MODULE - 3

TRANSIENT AND STEADY STATE RESPONSE ANALYSIS

LESSON STRUCTURE:

3.1. Introduction
3.2. Time Response
3.3. Steady State Response
3.4. Routh's-Hurwitz Criterion
3.5. Definition of root loci
3.6. Analysis using root locus plots
3.7. General rules for constructing root loci

OBJECTIVES:

- To analyse stability in complex domain and frequency domain systems.
- To educate static and transient behavior of a system.
- To demonstrate stability of the various control systems by applying Routh's stability criterion.
- To study stability by using Root locus plots.

3.1. Introduction:

Time is used as an independent variable in most of the control systems. It is important to analyse the response given by the system for the applied excitation, which is function of time. Analysis of response means to see the variation of output with respect to time. The output behavior with respect to time should be within these specified limits to have satisfactory performance of the systems. The stability analysis lies in the time response analysis that is when the system is stable output is finite

The system stability, system accuracy and complete evaluation is based on the time response analysis on corresponding results.

3.2. Time Response:

The response given by the system which is function of the time, to the applied excitation is called time response of a control system.

Practically, output of the system takes some finite time to reach to its final value. This time varies from system to system and is dependent on different factors. The factors like friction mass or inertia of moving elements some nonlinearities present etc. Example: Measuring instruments like Voltmeter, Ammeter.

Classification:

The time response of a control system is divided into two parts.

1 Transient response ct(t)
2 Steady state response css(t)

$$\therefore c(t)=c_t(t)+c_{SS}(t)$$

Where c(t)= Time Response
Total Response=Zero State Response +Zero Input Response.

3.3. Steady State Response:

It is defined the part of the response which remains after complete transient response vanishes from the system output.

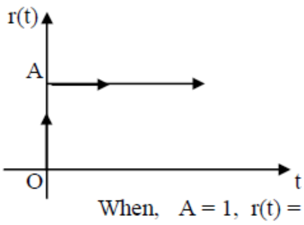
$$\text{i,e, } \lim_{t\to\infty} c_t(t)=c_{ss}(t)$$

The time domain analysis essentially involves the evaluation of the transient and Steady state response of the control system.

For the analysis point of view, the signals, which are most commonly used as reference inputs, are defined as **standard test inputs.**

- The performance of a system can be evaluated with respect to these test signals.
- Based on the information obtained the design of control system is carried out. The
- commonly used test signals are
  1. Step Input signals.
  2. Ramp Input Signals.
  3. Parabolic Input Signals.
  4. Impulse input signal.

1. **Step input signal (position function)**
   It is the sudden application of the input at a specified time as usual in the figure or instant any us change in the reference input
   Example :-
   a. If the input is an angular position of a mechanical shaft a step input represent the sudden rotation of a shaft.
   b. Switching on a constant voltage in an electrical circuit.
   c. Sudden opening or closing a valve.



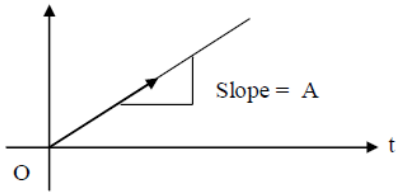
The step is a signal who‘s value changes from 1 value (usually 0) to another level A in Zero time.

In the Laplace Transform form R(s) = A / S
Mathematically r(t) = u(t)
= 1 for $t \geq 0$
= 0 for $t < 0$

2. **Ramp Input Signal (Velocity Functions):**
   It is constant rate of change in input that is gradual application of input as shown in fig (2 b).

r(t)

Ex:- Altitude Control of a Missile



The ramp is a signal, which starts at a value of zero and increases linearly with time.

Mathematically r (t) = At for $t \geq 0$

= 0 for $t \leq 0$.

In LT form $R(S) = \frac{A}{S^2}$

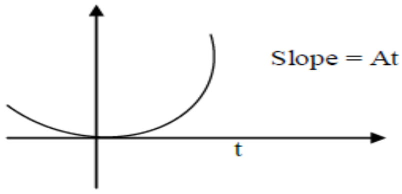
If A=1, it is called Unit Ramp Input

Parabolic Input Signal (Acceleration function):

- The input which is one degree faster than a ramp type of input as shown in fig (2 c) or it is an integral of a ramp.
- Mathematically a parabolic signal of magnitude

A is given by $r(t) = \frac{A t^2}{2} u(t)$

$$r(t) = \begin{cases} \frac{At^2}{2} & \text{for } t \geq 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$



In LT form $R(S) = \frac{A}{S^3}$

- If A = 1, a unit parabolic function is defined as $r(t) = \frac{t^2}{2} u(t)$

ie., $r(t) = \begin{cases} \frac{t^2}{2} & \text{for } t \geq 0 \\ 0 & \text{for } t \leq 0 \end{cases}$

In LT for $R(S) = \frac{1}{S} =$

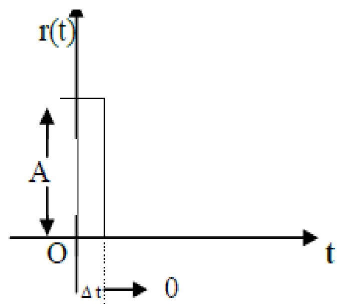
Impulse Input Signal :

It is the input applied instantaneously (for short duration of time) of very high amplitude as shown in fig 2(d)

Eg: Sudden shocks i e, HV due lightening or short circuit.

It is the pulse whose magnitude is infinite while its width tends to zero.

ie., $t \rightarrow 0$ (zero) applied momentarily



Area of impulse = Its magnitude

If area is unity, it is called **Unit Impulse Input** denoted as (δ)

Mathematically it can be expressed as

r(t) = A for t = 0

= 0 for t ≠ 0

In LT form R(S) = 1 if A = 1

3.4. Routh's-Hurwitz Criterion

E.J. Routh (1877) developed a method for determining whether or not an equation has roots with + ve real parts without actually solving for the roots.

A necessary condition for the system to be **STABLE** is that the real parts of the roots of the characteristic equation have - ve real parts. This insures that the impulse response will decay exponentially with time.

If the system has some roots with real parts equal to zero, but none with +ve real parts the system is said to be **MARGINALLY STABLE**.

It determines the poles of a characteristic equation with respect to the left and the right half of the S-plane without solving the equation.

The roots of this characteristic equation represent the closed loop poles. The stability of the system depends on these poles. The necessary, but not sufficient conditions for the system having no roots in the right half S-Plane are listed below.

i. All the co-efficients of the polynomial must have the same sign.
ii. All powers of S, must present in descending order.
iii. The above conditions are not sufficient.

In a vast majority of practical systems. The following statements on stability are quite useful.

i. If all the roots of the characteristic equation have –ve real parts the system is **STABLE**.
ii. If any root of the characteristic equation has a +ve real part or if there is a repeated root on the j -axis, the system is **unstable**.
iii. If condition (i) is satisfied except for the presence of one or more non repeated roots on the j -axis the system is limitedly **STABLE**

In this instance the impulse response does not decay to zero although it is bounded. Additionally certain inputs will produce outputs. Therefore **marginally stable** systems are **UNSTABLE**.

The Routh Stability criterion is a method for determining system stability that can be applied to an nth order characteristic equation of the form

$$s^n + an\text{-}1\ s^{n-1} + an\text{-}2\ s^{n-2} + an\text{-}3\ s^{n-3} + \ldots\ldots\ldots\ldots\ a1\ s^1 + a^0 = \mathbf{0}$$

The criterion is applied through the use of a Routh Array (Routh table) Defined as follows:

s^n	a_n	a_{n-2}	a_{n-4}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}
s^{n-2}	b_1	b_2	b_3
s^{n-3}	c_1	c_2	c_3
s^{n-4}	d_1	d_2	
.			
.			
s^2	e_1	a_0	
s^1	f_1		
s^0	a_0		

The ROUTH STABILITY CRITERION is stated as follows,

All the terms in the first column of Routh's Array should have same sign, and there should not be any change of sign.

This is a necessary and sufficient condition for the system to be stable. On the other hand any change of sign in the first column of Routh's Array indicates,

i. The System is Unstable, and
ii. The Number of changes of sign gives the number of roots lying in the right half of S-Plane

Example : find the stability of the system using Routh's criteria. For the equation

$3S^4+10S^3+5S^2+5S+2=0$

S^4	3	5	2
S^3	10	5	0
S^3	(2)	(1)	0
S^2	7/2	2	0
S^1	-1/7	0	
S^0	2		

Here two roots are +ve (2 changes of sign) and hence the system **is unstable.**

3.5. Definition of root loci

The root locus of a feedback system is the graphical representation in the complex *s*-plane of the possible locations of its closed-loop poles for varying values of a certain system parameter. The points that are part of the root locus satisfy the angle condition. The value of the parameter for a certain point of the root locus can be obtained using the magnitude condition.

In **root locus technique in control system** we will evaluate the position of the roots, their locus of movement and associated information. These information will be used to comment upon the system performance.

3.6. Analysis using root locus plots.

A designer can determine whether his design for a control system meets the specifications if he knows the desired time response of the controlled variable. By deriving the differential equations for the control system and solving them, an accurate solution of the system's performance can be obtained, but this approach is not feasible for other than simple systems. It is not easy to determine from this solution just what parameters in the system should be changed to improve the response. A designer wishes to be able to predict the performance by an analysis that does not require the actual solution of the differential equations.

The first thing that a designer wants to know about a given system is whether or not it is stable. This can be determined by examining the roots obtained from the characteristic equation

$$1 + G_0(s) = 0 \quad (3.1)$$

of the closed loop. The work involved in determining the roots of this equation can be avoided by applying the Hurwitz or Routh criterion. Determining in this way whether the system is stable or unstable does not satisfy the designer, because it does not indicate the degree of stability of the system, i.e., the amount of overshoot and the settling time of the controlled variable for a step input. Not only must the system be stable, but the overshoot must be maintained within prescribed bounds and transients must die out in a sufficiently short time.

The root-locus method described in this section not only indicates whether a system is stable or unstable but, for a stable system, also shows the degree of stability. The root locus is a plot of the roots of the characteristic equation of the closed loop as a function of the gain. This graphical approach yields a clear indication of the effect of gain adjustment with relatively small effort.

With this method one determines the closed-loop poles in the $s$plane - these are the roots of Eq.(5.1) - by using the known distribution of the poles and zeros of the open-loop transfer function $G_0(s)$. If for instance a parameter is varied, the roots of the characteristic equation will move on certain curves in the $s$plane as shown by the example in Figure 3.1. On these curves lie all

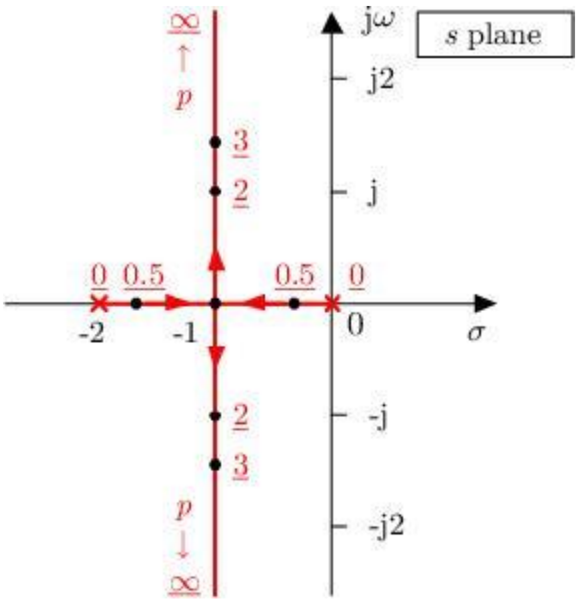


Figure 3.1: Plot of all roots of the characteristic equation $s^2 + 2s + p = 0$ for $0 \le p < \infty$. Values of p are red and underlined.

possible roots of the characteristic equation for all values of the varied parameter from zero to infinity. These curves are defined as the *root-locus plot* of the closed loop. Once this plot is obtained, the roots that best fit the system performance specifications can be selected. Corresponding to the selected roots there is a required value of the parameter which can be determined from the plot. When the roots have been selected, the time response can be obtained. Since the process of finding the root locus by calculating the roots for various values of a parameter becomes tedious, a simpler method of obtaining the root locus is desired. The graphical method for determining the root-locus plot is shown in the following.

An open-loop transfer function with k poles at the origin of the s plane is often described by

$$G_0(s) = \frac{K_0}{s^k} \frac{1 + \beta_1 s + \ldots + \beta_m s^m}{1 + \alpha_1 s + \ldots + \alpha_{n-k} s^{n-k}} \qquad m \le n\,, \tag{3.2}$$

where K_0 is the gain of the open loop. In order to represent this transfer function in terms of the open-loop poles and zeros it is rewritten as

$$G_0(s) = k_0 \frac{\prod_{\mu=1}^{m} (s - s_{z_\mu})}{\prod_{\nu=1}^{n} (s - s_{p_\nu})} = k_0 G(s) \tag{3.3}$$

or

$$G_0(s) = k_0 \frac{\prod_{\mu=1}^{m} (-s_{z_\mu})}{\prod_{\substack{\nu=1 \\ s_{p_\nu} \neq 0}}^{n-k} (-s_{p_\nu})} \frac{1}{s^k} \frac{\prod_{\mu=1}^{m} \left(1 + \frac{s}{-s_{z_\mu}}\right)}{\prod_{\substack{\nu=1 \\ s_{p_\nu} \neq 0}}^{n-k} \left(1 + \frac{s}{-s_{p_\nu}}\right)} \tag{3.4}$$

with $k_0 > 0$ and $s_{Z_\nu} \neq s_{P_\nu}$. The relationship between the *factor* k_0 and the *open-loop gain* K_0 is

$$K_0 = k_0 \frac{\prod_{\mu=1}^{m} (-s_{Z_\mu})}{\prod_{\substack{\nu=1 \\ s_{P_\nu} \neq 0}}^{n-k} (-s_{P_\nu})} \frac{1}{s^k} . \tag{3.5}$$

The characteristic equation of the closed loop using Eq. (5.3) is

$$1 + k_0 G(s) = 0 \tag{3.6}$$

or

$$G(s) = -\frac{1}{k_0} . \tag{3.7}$$

All complex numbers $s_i = s_i(k_0)$, which fulfil this condition for $0 \leq k_0 \leq \infty$, represent the root locus.

From the above it can be concluded that the magnitude of $k_0 G(s)$ must always be unity and its phase angle must be an odd multiple of π. Consequently, the following two conditions are formalised for the root locus for all positive values of k_0 from zero to infinity:

a)

Magnitude condition:

$$|G(s)| = \frac{1}{k_0} \tag{3.8}$$

b)

Angle condition

$$\varphi(s) = \arg G(s) = \pm 180°(2k+1) \quad \text{for} \quad \begin{matrix} k = 0, 1, 2, \ldots \\ k_0 \geq 0 \end{matrix} \tag{3.9}$$

In a similar manner, the conditions for negative values of k_0 ($-\infty \leq k_0 < 0$) can be determined. The magnitude conditions is the same, but the angle must satisfy the

c)

Angle condition

$$\varphi(s) = \arg G(s) = \pm k\, 360° \quad \text{for} \quad \begin{matrix} k = 0, 1, 2, \ldots \\ k_0 < 0 . \end{matrix} \tag{3.10}$$

Apparently the angle condition is independent of k_0. All points of the s plane that fulfil the angle condition are the loci of the poles of the closed loop by varying k_0. The calibration of the curves by the values of k_0 is obtained by the magnitude condition according to Eq. 8(3.8). Based upon this interpretation of the conditions the root locus can constructed in a graphical/numerical way.

Once the open-loop transfer function $G_0(s)$ has been determined and put into the proper form, the poles and zeros of this function are plotted in the s plane.

- The plot of the locus of the closed loop poles as a function of the open loop gain K, when K is varied from 0 to +00.
- When system gain K is varied from 0 to +oo, the locus is called direct root locus.
- When system gain K is varied from -oo to 0, the locus is called as inverse root locus.
- The root locus is always symmetrical about the real axis i.e. x-axis.
- The number of separate branches of the root locus equals either the number of open loop poles are number of open-loop zeros whichever is greater.
- A section of root locus lies on the real axis if the total number of open-loop poles and zeros to the right of the section is odd.
- If the root locus intersects the imaginary axis then the point of intersection are conjugate. From the open loop complex pole the root locus departs making an angle with the horizontal line.
- The root locus starts from open-loop poles.
- The root locus terminates either on open loop zero or infinity.
- The number of branches of roots locus are:

N if P>Z
and M if P<Z
where N —> No. of poles _P‘
M —> No. of zeros _Z‘

- Centroid is the centre of asymptotes. It is given by (an)

$$\sigma_c = \frac{\Sigma P - \Sigma Z}{N-M}$$

- Angle of asymptotes is denoted by _p

$$\phi = \frac{(2K+1)}{N-M} \times 180°$$

☞ Breakaway/saddle point is the point at which the root locus comes out of the real axis. To find breakaway point

Put $\frac{dK}{ds} = 0$

- Angle of departure is. tangent to root locus at complex pole

$$\phi_d = 180° - (\phi_P - \phi_Z).$$

Angle of arrival is tangent to the root locus at the complex zero.

$$\phi_a = 180° - (\phi_Z - \phi_P)$$

Where ϕ_Z = sum of all angles subtended by remaining zeros,

ϕ_P = sum of all angles subtended by remaining poles.

Based on the pole and zero distributions of an open-loop system the stability of the closed-loop system can be discussed as a function of one scalar parameter. The root-locus method shown in this module is a technique that can be used as a tool to design control systems. The basic ideas and its relevancy to control system design are introduced and illustrated. Ten general rules for constructing root loci for positive and negative gain are shortly presented such that they can be easily applied. This is demonstrated by some discussed examples, by a table with sixteen examples and by a comprehensive design of a closed-loop system of higher order.

Example Problems:

Q.1. Consider the example

$$G_0(s) = \frac{K_0}{s(s+2)} = \frac{k_0}{(s - s_{P_1})(s - s_{P_2})}$$

with $s_{P_1} = 0$, $s_{P_2} = -2$ and $k_0 = K_0$. The poles of the closed-loop transfer function

$$G_W(s) = \frac{K_0}{s^2 + 2s + K_0}$$

are the roots s_1 and s_2 of the characteristic equation

$$P(s) = s^2 + 2s + K_0 = 0$$

and are given by

$$s_{1,2} = -1 \pm \sqrt{1 - K_0}\ .$$

As $s_1 = s_{P_1} = 0$ and $s_2 = s_{P_2} = -2$ it can be seen that for $K_0 = 0$ the poles of the closed loop transfer function are identical with those of the open-loop transfer function $G_0(s)$. For other values K_0 the following two cases are considered:

a) $K_0 \leq 1$: Both roots s_1 and s_2 are real and lie on the real axis in the range of $-2 \leq \sigma \leq -1$ and $-1 \leq \sigma \leq 0$;

b) $K_0 > 1$: The roots s_1 and s_2 are conjugate complex with the real part $\text{Re}\, s_{1,2} = -1$, which does not depend on K_0, and the imaginary part Im $s_{1,2} = \pm\sqrt{K_0 - 1}$.

The curve has two branches as shown in Figure 6.2.

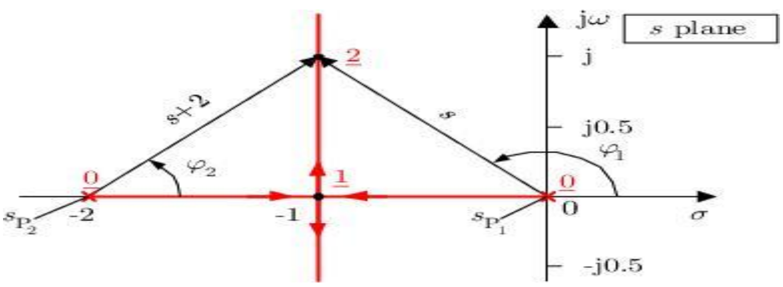


Figure 5.2: Root locus of a simple second-order system

At $(s_{P_1}+s_{P_2})/2=-1$ is the *breakaway point* of the two branches. Checking the angle condition the condition

$$\varphi(s)=\arg\{G(s)\}=\arg\left\{\frac{1}{s(s+2)}\right\}=-\arg s-\arg(s+2)\stackrel{!}{=}\pm 180°(2k+1)$$

must be valid. The complex numbers s and $(s+2)$ have the angles φ_1 and φ_2 and the magnitudes $|s|$ and $|s+2|$. The triangle ($-2,\ 0,\ -1+j$) in Figure 6.2 yields the angle condition. Evaluating the magnitude condition according to Eq. (6.8)

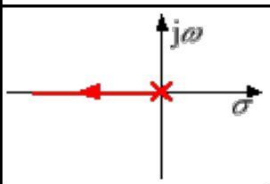
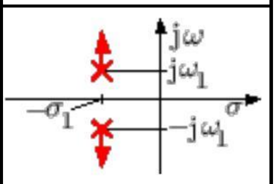
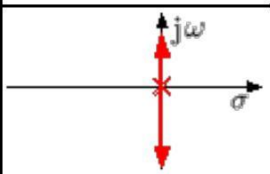
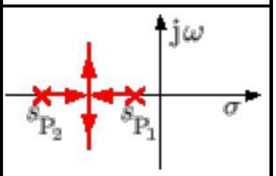
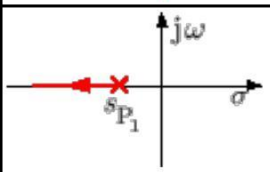
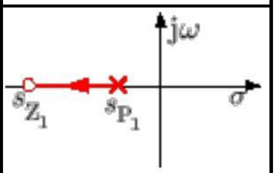
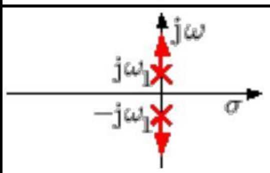
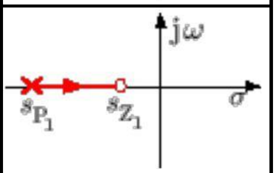
$$|G(s)|=\left|\frac{1}{s(s+2)}\right|=\frac{1}{K_0}$$

one obtains the value K_0 on the root locus. E.g. for $s=-1+j$ the gain of the open loop is $K_0=|s(s+2)|_{s=-1+j}=2$.

The value of K_0 at the breakaway point $s_B=-1$ is $K_0=|-1(-1+2)|=1$.

Table 5.1 shows further examples of some 1st- and 2nd-order systems.

Table 5.1: Root loci of 1st- and 2nd-order systems

$G_0(s)$	root locus	$G_0(s)$	root locus
$\frac{k_0}{s}$		$\frac{k_0}{(s+\sigma_1)^2+\omega_1^2}$	
$\frac{k_0}{s^2}$		$\frac{k_0}{(s-s_{P_1})(s-s_{P_2})}$	
$\frac{k_0}{s-s_{P_1}}$		$\frac{k_0(s-s_{Z_1})}{(s-s_{P_1})}$ $\lvert s_{Z_1}\rvert > \lvert s_{P_1}\rvert$	
$\frac{k_0}{s^2+\omega_1^2}$		$\frac{k_0(s-s_{Z_1})}{(s-s_{P_1})}$ $\lvert s_{Z_1}\rvert < \lvert s_{P_1}\rvert$	

3.7. General rules for constructing root loci

To facilitate the application of the root-locus method for systems of higher order than 2nd, rules can be established. These rules are based upon the interpretation of the angle condition and the analysis of the characteristic equation. The rules presented aid in obtaining the root locus by expediting the manual plotting of the locus. But for automatic plotting using a computer these rules provide checkpoints to ensure that the solution is correct.

Though the angle and magnitude conditions can also be applied to systems having dead time, in the following we restrict to the case of the open-loop rational transfer functions according to Eq. (3.3)

$$G_0(s) = k_0 \frac{(s - s_{Z_1})(s - s_{Z_2})\ldots(s - s_{Z_m})}{(s - s_{P_1})(s - s_{P_2})\ldots(s - s_{P_n})}, \quad k_0 \geq 0 \tag{3.11}$$

or

$$G_0(s) = k_0 \frac{b_0 + b_1 a + \ldots + b_{m-1}s^{m-1} + s^m}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n} = k_0 \frac{N_0(s)}{D_0(s)} . \tag{3.12}$$

As this transfer function can be written in terms of poles and zeros s_{P_ν} and s_{Z_μ} ($\nu = 1, 2, \ldots n$; $\mu = 1, 2, \ldots, m$) $G_0(s)$ can be represented by their magnitudes and angles

$$G_0(s) = k_0 \frac{|s - s_{Z_1}|\, e^{j\varphi_{Z_1}} |s - s_{Z_2}|\, e^{j\varphi_{Z_2}} \ldots |s - s_{Z_m}|\, e^{j\varphi_{Z_m}}}{|s - s_{P_1}|\, e^{j\varphi_{P_1}} |s - s_{P_2}|\, e^{j\varphi_{P_2}} \ldots |s - s_{P_n}|\, e^{j\varphi_{P_n}}}$$

or

$$G_0(s) = k_0 \frac{\prod_{\mu=1}^{m} |s - s_{Z_\mu}|}{\prod_{\nu=1}^{n} |s - s_{P_\nu}|} e^{j\left(\sum_{\mu=1}^{m} \varphi_{Z_\mu} - \sum_{\nu=1}^{n} \varphi_{P_\nu}\right)} . \tag{3.13}$$

From Eq. (3.8) the *magnitude condition*

$$\frac{\prod_{\mu=1}^{m} |s - s_{Z_\mu}|}{\prod_{\nu=1}^{n} |s - s_{P_\nu}|} = \frac{1}{k_0} \tag{3.14}$$

and from Eq. (3.9) the *angle condition*

$$\varphi(s) = \sum_{\mu=1}^{m} \varphi_{Z_\mu} - \sum_{\nu=1}^{n} \varphi_{P_\nu} = \pm 180^\circ (2\,k + 1) \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{3.15}$$

follows. Here φ_{Z_μ} and φ_{P_ν} denote the angles of the complex values $(s - s_{Z_\mu})$ and $(s - s_{P_\nu})$, respectively. All angles are considered positive, measured in the counterclockwise sense. If for each point the sum of these angles in the s plane is calculated, just those particular points that fulfil the condition in Eq. (3.15) are points on the root locus. This principle of constructing a root-locus curve - as shown in Figure 3.3 - is mostly used for automatic root-locus plotting.

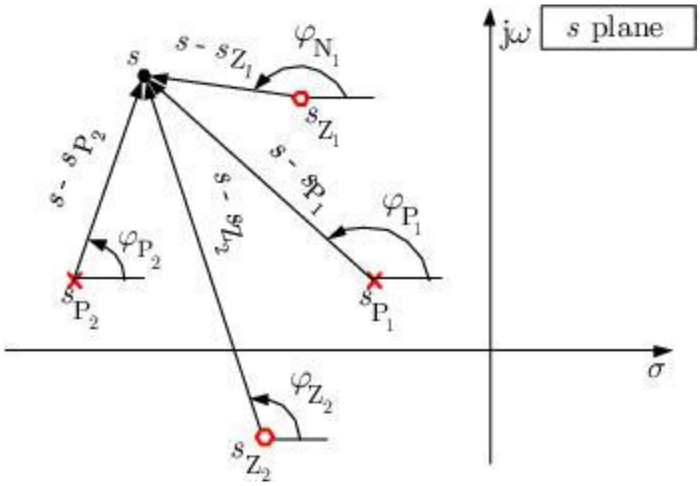


Figure 3.3: Pole-zero diagram for construction of the root locus

In the following the most important *rules for the construction of root loci* for $k_0 > 0$ are listed:

Rule 1 Symmetry

As all roots are either real or complex conjugate pairs so that the root locus is symmetrical to the real axis.

Rule 2 Number of branches

The number of branches of the root locus is equal to the number of poles n of the open-loop transfer function.

Rule 3 Locus start and end points

The locus starting points ($k_0 = 0$) are at the open-loop poles and the locus ending points ($k_0 = \infty$) are at the open-loop zeros. $(n - m)$ branches end at infinity. The number of starting branches from a pole and ending branches at a zero is equal to the multiplicity of the poles and zeros, respectively. A point at infinity is considered as an equivalent zero of multiplicity equal to $n - m$.

Rule 4 Real axis locus

If the total number of poles and zeros to the right of a point on the real axis is odd, this point lies on the locus.

Rule 5 Asymptotes

There are $n - m$ asymptotes of the root locus with a slope of

$$\alpha_k = \arg s = \frac{\pm 180°(2k+1)}{n-m}. \tag{3.16}$$

For $(n-m)=1,2,3$ and 4 one obtains the asymptote configurations as shown in Figure 3.4.

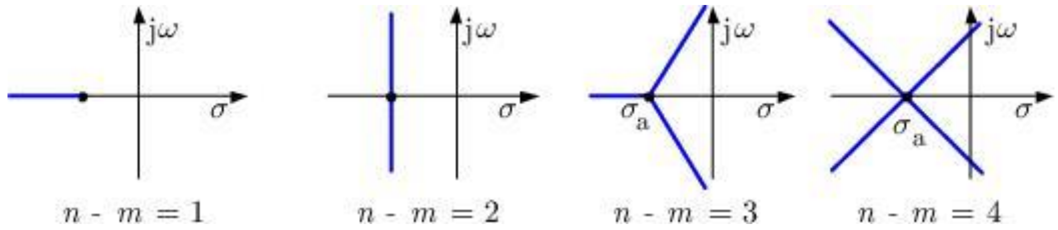


Figure 5.4: Asymptote configurations of the root locus

Rule 6 Real axis intercept of the asymptotes

The real axis crossing $(\sigma_a, j0)$ of the asymptotes is at

$$\sigma_a = \frac{1}{n-m}\left\{\sum_{\nu=1}^{n} \operatorname{Re} s_{P_\nu} - \sum_{\mu=1}^{m} \operatorname{Re} s_{Z_\mu}\right\} . \tag{3.17}$$

Rule 7 Breakaway and break-in points on the real axis

At least one breakaway or break-in point $(\sigma_B, j0)$ exists if a branch of the root locus is on the real axis between two poles or zeros, respectively. Conditions to find such real points are based on the fact that they represent multiple real roots. In addition to the characteristic equation for multiple roots the condition

$$\frac{d}{ds}[1+G_0(s)] = \frac{d}{ds}G_0(s) = 0 . \tag{3.18}$$

must be fulfilled, which is equivalent to

$$\sum_{\nu=1}^{n} \frac{1}{s-s_{P_\nu}} = \sum_{\mu=1}^{m} \frac{1}{s-s_{Z_\mu}} \tag{3.19}$$

for $s=\sigma_B$. If there are no poles or zeros, the corresponding sum is zero.

Rule 8 Complex pole/zero angle of departure/entry

The angle of departure of pairs of poles with multiplicity r_{P_ϱ} is

$$\varphi_{P_\varrho,D} = \frac{1}{r_{P_\varrho}}\left\{-\sum_{\substack{\nu=1\\ \nu\neq\varrho}}^{n} \varphi_{P_\nu} + \sum_{\mu=1}^{m} \varphi_{Z_\mu} \pm 180^\circ(2k+1)\right\} \tag{3.20}$$

and the angle of entry of the pairs of zeros with multiplicity r_{Z_ϱ}

$$\varphi_{Z_\varrho,E} = \frac{1}{r_{Z_\varrho}} \left\{ -\sum_{\substack{\mu=1 \\ \mu\neq\varrho}}^{m} \varphi_{Z_\mu} + \sum_{\nu=1}^{n} \varphi_{P_\nu} \pm 180^\circ(2k+1) \right\} \quad (3.21)$$

Rule 9 Root-locus calibration

The labels of the values of k_0 can be determined by using

$$k_0 = \frac{\prod_{\nu=1}^{n} |s - s_{P_\nu}|}{\prod_{\mu=1}^{m} |s - s_{Z_\mu}|} \quad (3.22)$$

For $m = 0$ the denominator is equal to one.

Rule 10 Asymptotic stability

The closed loop system is asymptotically stable for all values of k_0 for which the locus lies in the left-half s plane. From the imaginary-axis crossing points the critical values $k_{0_{crit}}$ can be determined.

The rules shown above are for positive values of k_0. According to the angle condition of Eq. (5.10) for negative values of k_0 some rules have to be modified. In the following these rules are numbered as above but labelled by a *.

Rule 3* Locus start and end points

The locus starting points ($k_0 = 0$) are at the open-loop poles and the locus ending points ($k_0 = -\infty$) are at the open-loop zeros. $(n-m)$ branches end at infinity. The number of starting branches from a pole and ending branches at a zero is equal to the multiplicity of the poles and zeros, respectively. A point at infinity is considered as an equivalent zero of multiplicity equal to $n-m$.

Rule 4* Real axis locus

If the total number of poles and zeros to the right of a point on the real axis is even including zero, this point lies on the locus.

Rule 5* Asymptotes

There are $n-m$ asymptotes of the root locus with a slope of

$$\alpha_k = \arg s = \frac{\pm 360^\circ k}{n-m} \quad (3.23)$$

Rule 8* Complex pole/zero angle of departure/entry

The angle of departure of pairs of poles with multiplicity r_{P_ϱ} is

$$\varphi_{P_\varrho,D} = \frac{1}{r_{P_\varrho}} \left\{ -\sum_{\substack{\nu=1 \\ \nu\neq\varrho}}^{n} \varphi_{P_\nu} + \sum_{\mu=1}^{m} \varphi_{Z_\mu} \pm 360^\circ k \right\} \tag{3.24}$$

and the angle of entry of the pairs of zeros with multiplicity r_{Z_ϱ}

$$\varphi_{Z_\varrho,E} = \frac{1}{r_{Z_\varrho}} \left\{ -\sum_{\substack{\mu=1 \\ \mu\neq\varrho}}^{m} \varphi_{Z_\mu} + \sum_{\nu=1}^{n} \varphi_{P_\nu} \pm 360^\circ k \right\} . \tag{3.25}$$

The root-locus method can also be applied for other cases than varying k_0. This is possible as long as $G_0(s)$ can be rewritten such that the angle condition according to Eq. (3.15) and the rules given above can be applied. This will be demonstrated in the following two examples.

Q.2. Given the closed-loop characteristic equation

$$a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + s^n = 0 ,$$

the root locus for varying the parameter a_1 is required. The characteristic equation is therefore rewritten as

$$1 + a_1 \frac{s}{a_0 + a_2 s^2 + \ldots + s^n} = 0 .$$

This form then correspondents to the standard form

$$1 + G_0(s) = 1 + a_1 \frac{N_0(s)}{D_0(s)} = 0$$

to which the rules can be applied. ■

Q.3.Given the closed-loop characteristic equation

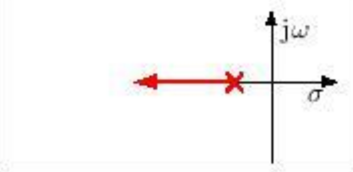
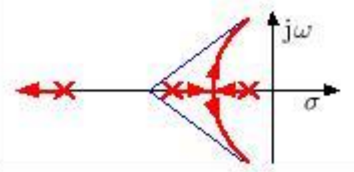
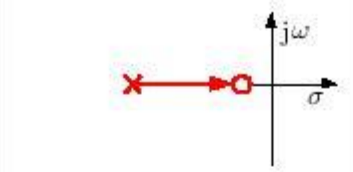
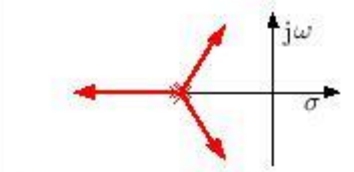
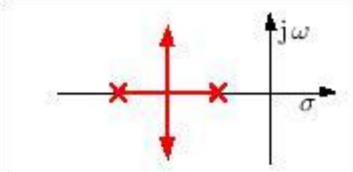
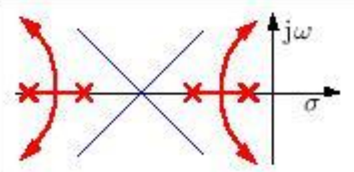
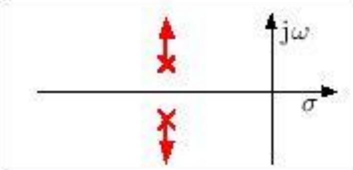
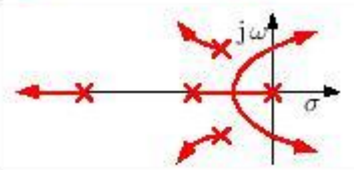
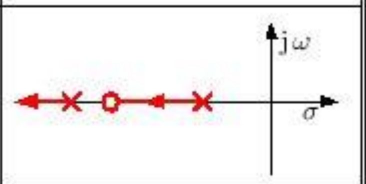
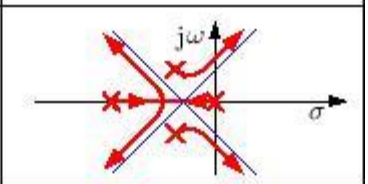
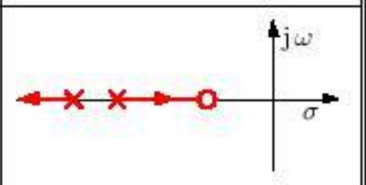
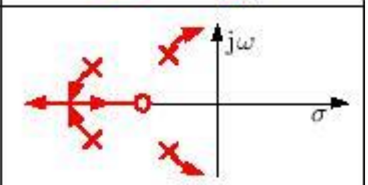
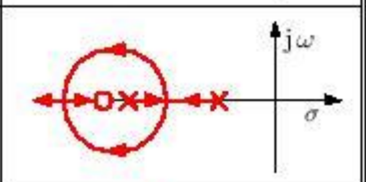
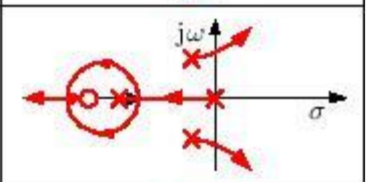
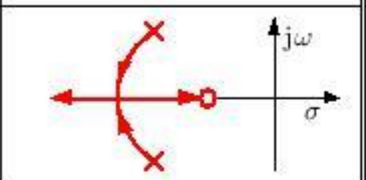
$$s^3 + (3+\alpha)\, s^2 + 2s + 4 = 0 ,$$

it is required to find the effect of the parameter α on the position of the closed-loop poles. The equation is rewritten into the desired form

$$1+\alpha\frac{s^2}{s^3+3s^2+2s+4}=0 .$$

Using the rules 1 to 10 one can easily predict the geometrical form of the root locus based on the distribution of the open-loop poles and zeros. Table 3.2 shows some typical distributions of open-loop poles and zeros and their root loci.

Table 3.2: Typical distributions of open-loop poles and zeros and the root loci

No.	root locus	No.	root locus
1		9	
2		10	
3		11	
4		12	
5		13	
6		14	
7		15	
8		16	

For the qualitative assessment of the root locus one can use a physical analogy. If all open-loop poles are substituted by a negative electrical charge and all zeros by a commensurate positive one and if a massless negative charged particle is put onto a point of the root locus, a movement is observed. The path that the particle takes because of the interplay between the repulsion of the poles and the attraction of the zeros lies just on the root locus. Comparing the root locus examples 3 and 9 of Table 3.2 the 'repulsive' effect of the additional pole can be clearly seen.

The systematic application of the rules from section 3.2 for the construction of a root locus is shown in the following non-trivial example for the open-loop transfer function

$$G_0(s) = \frac{k_0(s+1)}{s(s+2)(s^2+12s+40)} \qquad (3.26)$$

The degree of the numerator polynomial is $m = 1$. This means that the transfer function has one zero ($s_{Z_1} = -1$). The degree of the denominator polynomial is $n = 4$ and we have the four poles ($s_{P_1} = 0$, $s_{P_2} = -2$, $s_{P_3} = -6 + j2$, $s_{P_4} = -6 - j2$). First the poles (x) and the zeros (o) of the open loop are drawn on the s plane as shown in Figure 3.5. According to rule 3 these poles are just

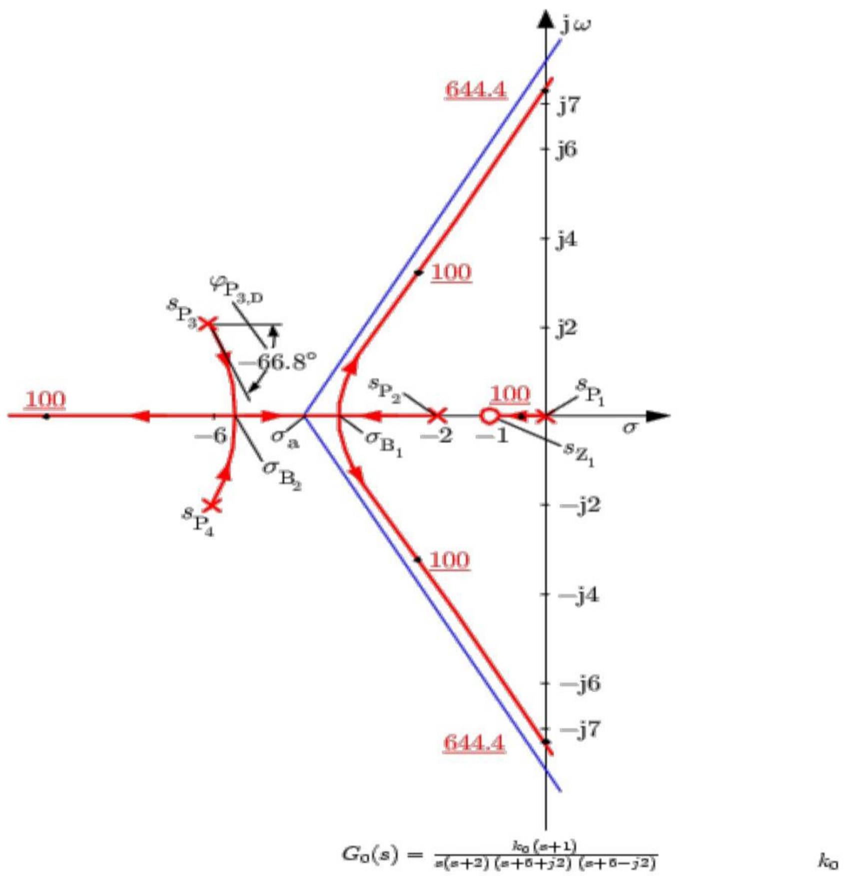


Figure 3.5: Root locus of . Values of are in red and underlined.

those points of the root locus where $k_0 = 0$ and the zeros where $k_0 \to \infty$. We have $(n-m) = 3$ branches that go to infinity and the asymptotes of these three branches are lines which intercept the real axis according to rule 6. From Eq. (3.17) the crossing is at

$$\sigma_a = \frac{(0-2-6-6)-(-1)}{3} = -\frac{13}{3} = -4.33 \qquad (3.27)$$

and the slopes of the asymptotes are according to Eq. (3.16)

$$\alpha_k = \frac{\pm 180^\circ(2k+1)}{3} = \pm 60^\circ(2k+1) \quad k = 0, 1, 2, \ldots \qquad (3.28)$$

i.e. $\alpha_0 = 60^\circ, \quad \alpha_1 = +180^\circ, \quad \alpha_2 = -60^\circ$.

The asymptotes are shown in Figure 3.5 as blue lines. Using Rule 4 it can be checked which points on the real axis are points on the root locus. The points σ with $-1 < \sigma < 0$ and $\sigma < -2$ belong to the root locus, because to the right of them the number of poles and zeros is odd. According to rule 7 breakaway and break-in points can only occur pairwise on the real axis to the left of -2. These points are real solutions of the Eq. (3.19). Here we have

$$\frac{1}{s} + \frac{1}{s+2} + \frac{1}{s+6-j2} + \frac{1}{s+6+j2} = \frac{1}{s+1} \qquad (3.29)$$

or

$$3s^4 + 32s^3 + 106s^2 + 128s + 80 = 0 .$$

This equation has the solutions $s_{B_1} = -3.68$, $s_{B_2} = -5.47$ and $s_{B_{3,4}} = -0.76 \pm j0.866$. The real roots $s_{B_1} = -3.68$ and $s_{B_2} = -5.47$ are the positions of the breakaway and the break-in point. The angle of departure $\varphi_{P_3,D}$ of the root locus from the complex pole at $s_{P_3} = -6 + j2$ can be determined from Figure 3.6 according to Eq. (3.20):

$$\begin{aligned}\varphi_{P_3,D} &= -90^\circ - 153.4^\circ - 161.6^\circ + 158.2^\circ \pm 180^\circ(2k+1) \\ &= -246.8^\circ + 180^\circ = -66.8^\circ .\end{aligned} \qquad (3.30)$$

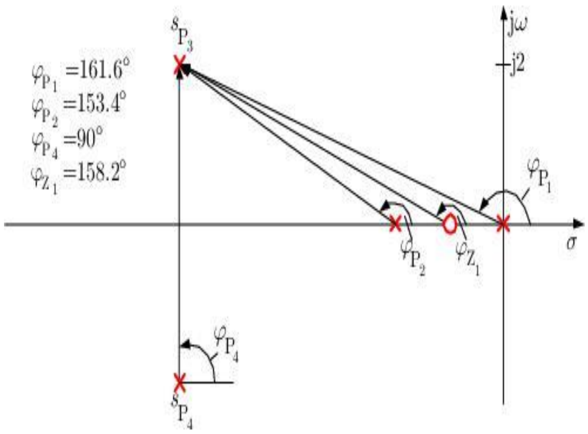


Figure 5.6: Calculating the angle of departure $\varphi_{P_3,D}$ of the complex pole $s_{P_3} = -6 + j2$

With this specifications the root locus can be sketched. Using rule 9 the value of k_0 can be determined for some selected points. The value at the intersection with the imaginary axis is

$$k_{0,crit} = \frac{7.2 \cdot 7.4 \cdot 7.9 \cdot 11.1}{7.25} = 644.4 .$$

OUTCOMES:

At the end of the module, the students are able to:

- Obtain the time response and steady-state error of the system.
- Knowledge about improvement of static and transient behaviour of a system.
- Determine stability of the various control systems by applying Routh's stability criterion.
- Construct root loci from open loop transfer functions of control systems and Analyze the behaviour of roots with system gain.
- Assess the stability of closed loop systems by means of the root location in s-plane and their effects on system performance.

SELF-TEST QUESTIONS:

1. Obtain an expression for time response of the first order system subject to step input.
2. Define
    1) Time response.
    2) Transient response.
    3) Steady state response.
    4) Steady state error.
3. Determine the stability of the system whose characteristic equation is given by $S^4+6S^3+23S^2+40S+50=0$, Using Routh's criterion.
4. Sketch the root locus for **G(S)H(S)=** $\frac{K}{S(S+2)(S+4)}$ show all details on it.
5. Sketch the root locus for **G(S)H(S)=** $\frac{10K}{S(S+2)(S+6)}$ show all details on it.

6. Sketch the root locus for $G(S)H(S)=\frac{K(S+1)}{S(S+2)(S+4)}$ show all details on it.

FURTHER READING:

1. **Control engineering,** Swarnakiran S, Sunstar publisher, 2018.
2. **Feedback Control System**, Schaum's series. 2001.