Module 3

Bending Moment and Shear Force

Objectives:

Determine the shear force, bending moment and draw shear force and bending moment diagrams, describe behaviour of beams under lateral loads. Stresses induced in beams, bending equation derivation & Deflection behaviour of beams

Learning Structure

- 3.1 Types Of Beams
- 3.2 Shear Force
- 3.3 Bending Moment
- 3.4 Shear Force Diagram And Bending Moment
- 3.5 Relations Between Load, Shear And Moment
- 3.6 Problems
- 3.7 Pure Bending
- 3.8 Effect Of Bending In Beams
- 3.9 Assumptions Made In Simple Bending Theory
- 3.10 Problems
- 3.11 Deflection Of Beams
- Outcomes
- Further Reading

3.1 TYPES OF BEAMS

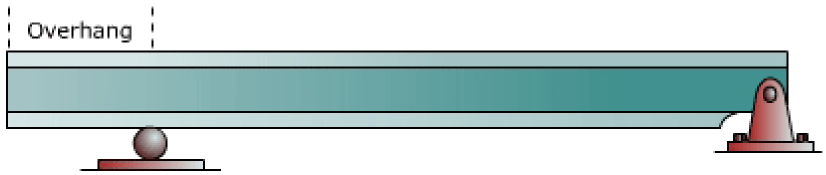
a) Simple Beam

A simple beam is supported by a hinged support at one end and a roller support at the other end.

b) Cantilever beam

A cantilever beam is supported at one end only by a fixed support.

c) Overhanging beam.



An overhanging beam is supported by a hinge and a roller support with either or both ends extending beyond the supports.

Note: All the beams shown above are the statically determinate beams.

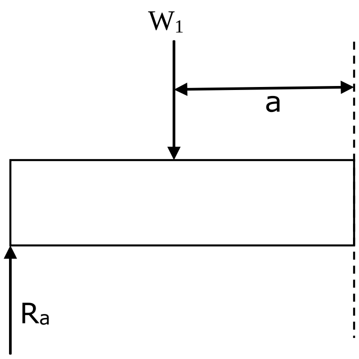


Fig 2 :Shear Force

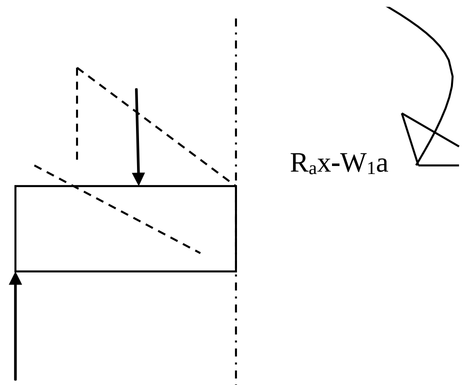


Fig 3 : Bending Moment

Consider a simply supported beam subjected to loads W_1 and W_2. Let R_A and R_B be the reactions at supports. To determine the internal forces at C pass a section at C. The effects of R_A and W_1 to the left of section are shown in Fig (b) and (c). In each case the effect of applied load has been transferred to the section by adding a pair of equal and opposite forces at that section. Thus at the section, moment $M = (W_1 a - R_a x)$ and shear force $F = (R_A - W_1)$, exists. The moment M which tend to bends the beam is called bending moment and F which tends to shear the beam is called shear force.

Thus the resultant effect of the forces at one side of the section reduces to a single force and a couple which are respectively the vertical shear and the bending moment at that section. Similarly, if the equilibrium of the right hand side portion is considered, the loading is reduced to a vertical force and a couple acting in the opposite direction. Applying these forces to a free body diagram of a beam segment, the segments to the left and right of section are held in equilibrium by the shear and moment at section.

Thus the shear force at any section can be obtained by considering the algebraic sum of all the vertical forces acting on any one side of the section

Bending moment at any section can be obtained by considering the algebraic sum of all the moments of vertical forces acting on any one side of the section.

3.2 Shear Force

It is a single vertical force developed internally at any point on the beam to balance the external vertical forces and keep the point in equilibrium. It is therefore equal to algebraic sum of all external forces acting to either left or right of the section.

3.3 Bending Moment

It is a moment developed internally at each point in a beam that balances the external moments due to forces and keeps the point in equilibrium. It is the algebraic sum of moments to section of all forces either on left or on right of the section.

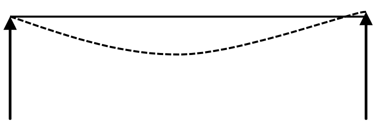
3.3.1 Types of Bending Moment

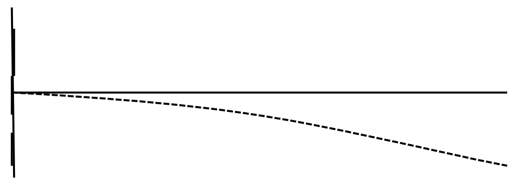
1) Sagging bending moment

The top fibers are in compression and bottom fibers are in tension.

2) Hogging bending moment

The top fibers are in tension and bottom fibers are in compression.





Sagging Bending Moment

Hogging Bending Moment

3.4 Shear Force Diagram and Bending Moment

3.4.1 Diagram Shear Forces Diagram (SFD)

The SFD is one which shows the variation of shear force from section to section along the length of the beam. Thus the ordinate of the diagram at any section gives the Shear Force at that section.

3.4.2 Bending Moment Diagram (BMD)

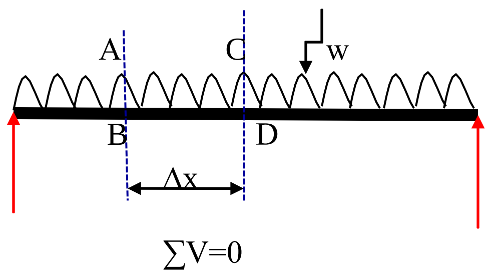
The BMD is one which shows the variation of Bending Moment from section to section along the length of the beam. The ordinate of the diagram at any section gives the Bending Moment at that section.

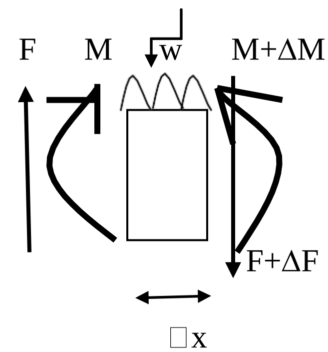
3.4.3 Point of Contraflexure

When there is an overhang portion, the beam is subjected to a combination of Sagging and Hogging moment. The point on the BMD where the nature of bending moment changes from hogging to sagging or sagging to hogging is known as point of contraflexure. Hence, at point of contraflexure BM is zero. The point corresponding to point of contraflexure on the beam is called as point of inflection.

3.5 RELATIONS BETWEEN LOAD, SHEAR AND MOMENT

Consider a simply supported beam subjected to a Uniformly Distributed Load w/m. Let us assume that a portion PQRS of length Δx is cut and taken out. Consider the equilibrium of this portion





$\sum V=0$

$F-(F+\Delta F)-w\,\Delta x=0$

Limit $\Delta x \to 0$, then <u>or</u> F =

Taking moments about section CD for equilibrium

$$M-(M+\Delta M)+F\,\Delta x-(w(\Delta x)^2/2)=0$$

Rate of change of Shear Force or slope of SFD at any point on the beam is equal to the intensity of load at that point.

Properties of BMD and SFD

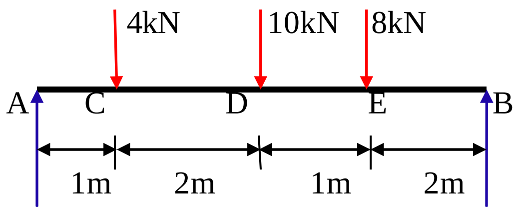
1) when the load intensity in the region is zero, Shear Force remains constant and Bending Moment varies linearly.

2) When there is Uniformly Distributed Load (UDL), Shear Force varies linearly and BM varies parabolically.

3) When there is Uniformly Varying Load (UVL), Shear Force varies parabolically and Bending Moment varies cubically.

3.6 Problems:

1. **A simply supported beam is carrying point loads, as shown in figure. Draw the SFD and BMD for the beam**.



11

7

+ve SFD

3

-ve SFD

11

25

22

11 +ve BMD

kN-m

To draw S.F.D. and B.M.D. we need R_A and R_B.

By taking moment of all the forces about point A, we get

$R_B \times 6 – (8 \times 4) – (10 \times 3) – (4 \times 1) = 0$

$R_B = 11$ kN

From condition of static equilibrium:

$R_A + 11 – 4 – 10 – 8 = 0$

$R_A = 11$ kN

Shear Force Calculations

SF at A $F_A = + R_A = + 11$ kN

SF left of C $F_C = + R_A = +11$ kN SF

right of C $F_C = + 11 – 4 = + 7$ kN

SF left of D $F_D = + 11 – 4 – 10 = 7$ kN

SF right of D $F_D = + 11 – 4 – 10 = – 3$ kN SF

left of E $F_E = + 11 – 4 – 10 = – 3$ kN

SF left of E $F_E =+ 11 – 4 – 10 – 8 = – 11$ kN SF

left of B $F_B = + 11 – 4 – 10 – 8 = – 11$ kN

Bending moment Calculations

At x = 0, $M_A = 0$

At x = 1 m; $M_C = + R_A.1 = 11 \times 1 = 11$ kN m At

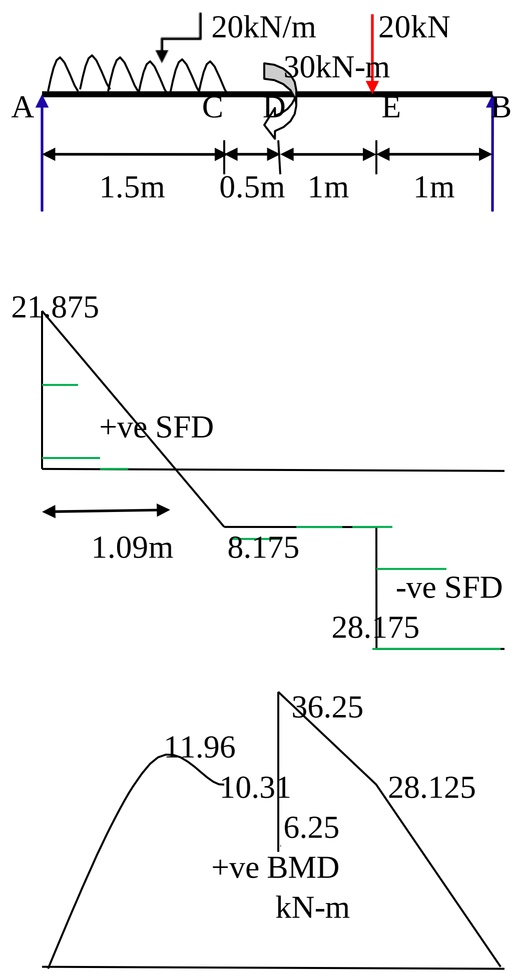
x = 3 m; $M_D = 11 \times 3 – 4 (3 – 1) = 25$ kN m

At x= 4m

$M_E = 11 \times 4 – 4 (4 – 1) – 10 (4 – 3) = 22$ kN m At

X= 6m $M_B = 0$

2. Draw the SF and BM diagram for the simply supported beam loaded as shown in fig.



To draw S.F.D. and B.M.D. we need R_A and R_B.

By taking moment of all the forces about point A, we get

$R_B \times 4 - (20 \times 3) - (20 \times 1.5^2/2) - 30 = 0$

$R_B = 28.125$ kN

From condition of static equilibrium:

$R_A + 28.125 - 30 - 20 = 0$

$R_A = 21.875$ kN

Shear Force Calculations

SF at A	$F_A = + R_A = + 21.875$ kN
SF left of C	$F_C = +21.875 - 30 = -8.175$ kN SF
right of C	$F_C = +21.875 - 30 = -8.175$ kN SF
left of D	$F_D = +21.875 - 30 = -8.175$ kN SF
right of D	$F_D = +21.875 - 30 = -8.175$ kN SF
left of E	$F_E = +21.875 - 30 = -8.175$ kN SF
left of E	$F_E = +21.875-30-20 = -28.125$ kN SF
left of B	$F_B = +21.875-30-20 = -28.125$ kN

SF is zero, between A & C at x = 1.09m

Bending moment Calculations

At x = 0, $M_A = 0$

At x = 1.5 m;

$M_C = 21.875 \times 1.5 - 20 \times 1.5^2/2 = 10.31$ kN m At x = 1.09 m;

Mmax $= 21.875 \times 1.09 - 20 \times 1.09^2/2 = 11.96$ kN m

At x = 2 m;

$M_D = 21.875 \times 2 - 20 \times 1.5(0.75+0.5) = 6.25$ kN m

At x = 2 m;

$M_D = 21.875 \times 2 - 20 \times 1.5(0.75+0.5)+30$

$= 36.25$ kN m

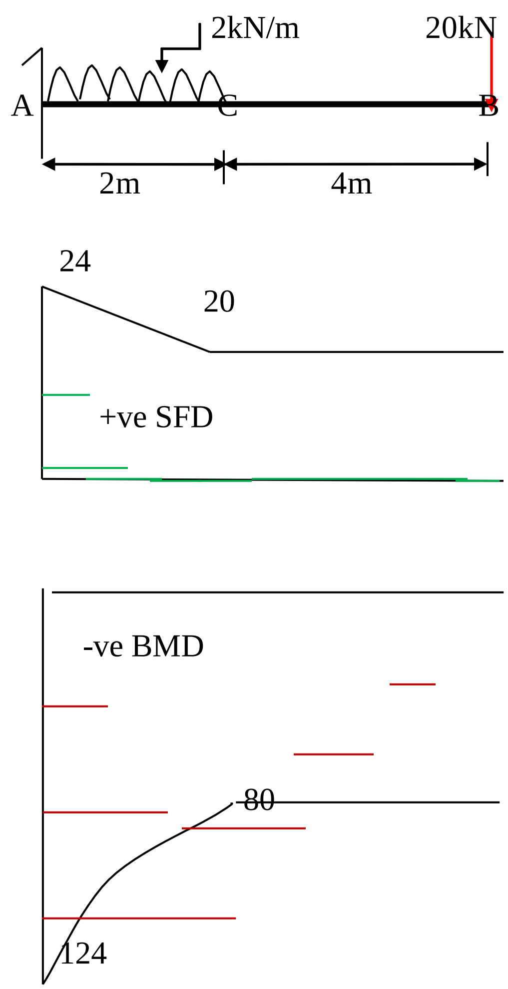
At x= 3m

$M_E = 21.875 \times 3 - 20 \times 1.5(0.75+1.5)+30$

$= 28.125$ kN m

At X= 4m $M_B = 0$

3. A cantilever is shown in fig. Draw the BMD and SFD. What is the reaction at supports?



To draw S.F.D. and B.M.D. we need R_A and R_B.

By taking moment of all the forces about point A, we get

$R_B \times 4 - (20 \times 3) - (20 \times 1.5^2/2) - 30 = 0$

$R_B = 28.125$ kN

From condition of static equilibrium:

$R_A + 28.125 - 30 - 20 = 0$

$R_A = 21.875$ kN

Shear Force Calculations

SF at B $\quad F_B = +20$ kN

SF right of C $\quad F_C = +20$ kN

SF left of C $\quad F_C = +20$ kN

SF right of A $\quad F_A = +20 + 2x\ 2 = 24$ kN

Since the moment is considered from right to left the anticlockwise moments are considered to be +ve and clockwise moments are considered to be -ve

Bending moment Calculations

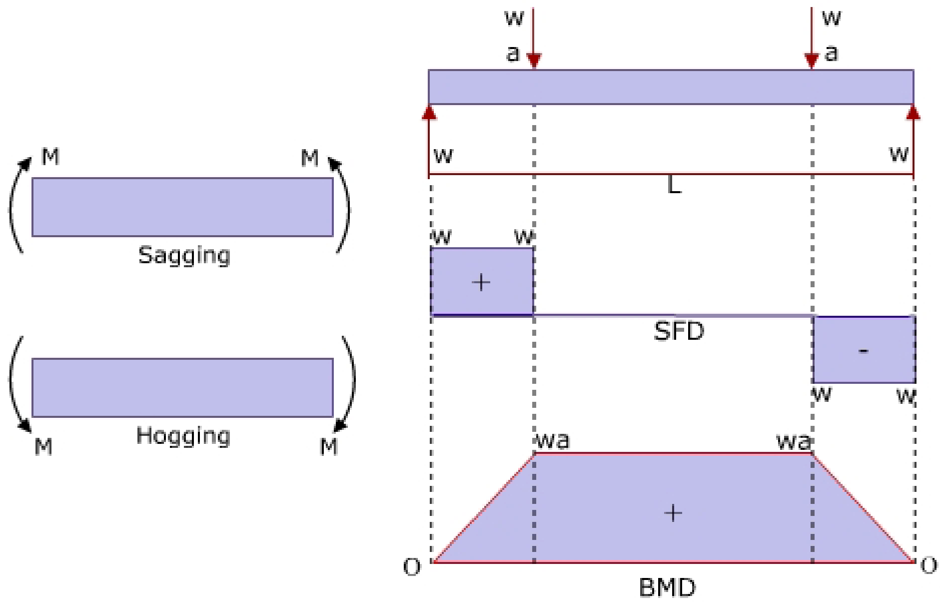
At x = 0, $\quad M_B = 0$

At x = 4 m; $M_c = -20x\ 4 =- 80$ kN-m At x = 6 m;

$M_A = -(20 \times 6 + 2 \times 2^2/2) = 124$ kN m

Stresses in Beams

3.7 Pure Bending



A beam or a part of a beam is said to be under pure bending if it is subjected to only Bending Moment and no Shear Force.

3.8 Effect of Bending in Beams

The figure shows a beam subjected to sagging Bending Movement. The topmost layer is under maximum compressive stress and bottom most layer is under maximum tensile stress. In between there should be a layer, which is neither subjected to tension nor to compression. Such a layer is called "Neutral Layer". The projection of Neutral Layer over the cross section of the beam is called "Neutral Axis".

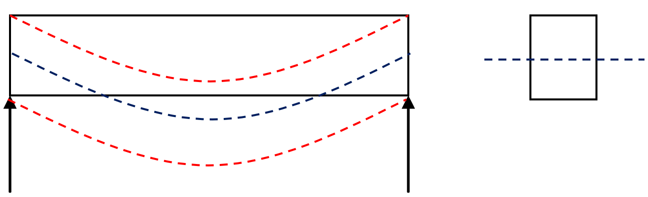


Fig-1

When the beam is subjected to sagging, all layers below the neutral layer will be under tension and all layers above neutral layer will be under compression. When the beam is subjected to hogging, all layers above the neutral layer will be under tension and all the layers below neutral layer will be under compression and vice versa if it is hogging bending moment

3.9 Assumptions made in simple bending theory

- The material is isotropic and homogenous.
- The material is perfectly elastic and obeys Hooke's Law i.e., the stresses are within the limit of proportionality.
- Initially the beam is straight and stress free.
- Beam is made up of number of layers and they undergo bending independently.
- Bending takes place over an arc of a circle and the radius of curvature is very large when compared to the dimensions of the beam.
- Normal plane sections before bending remain normal and plane even after bending.
- Young's Modulus of Elasticity is same under tension a ndcompression.

3.9.1 Euler- Bernoulli bending Equation (Flexure Formula)

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$$

where,

M = Resisting moment developed inside the material against applied bending movement and is numerically equal to bending moment applied (Nmm)

I = Moment of Inertia of cross section of beam about the Neutral Angle. (mm^4)

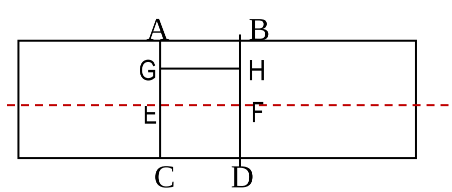
F = Direct Stress (Tensile or Compression) developed in any layer of the beam (N/mm^2)

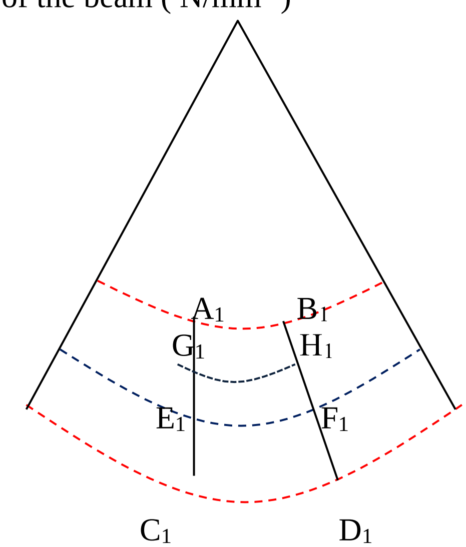
Y = Distance of the layer from the neutral axis (mm)

E = Young's Modulus of Elasticity of the material of the beam (N/mm^2)

R = Radius of curvature of neutral layer (mm)

Euler- Bernoulli's Equation





Consider two section very close together (AB and CD). After bending the sections will be at A_1 B_1 and C_1 D_1 and are no longer parallel. AC will have extended to A_1 C1 and B_1 D_1 will have compressed to B_1D_1. The line EF will be located such that it will not change in length. This surface is called neutral surface and its intersection with Z-Z is called the neutral axis.

The development lines of A'B' and C'D' intersect at a point 0 at an angle of θ radians and the radius of $E_1F_1 = R$.

Let y be the distance(E'G') of any layer H_1G_1 originally parallel to EF.

Then $H_1G_1/ E_1F_1 =(R+y)\theta /R \theta = (R+y)/R$

and the strain at layer H_1G_1= = $(H_1G_1'- HG) / HG = (H_1G_1- HG) / EF$

$$= [(R+y)\theta - R \theta] /R \theta$$

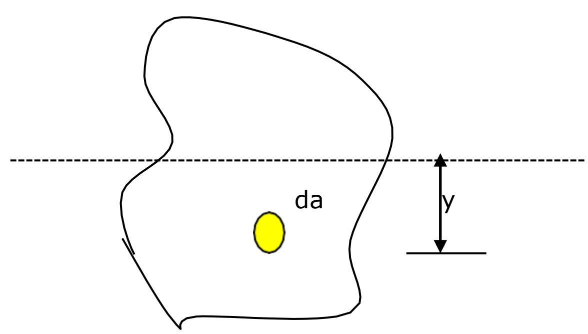
$$= y /R.$$

The relation between stress and strain is σ= E. Therefore

$$\sigma = E. = E. y /R$$

$$\sigma / E = y / R$$

Let us consider an elemental area 'da 'at a distance y, from the Neutral Axis.



Section Modulus(Z)

$$F = \frac{M}{I}.y$$

$$\Rightarrow f_{max} = \frac{M_{max}}{I}.y_{max}$$

$$\text{i,e., } M_{max} = f_{max}.\frac{I}{y_{max}}$$

$$\text{Therefore,} \quad M_{max} = f_{max}.Z$$

Section modulus of a beam is the ratio of moment of inertia of the cross section of the bea m about the neutral axis to the distance of the farthest fiber from neutral axis.

$$\text{Therefore,} \quad Z = \frac{I}{y_{max}} \qquad \text{unit} = mm^3$$

More the section modulus more will be the moment of resistive (or) moment carrying capacity of the beam. For the strongest beam, the section modulus must be maximum.

3.10 Problems

1. **A steel bar 10 cm wide and 8 mm thick is subjected to bending mome nt. The radius of neutral surface is 100 cm. Determine maximum and minimum bending stress in the beam.**

Solution : Assume for steel bar $E = 2 \times 10^5$ N/mm^2

$y_{max} = 4$mm
R = 1000mm
$f_{max} = E.y_{max}/R = (2 \times 105 \times 4)/1000$

We get maximum bending moment at lower most fiber, Because for a simply supported beam tensile stress (+ve value) is at lower most fiber, while compressive stress is at top most fiber (–ve value).

F_{max} **= 800 N/mm^2**
f_{min} occurs at a distance of – 4mm
R = 1000mm
$f_{min} = E.y_{min}/R = (2 \times 105\ x - 4$
$)/1000$ f_{min} **= –800 N/mm^2**

2. **A simply supported rectangular beam with symmetrical section 200mm in dept h has mome nt of inertia of 2.26 x 10^{-5} m^4 about its neutral axis. Determine the longest span over which the beam would carry a uniformly distributed load of 4kN/m run such that the stress due to bending does not exceed 125 MN/m^2.**

Solution: Given data:

Depth d = 200mm = 0.2m

I = Moment of inertia = 2.26 × 10-5 m4

UDL = 4kN/m

Bending stress s = 125 MN/m^2 = 125 × 10^6 N/m^2

Span = ?

Since we know that Maximum bending moment for a simply supported beam with UDL on its entire span is given by = $WL^2/8$

i.e; $M = WL^2/8$ -------------(A)

From bending equation $M/I = f/y_{max}$

$y_{max} = d/2 = 0.2/2 = 0.1$m

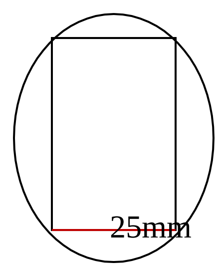
M = f.I/$y$max = [(125 × 106) × (2.26 × 10^{-5})]/ 0.1 = 28250 Nm

Substituting this value in equation (A); we get

28250 = (4 × 103)L^2/8

***L* = 7.52m**

3. **Find the dime nsion of the strongest rectangular beam that can be cut out of a log of 25 mm diameter.**



Solution: $b^2 + d^2 = 25^2$

$d^2 = 25^2 - b^2$

we Know

$M = f\,(I/y) = f.Z$

M will be maximum when Z will be maximum

$Z = I/y = (bd^3/12)/(d/2) = bd^2/6 = b.(25^2 - b^2)/6$

The value of Z maximum at dZ/db = 0;

i.e.; $d/db[25^2b/6 - b^3/6] = 0$

$25^2/6 - 3b^2/6 = 0$

$b^2 = 25^2/3$

b = 14.43 mm

d = 20.41 mm

3.11 Deflection of Beams

3.11.1 INTRODUCTION

Under the action of external loads, the beam is subjected to stresses and deformation at various points along the length. The deformation is caused due to bending moment and shear force. Since the deformation caused due to shear force in shallow beams is very small, it is generally neglected.

3.11.1.1 Elastic Line:

It is a line which represents the deformed shape of the beam. Hence, it is the line along which the longitudinal axis of the beam bends.

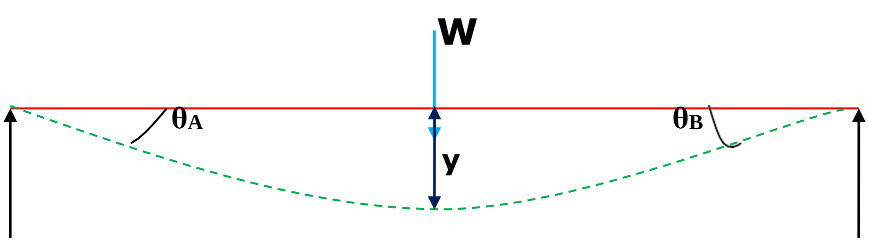
3.11.1.2 Deflection:

Vertical displacement measured from original neutral surface (refer to earlier chapter) to the neutral surface of the deformed beam.

3.11.1.3 Slope:

Angle made by the tangent to the elastic curve with respect to horizontal

The designers have to decide the dimensions of beam not only based on strength requirement but also based on considering deflection. In mechanical components excessive deflection causes mis-alignment and non performance of machine. In building it give rise to psychological unrest and sometimes cracks in roofing materials. Deflection calculations are required to impose consistency conditions in the analysis of indeterminate structures.



3.11.1.4 Strength:

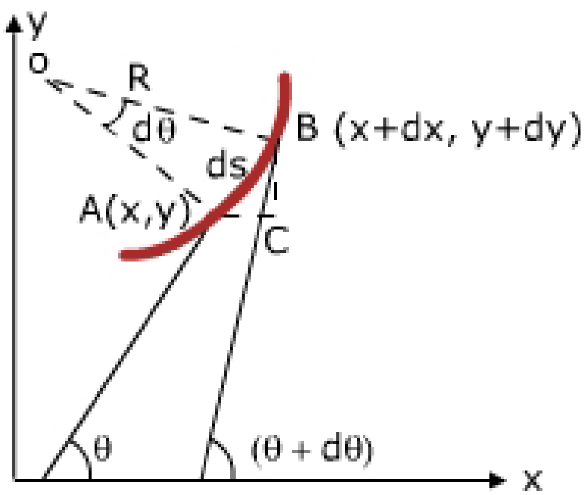
It is a measure of the resistance offered by the beam to load

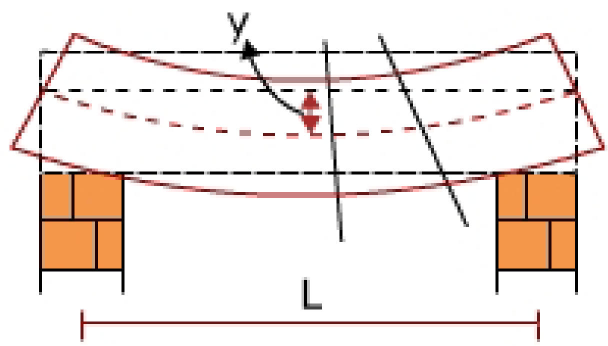
3.11.1.5 Stiffness:

It is a measure at the resistance offered by the beam to deformation. Usually span / deflection is used to denote the stiffness. Greater the stiffness, smaller will be the deflection. The term (EI) called “flexural rigidity” and is used to denote the stiffness.

3.11.2 Flexural Rigidity

The product of Young's modulus and moment of inertia (EI) is used to denote the flexural rigidity.





Let AB be the part of the beam which is bent into an arc of the circle. Let (x,y) be co- ordinates of A and (x + dx, y + dy) be the co-ordinates of B. Let the length of arc AB = ds. Let the tangents at A and B make angles q and (q + d q) with respect to x-axis.

We have __

Differentiating both sides with respect of x;

$$\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\sec^2\theta \quad \frac{d\theta}{ds} \quad \frac{ds}{dx} = \frac{d^2y}{dx^2} \qquad \text{----- (1)}$$

we have from figure $ds = Rd\theta$; $\frac{d\theta}{ds} = \frac{1}{R}$

again in Δ^{le} ABC, $\frac{ds}{dx} = \sec\theta$

From eq. 1; $\frac{d^2y}{dx^2} = \sec^2\theta \, \frac{1}{R} \, \sec\theta$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\sec^2\theta \sec\theta} = \frac{\frac{d^2y}{dx^2}}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

Since dy/dx is small, its square is still small, neglecting $(dy/dx)^2$; we have

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

From bending theory $\frac{M}{I} = \frac{E}{R}$

$$\frac{M}{EI} = \frac{1}{R} \qquad \text{or}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$M = EI \, \frac{d^2y}{dx^2}$$

This is also known as Euler - Bernoulli's equation.

NOTE:

- While deriving Y-axis is taken upwards
- Curvature is concave towards the positive y axis.
- This occurs for sagging BM, which is positive.

Sign Convention

Bending moment 

Sagging +vc

If Y is $+^{ve}$ - Deflection is upwards

Y is $-^{ve}$ - Deflection is downwards

If θ is $+^{ve}$ – Slope is Anticlockwise

θ is $-^{ve}$ – Slope is clockwise

Methods of Calculating Deflection and Slope

- Double Integration method
- Macaulay's method
- Strain energy method
- Moment area method
- Conjugate Beam method

Each method has certain advantages and disadvantages.

Relationship between Loading, S.F, BM, Slope and Deflection

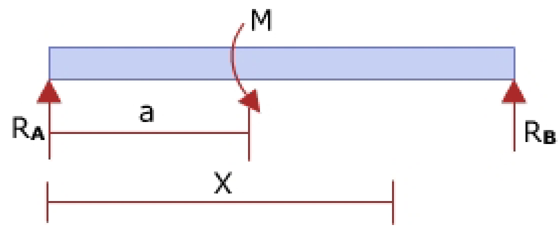
If	Y	-	deflection
Differentiating	$\frac{dy}{dx}$	-	Slope (θ)
Differentiating	$\frac{d^2y}{dx^2}$	-	M. Bending moment
Differentiating	$\frac{dM}{dx}$	=	$\frac{d^3y}{dx^3}$ = Shear force (F)
Differentiating	$\frac{dF}{dx}$	=	$\frac{d^4y}{dx^4}$ = Loading (W)

3.11.3 Macaulay's Method

1. Take the origin on the extreme left.
2. Take a section in the last segment of the beam and calculate BM by considering left portion.
3. Integrate (x-a) using the formula

$$\int (x-a)\,dx = \frac{(x-a)^2}{2}$$

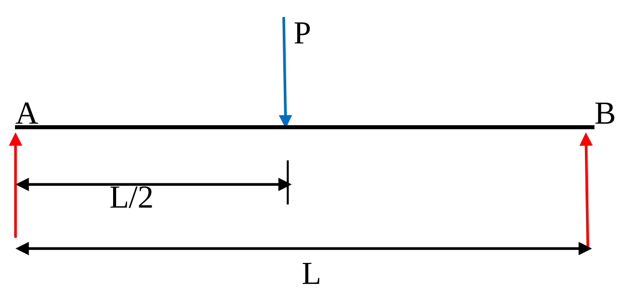
4. If the expression $(x-a)^n$ becomes negative on substituting the value of x, neglect the terms containing the factor $(x-a)^n$
5. If the beam carries UDL and if the section doesn't cuts the UDL, extend the UDL upto the section and impose a UDL in the opposite direction to counteract it.
6. If a couple is acting, the BM equation is modified as; $M = R_A x + M(x-a)^0$.



7. The constant C_1 and C_2 all determined using boundary conditions.
a) S.S. Beam – Deflection is zero at supports
b) Cantilever – Deflection and slope are zero at support.

3.11.4 Problems:

1. **Determine the maximum deflection in a simply supported beam of length L carrying a concentrated load P at its midspan**.



$$EI\,y'' = \tfrac{1}{2}Px - P\langle x - \tfrac{1}{2}L\rangle$$

$EI\, y' = \frac{1}{4}Px^2 - \frac{1}{2}P\langle x - \frac{1}{2}L\rangle^2 + C_1$…………………(1)

$EI\, y = \frac{1}{12}Px^3 - \frac{1}{6}P\langle x - \frac{1}{2}L\rangle^3 + C_1 x + C_2$ …………………(2)

At x =0; y =0 □ C_2 =0
At x = L y =0

$0 = \frac{1}{12}PL^3 - \frac{1}{48}PL^3 + C_1 L$
$C_1 = -\frac{1}{16}PL^2$
Maximum deflection occurs at x = L/2
Substituting the values of x and C_1 in equation…. (2)

$$EI\, y_{max} = \frac{1}{12}P(\frac{1}{2}L)^3 - \frac{1}{6}P(\frac{1}{2}L - \frac{1}{2}L)^3 - \frac{1}{16}PL^2(\frac{1}{2}L)$$

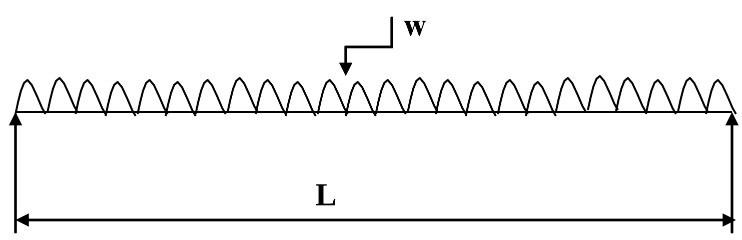
$$y_{max} = -\frac{PL^3}{48EI}$$

The negative sign indicates that the deflection is below the undeformed neural axis

$$\delta_{max} = \frac{PL^3}{48EI}$$

3. Determine the maximum deflection in a simply supported beam of length L carrying a uniformly distributed load ‘w’ for the entire length of the beam.

Solution : From the following fig



$EI\, y'' = \frac{1}{2}w_o Lx - \frac{1}{2}w_o x^2$
$EI\, y' = \frac{1}{4}w_o Lx^2 - \frac{1}{6}w_o x^3 + C_1$ **………………..(1)**
$EI\, y = \frac{1}{12}w_o Lx^3 - \frac{1}{24}w_o x^4 + C_1 x + C_2$**…………(2)**

At x =0 y=0 and C_2 =0

At x =L y =0

$0 = \frac{1}{12} w_o L^4 - \frac{1}{24} w_o L^4 + C_1 L$

$C_1 = -\frac{1}{24} w_o L^3$

Substituting the C_1 values in equation 2 we get

$EI\, y = \frac{1}{12} w_o L x^3 - \frac{1}{24} w_o x^4 - \frac{1}{24} w_o L^3 x$

x = L/2, y is maximum due to symmetric loading

$EI\, y_{max} = \frac{1}{12} w_o L (\frac{1}{2} L)^3 - \frac{1}{24} w_o (\frac{1}{2} L)^4 - \frac{1}{24} w_o L^3 (\frac{1}{2} L)$

$EI\, y_{max} = -\frac{5}{384} w_o L^4$

$$\delta_{max} = \frac{5 w_o L^4}{384 EI}$$