_{cu} = 3\sigma_{cu}$$

iii. Using force equilibrium, the stress in copper and steel wire can be calculated as:

$$P = P_s + P_{cu} = \sigma_s A_s + \sigma_{cu} A_{cu} = 3\,\sigma_{cu} A_s + \sigma_{cu} A_{cu} = \sigma_{cu}(3A_s + A_{cu})$$

$$\sigma_{cu} = \frac{P}{(A_{cu} + 3A_S)} = \frac{500}{(4.91 + 3 \times 4.91)} = 25.46\ MPa$$

$$\sigma_s = 3 \times 25.46 = 76.37\ MPa$$

iv. Downward movement of rigid bar = elongation of wires

$$\delta L_s = \frac{\sigma_s}{E_s}L = \frac{76.37}{201000} \times 1200 = 0.456\ mm$$

v. Position of load on the rigid bar is computed by equating moments of forces carried by steel and copper wires about the point of application of load on the rigid bar.

$$P_s\, x = P_c\,(150 - x)$$
$$(76.37 \times 4.91)x = (25.46 \times 4.91)\,(150 - x)$$
$$\frac{x}{150 - x} = 0.333$$

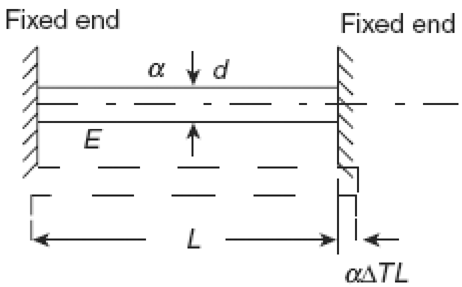
$$x = \mathbf{37.47mm}\ \textit{from steel wire}$$

Note:

If the load is suspended at the centre of rigid bar, then both steel and copper wire carry the same load. Hence the stress in the wires is also same. As the moduli of elasticity of wires are different, strains in the wires will be different. This results in unequal elongation of wires causing the rigid bar to rotate by some magnitude. This can be prevented by offsetting the load or with wires having different length or with different diameter such that elongation of wires will be same.

1.13 Temperature stresses in a single bar

If a bar is held between two unyielding (rigid) supports and its temperature is raised, then a compressive stress is developed in the bar as its free thermal expansion is prevented by the rigid supports. Similarly, if its temperature is reduced, then a tensile stress is developed in the bar as its free thermal contraction is prevented by the rigid supports. Let us consider a bar of diameter ***d*** and length ***L*** rigidly held between two supports as shown in the following figure. Let ***α*** be the coefficient of linear expansion of the bar and its temperature is raised by **Δ*T*** (°C)



• Free thermal expansion in the bar = $\alpha\, \Delta T\, L$.

• Since the supports are rigid, the final length of the bar does not change. The fixed ends exert compressive force on the bar so as to cause shortening of the bar by $\alpha\, \Delta T\, L$.

• Hence the compressive strain in the bar = $\alpha\, \Delta T\, L / L = \alpha\, \Delta T$

• Compressive stress = $\alpha\, \Delta T\, E$

• Hence the thermal stresses introduced in the bar = $\alpha\, \Delta T\, E$

Note:

The bar can buckle due to large compressive forces generated in the bar due to temperature increase or may fracture due to large tensile forces generated due to temperature decrease.

Problem

A rail line is laid at an ambient temperature of 30°C. The rails are 30 m long and there is a clearance of 5 mm between the rails. If the temperature of the rail rises to 60°C, what is the stress developed in the rails?. Assume $\alpha = 11.5 \times 10^{-6}$/°C, E = *2,10,000 N/mm*^{2}

• L = 30,000 mm, α = 11.5×10^{-6}/°C, Temperature rise ΔT = 60-30 = 30^{o}C

• Free expansion of rails = $\alpha\, \Delta T\, L = 11.5 \times 10^{-6} \times 30 \times 30000 = 10.35$mm

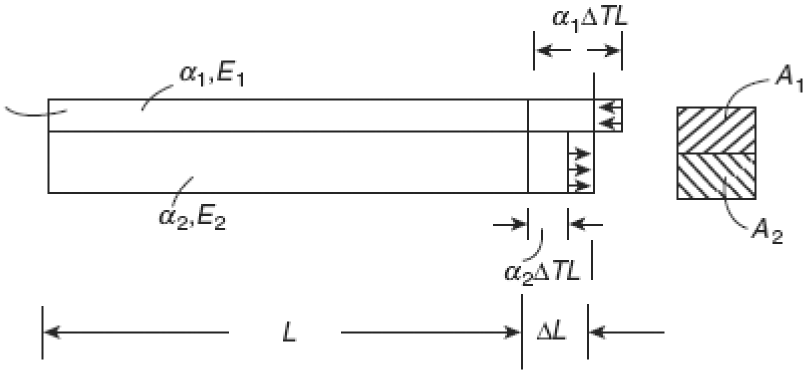
• Thermal expansion prevented by rails = Free expansion – clearance = 10.35 – 5 = 5.35mm

• Strain in the rails ε = 5.35/30000 = 0.000178

• Compressive stress in the rails = ε x E = 0.000178 x 210000 =**37.45N/mm^2**.

1.14 Temperature Stresses in a Composite Bar

A composite bar is made up of two bars of different materials perfectly joined together so that during temperature change both the bars expand or contract by the same amount. Since the coefficient of expansion of the two bars is different thermal stresses are developed in both the bars. Consider a composite bar of different materials with coefficients of expansion and modulus of elasticity, as $\boldsymbol{\alpha_1}$, $\boldsymbol{E_1}$ and $\boldsymbol{\alpha_2}$, $\boldsymbol{E_2}$, respectively, as shown in the following figure. Let the temperature of the bar is raised by $\boldsymbol{\Delta T}$ *and* $\boldsymbol{\alpha_1} > \boldsymbol{\alpha_2}$



Free expansion in bar 1 = $\alpha_1 \Delta T\, L$ and Free expansion in bar 2 = $\alpha_2 \Delta T\, L$. Since both the bars expand by ΔL together we have the following conditions:

- Bar 1: $\Delta L < \alpha_1 \Delta T\, L$. The bar gets compressed resulting in compressive stress
- Bar 2: $\Delta L > \alpha_2 \Delta T\, L$. The bar gets stretched resulting in tensile stress.

$$\text{Compressive strain in Bar 1 : } \varepsilon_1 = \frac{\alpha_1 \Delta T L - \Delta L}{L}$$

$$\text{Tensile strain in Bar 2 : } \varepsilon_2 = \frac{\Delta L - \alpha_2 \Delta T L}{L}$$

$$\varepsilon_1 + \varepsilon_2 = \frac{\alpha_1 \Delta T L - \Delta L}{L} + \frac{\Delta L - \alpha_2 \Delta T L}{L} = (\alpha_1 - \alpha_2)\Delta T$$

Let σ_1 and σ_2 be the temperature stresses in bars. The above equation can be written as:

$$\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2} = (\alpha_1 - \alpha_2)\Delta T$$

In the absence of external forces, for equilibrium, compressive force in Bar 1 = Tensile force in Bar 2. This condition leads to the following relation

$$\sigma_1 A_1 = \sigma_2 A_2$$

Using the above two equations, temperature stresses in both the bars can be computed. This is illustrated in the following example.

Note:

If the temperature of the composite bar is reduced, then a tensile stress will be developed in bar 1 and a compressive stress will be developed in bar 2 , since $\alpha_1 > \alpha_2$.

Problems

1 A steel flat of 20 mm × 10 mm is fixed with aluminium flat of 20 mm × 10 mm so as to make a square section of 20 mm × 20 mm. The two bars are fastened together at their ends at a temperature of 26°C. Now the temperature of whole assembly is raised to 55°C. Find the stress in each bar. E_s = 200 GPa, E_a = 70 GPa, α_s = 11.6 × 10^{-6}/°C, α_a = 23.2 × 10^{-6}/°C.

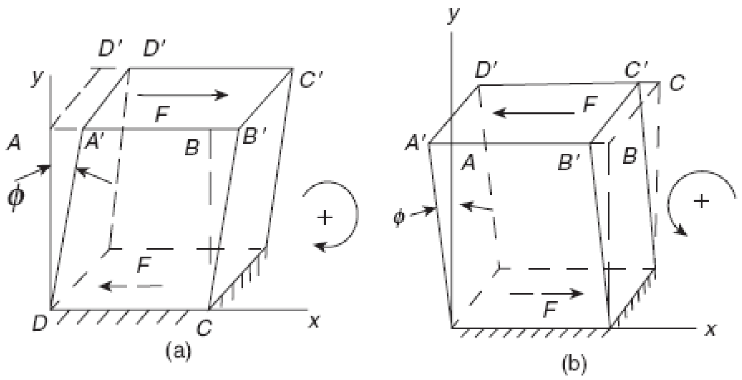
- Net temperature rise, $\Delta T = 55 - 26 = 29°C$.
- Area of Steel flat (As) = Area of Aluminium flat (Aa) = 20 x10 =200 mm2
- For equilibrium, $\sigma_s A_s = \sigma_a A_a$; $\sigma_s = \sigma_a$ will be one of the conditions to be satisfied by the composite assembly.
- $But\ \frac{\sigma_a}{E_a} + \frac{\sigma_s}{E_s} = (\alpha_a - \alpha_s)\Delta T = (23.2 - 11.6) \times 29 \times 10^{-6} = 0.000336$
- $\frac{\sigma_s}{200000} + \frac{\sigma_a}{70000} = 0.000336$
- $270000\,\sigma_s = 4709600$;
- $\sigma_s(tensile) = \sigma_a(compressive) = 17.44MPa$ as $\alpha_a > \alpha_s$

2. A flat steel bar of 20 mm × 8 mm is placed between two copper bars of 20 mm × 6 mm each so as to form a composite bar of section of 20 mm × 20 mm. The three bars are fastened together at their ends when the temperature of each is 30°C. Now the temperature of the whole assembly is raised by 30°C. Determine the temperature stress in the steel and copper bars. $E_s = 2E_{cu}$= 210 kN/mm^2, α_s = 11 × 10^{-6}/°C, α_{cu} = 18 × 10^{-6}/°C.

- Net temperature rise, $\Delta T = 30°C$.
- Area of Steel flat (A_s) = 20 x 8 = 160 mm^2
- Area of Copper flats (A_{cu}) = 2 x 20 x 6 =240 mm^2
- For equilibrium, $\sigma_s A_s = \sigma_{cu} A_{cu}$; $\sigma_s = 1.5\, \sigma_{cu}$ will be one of the conditions to be satisfied by the composite assembly.
- $But\ \frac{\sigma_{cu}}{E_{cu}} + \frac{\sigma_s}{E_s} = (\alpha_{cu} - \alpha_s)\Delta T = (18-11) \times 30 \times 10^{-6} = 0.00021$
- $\frac{\sigma_{cu}}{105000} + \frac{1.5\sigma_{cu}}{210000} = 0.00021$
- σ_{cu} = 12.6MPa (compressive) and σ_s = 18.9MPa (tensile) as $\alpha_{cu} > \alpha_s$

1.15 Simple Shear stress and Shear Strain

Consider a rectangular block which is fixed at the bottom and a force *F* is applied on the top surface as shown in the figure (a) below.



Equal and opposite reaction *F* develops on the bottom plane and constitutes a couple, *tending to rotate the body in a clockwise direction*. This type of shear force is a *positive shear force* and the shear force per unit surface area on which it acts is called *positive shear stress* (τ). If force is applied in the opposite direction as shown in Figure (b), then they are termed as negative shear force and shear stress.

The *Shear Strain* (ϕ) = AA''/AD = tanϕ. Since ϕ is a very small quantity, $\tan\phi \approx \phi$. Within the elastic limit, $\tau \propto \phi$ *or* $\tau = \boldsymbol{G}\, \phi$

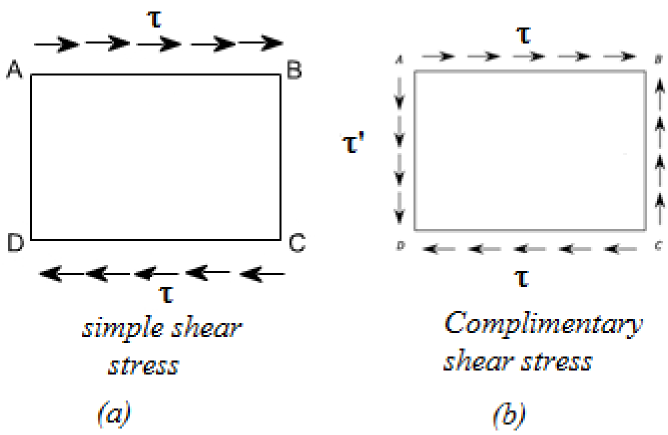
The constant of proportionality ***G*** is called *rigidity modulus or shear modulus.*

Note:

Normal stress is computed based on area perpendicular to the surface on which the force is acting, while, the shear stress is computed based on the surface area on which the force is acting. Hence shear stress is also called tangential stress.

1.16 Complementary Shear Stresses

Consider an element ABCD subjected to shear stress (τ) as shown in figure (a). We cannot have equilibrium with merely equal and opposite tangential forces on the faces AB and CD as these forces constitute a couple and induce a turning moment. The statical equilibrium demands that there must be tangential components (τ'') along AD and CB such that that can balance the turning moment. These tangential stresses (τ'') is termed as *complimentary shear stress*.



Let t be the thickness of the block. Turning moment due to τ will be $(\tau \text{ x } t \text{ x } L_{AB})\, L_{BC}$ and Turning moment due to τ' will be $(\tau'' \text{ x } t \text{ x } L_{BC})\, L_{AB}$. Since these moments have to be equal for equilibrium we have:

$$(\tau \text{ x } t \text{ x } L_{AB})\, L_{BC} = (\tau'' \text{ x } t \text{ x } L_{BC})\, L_{AB}.$$

From which it follows that $\tau = \tau''$, that is, intensities of shearing stresses across two mutually perpendicular planes are equal.

1.17 Volumetric strain

This refers to the slight change in the volume of the body resulting from three mutually perpendicular and equal direct stresses as in the case of a body immersed in a liquid under pressure. This is defined as the *ratio of change in volume to the original volume* of the body. Consider a cube of side „**a**" strained so that each side becomes „**a** ± δ**a'.**

- Hence the linear strain = δa/a.
- Change in volume = $(a \pm \delta a)^3 - a^3 = \pm 3a^2\delta a$. (ignoring small higher order terms)
- Volumetric strain $\varepsilon_v = \pm 3a^2\, \delta a/a^3 = \pm 3\, \delta a/a$
- The volumetric strain is three times the linear strain

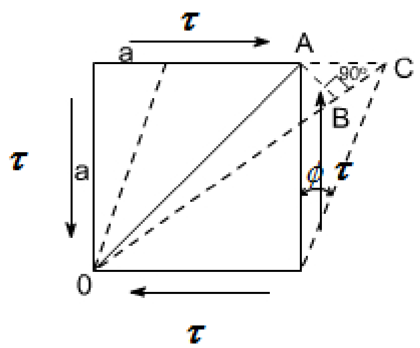
1.18 Bulk Modulus

This is defined as the ratio of the normal stresses (p) to the volumetric strain (ε_v) and denoted by **'K'.** Hence $\mathbf{K = p/\varepsilon_v}$. This is also an elastic constant of the material in addition to E, G and ν.

1.19 Relation between elastic constants

1.19.1 Relation between E,G and ν

Consider a cube of material of side „a' subjected to the action of the shear and complementary shear stresses and producing the deformed shape as shown in the figure below.



- Since, within elastic limits, the strains are small and the angle ACB may be taken as 45^0.
- Since angle between OA and OB is very small hence OA ≈ OB. BC, is the change in the length of the diagonal OA
- Strain on the diagonal OA = Change in length / original length = BC/OA

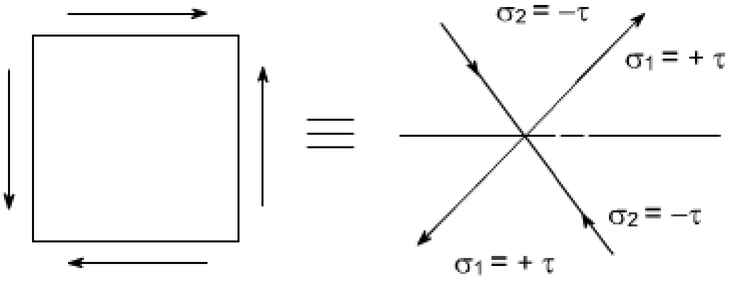
= AC cos45/ (a/sin45) = AC/ 2a = a φ / 2 a = φ / 2

- It is found that *strain along the diagonal is numerically half the amount of shear stain*.

• But from definition of rigidity modulus we have, $G = \tau / \phi$

• Hence, Strain on the diagonal OA = $\tau / 2G$

The shear stress system is equivalent or can be replaced by a system of direct stresses at 45° as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear stress.



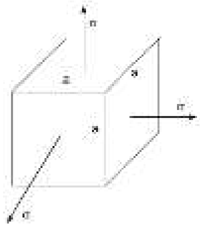
Strain in diagonal OA due to direct stresses $= \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{\tau}{E} + \nu\frac{\tau}{E} = \frac{\tau}{E}(1 + \nu)$

Equating the strain in diagonal OA we have $\frac{\tau}{2G} = \frac{\tau}{E}(1 + \nu)$

Relation between E,G and ν can be expressed as : $\boldsymbol{E = 2G(1+\nu)}$

1.19.2 Relation between E,K and ν

Consider a cube subjected to three equal stresses a shown in the figure below.



Strain in any one direction $= \frac{\sigma}{E} - \nu\frac{\sigma}{E} - \nu\frac{\sigma}{E} = \frac{\sigma}{E}\,(1 - 2\nu)$

Since the volumetric strain is three times the linear strain: $\varepsilon_v = 3\frac{\sigma}{E}\,(1 - 2\nu)$

From definition of bulk modulus : $\varepsilon_v = \frac{\sigma}{K}$

$$3\frac{\sigma}{E}\,(1 - 2\nu) = \frac{\sigma}{K}$$

Relation between E,K and ν can be expressed as : $\boldsymbol{E = 3K(1 - 2\nu)}$

Note: Theoretically ν < 0.5 as E cannot be zero

1.19.3 Relation between E, G and K

We have E = 2G(1+ν) from which ν = (E - 2G) / 2G
We have E = 3K(1-2ν) from which ν = (3K -E) / 6K

(E - 2G) / 2G = (3K -E) / 6K or (6EK - 12GK) = (6GK - 2EG) or 6EK+2EG = (6GK +12GK)

Relation between E,G and K can be expressed as: $\boldsymbol{E = \frac{9GK}{(3K+G)}}$

1.20 Exercise problems

1. A steel bar of a diameter of 20 mm and a length of 400 mm is subjected to a tensile force of 40 kN. Determine (a) the tensile stress and (b) the axial strain developed in the bar if the Young‟s modulus of steel $E = 200$ kN/mm2

Answer: (a) Tensile stress = 127.23MPa, (b) Axial strain = 0.00064

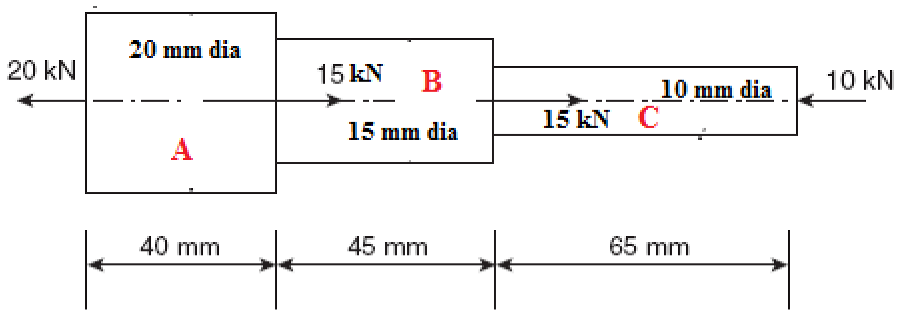
2. A 100 mm long bar is subjected to a compressive force such that the stress developed in the bar is 50 MPa. (a) If the diameter of the bar is 15 mm, what is the axial compressive force? (b) If E for bar is 105 kN/mm2, what is the axial strain in the bar?

Answer: (a) Compressive force = 8.835 kN, (b) Axial strain = 0.00048

3. A steel bar of square section 30 × 30 mm and a length of 600 mm is subjected to an axial tensile force of 135 kN. Determine the changes in dimensions of the bar. $E = 200$ kN/mm2, $v = 0.3$.

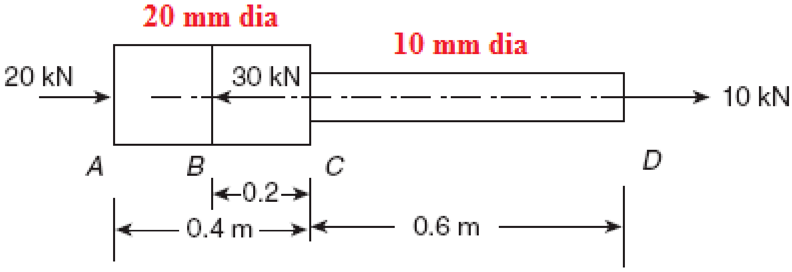
Answer: Increase in length δl = 0.45 mm, Decrease in breadth $\delta b = 6.75 \times 10^{-3}$ mm

4. A stepped circular steel bar of a length of 150 mm with diameters 20, 15 and 10 mm along lengths 40, 50 and 65 mm, respectively, subjected to various forces is shown in figure below. If E = 200 kN/mm2, determine the total change in its length.



Answer : Total decrease in length = 0.022mm

5. A stepped bar is subjected to axial loads as shown in the figure below. If *E* = 200 GPa, calculate the stresses in each portion *AB*, *BC* and *CD*. What is the total change in length of the bar?



Answer: Total increase in length = 0.35mm

6. A 400-mm-long aluminium bar uniformly tapers from a diameter of 25 mm to a diameter of 15 mm. It is subjected to an axial tensile load such that stress at middle section is 60 MPa. What is the load applied and what is the total change in the length of the bar if E = 67,000 MPa? (*Hint: At the middle diameter = (25+15)/2 = 20 mm*).

Answer: Load = 18.85kN, Increase in length = 0.382 mm

7. A short concrete column of 250 mm × 250 mm in section strengthened by four steel bars near the corners of the cross-section. The diameter of each steel bar is 30 mm. The column is subjected to an axial compressive load of 250 kN. Find the stresses in the steel and the concrete. Es = 15 Ec = 210 GPa. If the stress in the concrete is not to exceed 2.1 N/mm^2, what area of the steel bar is required in order that the column may support a load of 350 kN?

Answer: Stress in concrete = 2.45N/mm^2, Stress in steel = 36.75N/mm^2, Area of steel = 7440 mm^2

8. Two aluminium strips are rigidly fixed to a steel strip of section 25 mm × 8 mm and 1 m long. The aluminium strips are 0.5 m long each with section 25 mm × 5 mm. The composite bar is subjected to a tensile force of 10 kN as shown in the figure below. Determine the deformation of point B. Es = 3EA = 210 kN/mm2. *Answer: 0.203mm*

(*Hint: Portion CB is a single bar, Portion AC is a composite bar. Compute elongation separately for both the portions and add*)