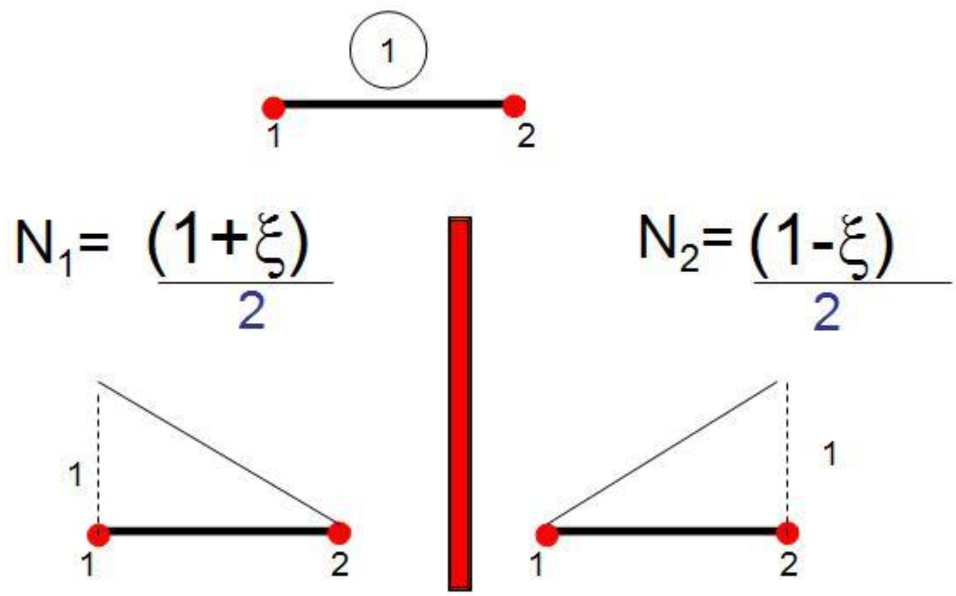
Module 2

Body force distribution for 2 noded bar element

We derived shape functions for 1D bar, variation of these shape functions is shown below .As a property of shape function the value of N_1 should be equal to 1 at node 1 and zero at rest other nodes (node 2).



From the potential energy of an elastic body we have the expression of work done by body force as

$$\int_V u^T f_b \, dv$$

$$U = N_1 q_1 + N_2 q_2$$

For an element

$$\int_e u^T f_b A \, dx$$

Where f_b is the body acting on the system. We know the displacement function $U = N_1q_1 + N_2q_2$ substitute this U in the above equation we get

$$= A f_b \int_e (N_1q_1 + N_2q_2)\, dx$$

$$= A f_b \int_e [N_1 \ \ N_2] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} dx$$

$$= A f_b \int_e \underbrace{[q_1 \ \ q_2]}_{qT} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$$

$$= A f_b \ qT \int_e \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$$

$$= qT \begin{bmatrix} A f_b \int_e N_1 \, dx \\ A f_b \int_e N_2 \, dx \end{bmatrix}$$

Now

$$\int_e N_1 \, dx = \int_e \frac{1-\xi}{2} \, dx$$

but $\frac{dx}{d\xi} = l_{e/2}$

$$= \int_{-1}^{+1} \frac{1-\xi}{2} \frac{l_e}{2} \, d\xi = \frac{l_e}{2}$$

Similarly

$$\int_e N_2 \, dx = \frac{l_e}{2}$$

Therefore

$$\int u^T f_b A \, dx = qT \, A f_b \frac{l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

f_e

This amount of body force will be distributed at 2 nodes hence the expression as 2 in the denominator.

Surface force distribution for 2 noded bar element

Now again taking the expression of work done by surface force from potential energy concept and following the same procedure as that of body we can derive the expression of surface force as

$$\int_s u^T T\, ds = \int_e u^T T\, dx$$

$$= q^T \underbrace{\frac{l_e T}{2}\begin{bmatrix}1\\1\end{bmatrix}}_{T_e}$$

Where T_e is element surface force distribution.

Methods of handling boundary conditions

We have two methods of handling boundary conditions namely Elimination method and penalty approach method. Applying BC's is one of the vital role in FEM improper specification of boundary conditions leads to erroneous results. Hence BC's need to be accurately modeled.

Elimination Method: let us consider the single boundary conditions say $Q_1 = a_1$.Extremising Π results in equilibrium equation.

$Q = [Q_1, Q_2, Q_3 \ldots\ldots\ldots Q_N]^T$ be the displacement vector and

$F = [F_1, F_2, F_3 \ldots\ldots\ldots\ldots F_N]^T$ be load vector

Say we have a global stiffness matrix as

$$K = \begin{bmatrix} K_{11} & K_{12} & \ldots\ldots\ldots\ldots & K_{1N} \\ K_{21} & K_{22} & \ldots\ldots\ldots\ldots & K_{2N} \\ . & & & \\ . & & & \\ . & & & \\ K_{N1} & K_{N2} & \ldots\ldots\ldots\ldots & K_{NN} \end{bmatrix}$$

Now potential energy of the form $\Pi = \frac{1}{2} Q^T K Q - Q^T F$ can written as

$$\Pi = \frac{1}{2}(Q_1K_{11}Q_1 + Q_1K_{12}Q_2 + \ldots + Q_1K_{1N}Q_N$$
$$+ Q_2K_{21}Q_1 + Q_2K_{22}Q_2 + \ldots + Q_2K_{2N}Q_N$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ Q_NK_{N1}Q_1 + Q_NK_{N2}Q_2 + \ldots + Q_NK_{NN}Q_N)$$
$$- (Q_1F_1 + Q_2F_2 + \ldots + Q_NF_N)$$

Substituting $Q_1 = a_1$ we have

$$\Pi = \frac{1}{2}(a_1K_{11}a_1 + a_1K_{12}Q_2 + \ldots + a_1K_{1N}Q_N$$
$$+ Q_2K_{21}a_1 + Q_2K_{22}Q_2 + \ldots + Q_2K_{2N}Q_N$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ Q_NK_{N1}a_1 + Q_NK_{N2}Q_2 + \ldots + Q_NK_{NN}Q_N)$$
$$- (a_1F_1 + Q_2F_2 + \ldots + Q_NF_N)$$

Extremizing the potential energy

ie $d\Pi/dQi = 0$ gives

Where $i = 2, 3 \ldots N$

$$K_{22}Q_2 + K_{23}Q_3 + \ldots + K_{2N}Q_N = F_2 - K_{21}a_1$$
$$K_{32}Q_2 + K_{33}Q_3 + \ldots + K_{3N}Q_N = F_3 - K_{31}a_1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$K_{N2}Q_2 + K_{N3}Q_3 + \ldots + K_{NN}Q_N = F_N - K_{N1}a_1$$

Writing the above equation in the matrix form we get

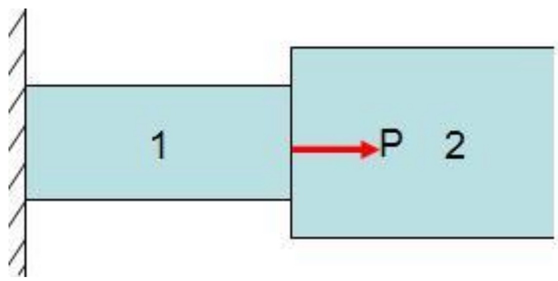
$$\begin{pmatrix} K_{22} & K_{23} & \ldots & K_{2N} \\ K_{32} & K_{33} & \ldots & K_{2N} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ K_{N2} & K_{N3} & \ldots & K_{NN} \end{pmatrix} \begin{pmatrix} Q_2 \\ Q_3 \\ \cdot \\ \cdot \\ \\ Q_N \end{pmatrix} = \begin{pmatrix} F_2 - K_{21}a_1 \\ F_3 - K_{31}a_1 \\ \\ \\ \\ F_N - K_{N1}a_1 \end{pmatrix}$$

Now the N X N matrix reduces to N-1 x N-1 matrix as we know $Q_1=a_1$ ie first row and first column are eliminated because of known Q_1. Solving above matrix gives displacement components. Knowing the displacement field corresponding stress can be calculated using the relation $\sigma = \varepsilon Bq$.

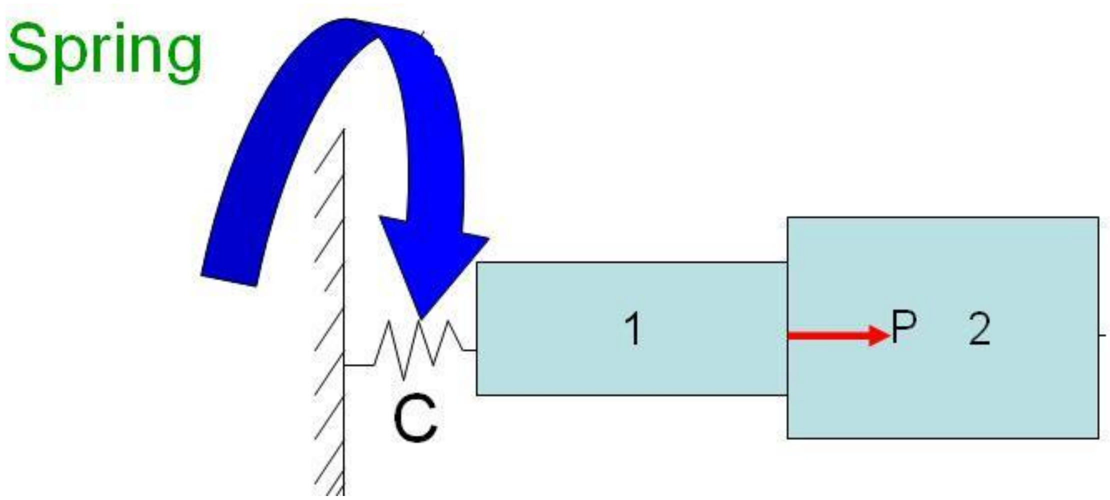
Reaction forces at fixed end say at node1 is evaluated using the relation

$$R_1= K_{11}Q_1+K_{12}Q_2+\dots\dots\dots\dots+K_{1N}Q_N-F_1$$

Penalty approach method: let us consider a system that is fixed at both the ends as shown



In penalty approach method the same system is modeled as a spring wherever there is a support and that spring has large stiffness value as shown.



Let a_1 be the displacement of one end of the spring at node 1 and a_3 be displacement at node 3. The displacement Q_1 at node 1 will be approximately equal to a_1, owing to the relatively small resistance offered by the structure. Because of the spring addition at the support the strain energy also comes into the picture of Π equation .Therefore equation Π becomes

$$\Pi = \frac{1}{2} Q^T K Q + \frac{1}{2} C (Q_1 - a_1)^2 - Q^T F$$

The choice of C can be done from stiffness matrix as

$$C = \max [K_{ij}] \times 10^4$$

We may also choose 10^5 &10^6 but 10^4 found more satisfactory on most of the computers.

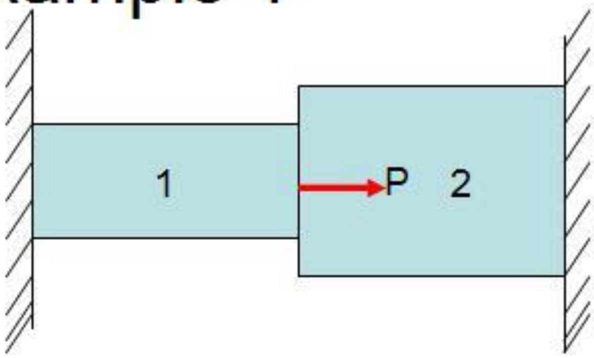
Because of the spring the stiffness matrix has to be modified ie the large number c gets added to the first diagonal element of K and Ca_1 gets added to F_1 term on load vector. That results in.

$$\begin{pmatrix} K_{11} + C & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} = \begin{pmatrix} F1 + C\,a1 \\ F2 \\ F3 \end{pmatrix}$$

A reaction force at node 1 equals the force exerted by the spring on the system which is given by

$$\text{Reaction forces} = - C (Q_1 - a_1)$$

Example 1



A_1 = 900mm^2

A_2 = 1200mm^2

E_1= 70 x 10^9 N/m^2

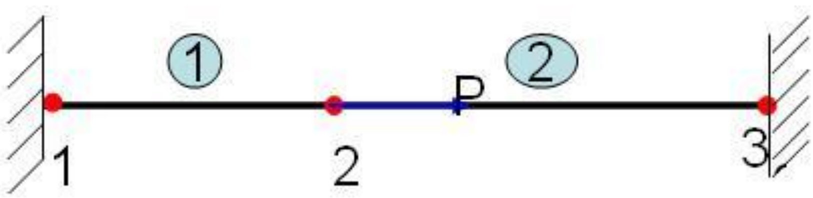
E_2= 200 X 10^9 N/m^2

L_1 = 200mm

L_2 = 300mm

P = 300 KN

To solve the system again the seven steps of FEM has to be followed, first 2 steps contain modeling and discretization. this result in



Third step is finding stiffness matrix of individual elements

$$K_1 = \frac{A_1 E_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{900 \times 0.75 \times 10^5}{200}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix}3.15 & -3.15\\ -3.15 & 3.15\end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

Similarly

$$K_2 = \frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

(rows and columns of the 10^5 matrix labelled 2, 3)

Next step is assembly which gives global stiffness matrix

$$K = \begin{bmatrix} 3.15 & -3.15 & 0 \\ -3.15 & 3.15+8 & -8 \\ 0 & -8 & 8 \end{bmatrix} 10^5$$

(rows and columns labelled 1, 2, 3)

Now determine global load vector

$$F = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 300 \times 10^3 \\ R_3 \end{Bmatrix}$$

We have the equilibrium condition KQ=F

$$10^5 \begin{bmatrix} 3.15 & -3.15 & 0 \\ -3.15 & 3.15+8 & -8 \\ 0 & -8 & 8 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 300 \times 10^3 \\ R_3 \end{Bmatrix} \begin{matrix} \\ -(-3.15 \times 10^5 \times Q1) \\ -(0 \times Q1) \end{matrix}$$

After applying elimination method we have Q2 = 0.26mm

Once displacements are known stress components are calculated as follows

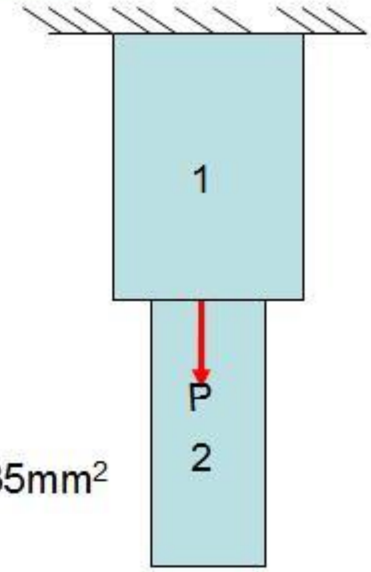
For element 1

$$\sigma_1 = E_1 \frac{1}{L_1} [-1 \quad 1] \begin{Bmatrix} Q1 \\ Q2 \end{Bmatrix} = 94.17 \text{ N/mm}^2$$

For element 2

$$\sigma_2 = E_2 \frac{1}{L_2} [-1 \quad 1] \begin{Bmatrix} Q2 \\ Q3 \end{Bmatrix} = -179.34 \text{ N/mm}^2$$

Example 2



$E_1 = 2.06 \times 10^5$ MPa

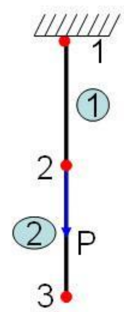
$A_1 = 3387.09 mm^2 \quad A_2 = 2419.35 mm^2$

$L_1 = L_2 = 304.8 mm$

$P = 444.8 N$

Body force = $f_b = 7.69 \times 10^{-5} N/mm^3$

Solution:



$$K_1 = \frac{A_1 E_1}{L_1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 10^6 \begin{pmatrix} 2.28 & -2.28 \\ -2.28 & 2.28 \end{pmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

(columns: 1, 2)

$$K_2 = \frac{A_2 E_2}{L_2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 10^6 \begin{pmatrix} 1.63 & -1.63 \\ -1.63 & 1.63 \end{pmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

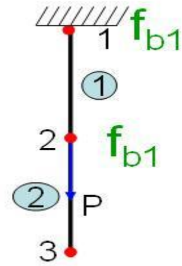
(columns: 2, 3)

$$K = \begin{pmatrix} 2.28 & -2.28 & 0 \\ -2.28 & 2.28+1.63 & -1.63 \\ 0 & -1.63 & 1.63 \end{pmatrix} 10^6 \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

(columns: 1, 2, 3)

Body force terms

Element 1



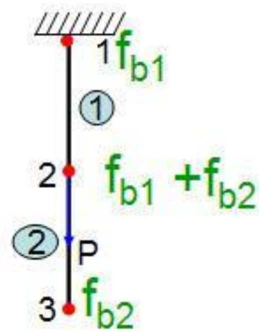
$$f_{b1} = \frac{A_1 f_b L_1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{matrix}1\\2\end{matrix}$$

$$= \frac{3387.09 \times 7.69 \times 10^{-5} \times 304.8}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{matrix}1\\2\end{matrix}$$

$$= \begin{pmatrix}39.69\\39.69\end{pmatrix}\begin{matrix}1\\2\end{matrix}$$

Body force terms

Element 2

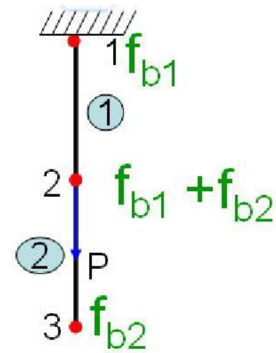


$$f_{b2} = \frac{A_2 f_b L_2}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{matrix}2\\3\end{matrix}$$

$$= \frac{2419.35 \times 7.69 \times 10^{-5} \times 304.8}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{matrix}2\\3\end{matrix}$$

$$= \begin{pmatrix}28.3\\28.3\end{pmatrix}\begin{matrix}2\\3\end{matrix}$$

Global load vector:



$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} f_{b1} \\ p + f_{b1} + f_{b2} \\ fb2 \end{pmatrix} = \begin{pmatrix} 39.69 \\ 512.8 \\ 28.3 \end{pmatrix}$$

We have the equilibrium condition KQ=F

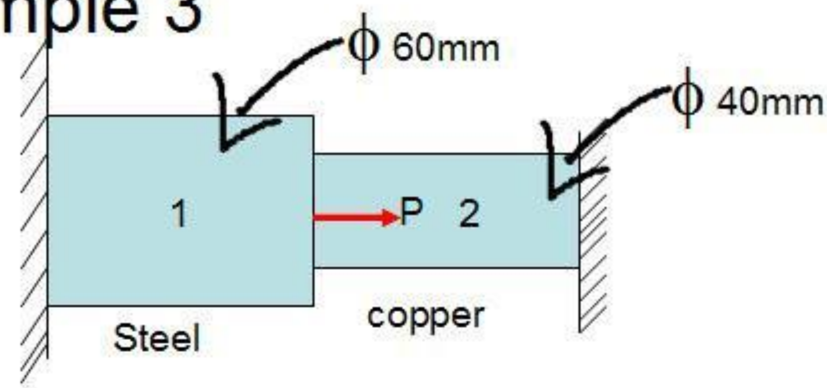
$$10^6 \begin{pmatrix} 2.28 & -2.28 & 0 \\ -2.28 & 6.92 & -16.3 \\ 0 & -1.63 & 1.63 \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} = \begin{pmatrix} 39.69 + R1 \\ 512.8 \\ 28.3 \end{pmatrix}$$

$Q2 = 0.23 \times 10^{-3}$mm

$Q3 = 2.5 \times 10^{-4}$mm

After applying elimination method and solving matrices we have the value of displacements as $Q2 = 0.23 \text{ X } 10^{-3}$mm & $Q3 = 2.5\text{X}10^{-4}$mm

Example 3

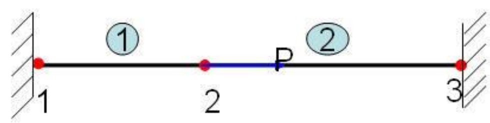


$E_1 = 2 \times 10^5$ MPa $\quad E_2 = 1 \times 10^5$ MPa

$L_1 = 800$ mm $\quad L_2 = 500$mm

$P = 100$ KN

Solution:



$$A_1 = \pi/4\,(60)^2 = 2827.43\text{mm}^2$$

$$A_2 = \pi/4\,(40)^2 = 1256.63\text{mm}^2$$

$$K_1 = \frac{A_1E_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{2827.43\text{x}2\text{x}\,10^5}{800}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^5\begin{bmatrix}7.06 & -7.06\\ -7.06 & 7.06\end{bmatrix}$$

(columns 1, 2; rows 1, 2)

$$K_2 = \frac{A_2E_2}{L_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^5\begin{bmatrix}2.51 & -2.51\\ -2.51 & 2.51\end{bmatrix}$$

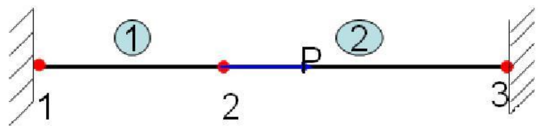
(columns 2, 3; rows 2, 3)

Global stiffness matrix

$$K = \begin{array}{c} \\ \end{array}\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$K = \begin{pmatrix} 7.07 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 \end{pmatrix} 10^5 \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Global load vector:



$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \times 10^3 \\ 0 \end{pmatrix}$$

Equilibrium Equation

$$K\,Q = F$$

$$K = \begin{pmatrix} 7.07 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 \end{pmatrix} 10^5 \begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \times 10^3 \\ 0 \end{pmatrix}$$

$$C = \max\,[\,K_{ij}\,] \times 10^4 = 9.583 \times 10^5 \times 10^4$$

Modification required

$$\begin{bmatrix} 7.07 + C & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 + C \end{bmatrix} 10^5 \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} = \begin{Bmatrix} 0 + C\,a1 \\ 100 \times 10^3 \\ 0 + C\,a3 \end{Bmatrix}$$

After Modification

$$\begin{bmatrix} 9.583 \times 10^4 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 9.583 \times 10^4 \end{bmatrix} 10^5 \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \times 10^3 \\ 0 \end{Bmatrix}$$

Solving the matrix we have

$Q1 = 7.698 \times 10^{-6}$mm, $Q2 = 0.104$mm, $Q3 = 2.736 \times 10^{-6}$mm

Reaction forces

@ node 1

$R_1 = C(Q1 - a1) = -73597.44$N

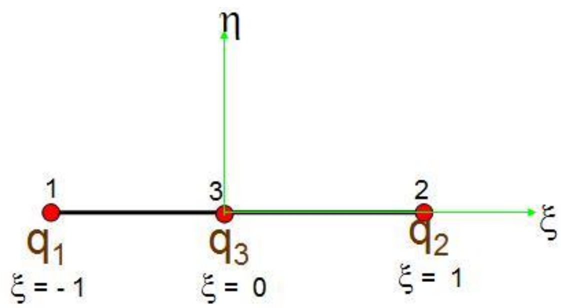
@ node 3

$R_3 = C(Q3 - a3) = -26219.08$N

Quadratic 1D bar element

In the previous sections we have seen the formulation of 1D linear bar element , now lets move a head with quadratic 1D bar element which leads to for more accurate results . linear element has two end nodes while quadratic has 3 equally spaced nodes ie we are introducing one more node at the middle of 2 noded bar element.

Consider a quadratic element as shown and the numbering scheme will be followed as left end node as 1, right end node as 2 and middle node as 3.



Let's assume a polynomial as

$$U = \alpha_o + \alpha_1 \xi + \alpha_2 \xi^2$$

Now applying the conditions as

@ node 1	$u = q_1$	$\xi = -1$
@ node 2	$u = q_2$	$\xi = 1$
@ node 3	$u = q_3$	$\xi = 0$

ie

$$q_1 = \alpha_o - \alpha_1 + \alpha_2$$

$$q_2 = \alpha_o + \alpha_1 + \alpha_2$$

$$q_3 = \alpha_o$$

Solving the above equations we have the values of constants

$$\alpha_1 = \frac{q_2 - q_1}{2} \qquad \alpha_2 = \frac{q_1 + q_2 - 2q_3}{2}$$

And substituting these in polynomial we get

$$U = \alpha_o + \alpha_1 \xi + \alpha_2 \xi^2$$

$$= q_3 + \left(\frac{q_2 - q_1}{2}\right)\xi + \left(\frac{q_1 + q_2 - 2q_3}{2}\right)\xi^2$$

$$= \frac{\xi(\xi-1)}{2} q_1 + \frac{\xi(\xi+1)}{2} q_2 + (1-\xi^2) q_3$$

Or

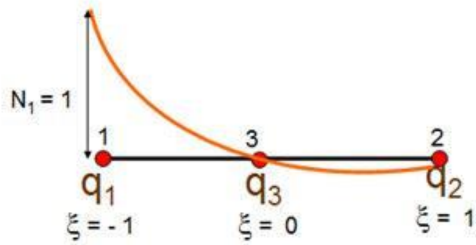
$$U = N_1 q_1 + N_2 q_2 + N_3 q_3$$

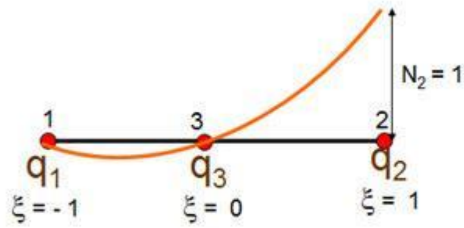
Where N1 N_2 N_3 are the shape functions of quadratic element

$$N_1 = \frac{\xi(\xi-1)}{2}$$

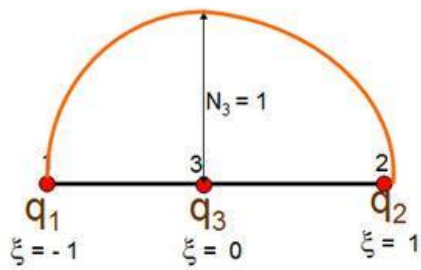
$$N_2 = \frac{\xi(\xi+1)}{2}$$

$$N_3 = (1-\xi^2)$$





$$N_1 = \frac{\xi(\xi-1)}{2} \qquad N_2 = \frac{\xi(\xi+1)}{2}$$



$$N_3 = (1-\xi^2)$$

Graphs show the variation of shape functions within the element .The shape function N_1 is equal to 1 at node 1 and zero at rest other nodes (2 and 3). N_2 equal to 1 at node 2 and zero at rest other nodes(1 and 3) and N_3 equal to 1 at node 3 and zero at rest other nodes(1 and 2)

Element strain displacement matrix If the displacement field is known its derivative gives strain and corresponding stress can be determined as follows

WKT

$$U = N_1 q_1 + N_2 q_2 + N_3 q_3$$

$$\varepsilon = \frac{du}{dx}$$

$$= \frac{du}{d\xi} \frac{d\xi}{dx} \qquad \text{By chain rule}$$

Now

$$\frac{du}{d\xi} = \frac{d[N_1 q_1 + N_2 q_2 + N_3 q_3]}{d\xi}$$

Splitting the above equation into the matrix form we have

$$\frac{du}{d\xi} = \frac{d[N_1 \; N_2 \; N_3]}{d\xi} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$\frac{du}{d\xi} = \begin{pmatrix} \frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi \end{pmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Therefore

$$\varepsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx}$$

$$= \left[\frac{(2\xi-1)}{2} \quad \frac{(2\xi-1)}{2} \quad -2\xi\right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \frac{d\xi}{dx}$$

$$= \frac{2}{le}\left[\frac{(2\xi-1)}{2} \quad \frac{(2\xi+1)}{2} \quad -2\xi\right] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$\varepsilon = Bq$$

B is element strain displacement matrix for 3 noded bar element

Stiffness matrix:

We know the stiffness matrix equation

$$K = \int_V B^T E\, B\, dv$$

For an element

$$K = \int_e B^T E\, B\, A\, dx$$

$$= \int_e B^T E\, B\, A \frac{L_e}{2} d\xi$$

Taking the constants outside the integral we get

$$K = \frac{E\,A\,L_e}{2} \int_e B^T B\, d\xi$$

Where

$$B = \frac{2}{le} \begin{pmatrix} \frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi \end{pmatrix}$$

and B^T

$$B^T = \frac{2}{le} \begin{pmatrix} \frac{(2\xi-1)}{2} \\ \frac{(2\xi+1)}{2} \\ -2\xi \end{pmatrix}$$

Now taking the product of B^T X B and integrating for the limits -1 to +1 we get

$$K = \frac{E\,A\,L_e}{2} \int_e B^T B\, d\xi$$

$$= \frac{E\,A\,L_e}{2} \int_{-1}^{+1} \frac{4}{L_e^2} \begin{pmatrix} \frac{1}{4}(2\xi-1)^2 & \frac{1}{4}(2\xi-1)(2\xi+1) & -(2\xi-1)\xi \\ \frac{1}{4}(2\xi-1)(2\xi+1) & \frac{1}{4}(2\xi+1)^2 & -(2\xi+1)\xi \\ -(2\xi-1)\xi & -(2\xi+1)\xi & 4\xi^2 \end{pmatrix}$$

Integration of a matrix results in

$$K = \frac{EA}{3L_e}\begin{pmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{pmatrix}$$

Body force term & surface force term can be derived as same as 2 noded bar element and for quadratic element we have

Body force:

$$f_e = A f_b l_e \begin{pmatrix} 1/6 \\ 1/6 \\ 2/3 \end{pmatrix}$$

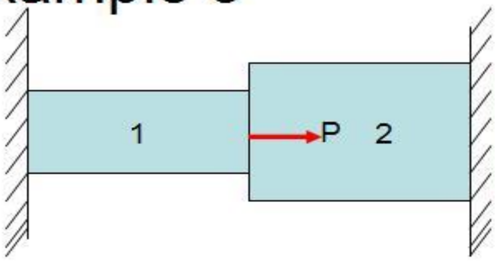
Surface force term:

$$T_e = T \; l_e \begin{pmatrix} 1/6 \\ 1/6 \\ 2/3 \end{pmatrix}$$

This amount of body force and surface force will be distributed at three nodes as the element as 3 equally spaced nodes.

Problems on quadratic element

Example 5



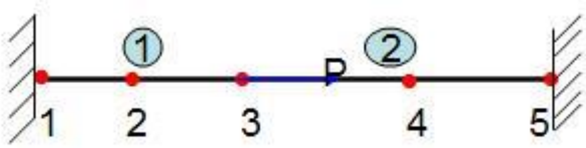
$A_1 = 600mm^2$ $A_2 = 800mm^2$

$E= 2 \times 10^5\ N/mm^2$

$L_1 = 150mm$ $L_2 = 220mm$

$P = 30\ KN$

Solution:



$$K_1 = 10^5 \begin{array}{c} \\ \end{array} \begin{array}{ccc} 1 & 3 & 2 \\ \end{array}$$

$$K_1= 10^5 \begin{pmatrix} 18.6 & 2.6 & -21.3 \\ 2.6 & 18.6 & -21.3 \\ -21.3 & -21.3 & 42.6 \end{pmatrix} \begin{array}{c} 1 \\ 3 \\ 2 \end{array}$$

$$\begin{array}{ccc} 3 & 5 & 4 \end{array}$$

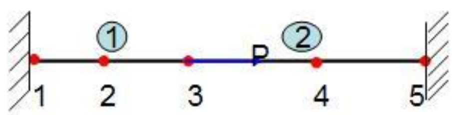
$$K_2= 10^5 \begin{pmatrix} 16.9 & 2.42 & -19.3 \\ 2.42 & 16.9 & -19.3 \\ -19.3 & -19.3 & 38.7 \end{pmatrix} \begin{array}{c} 3 \\ 5 \\ 4 \end{array}$$

Global stiffness matrix

$$
K = 10^5 \begin{pmatrix} 18.6 & -21.3 & 2.6 & 0 & 0 \\ -21.3 & 42.6 & -21.3 & 0 & 0 \\ 2.6 & -21.3 & 35.5 & -19.3 & 2.4 \\ 0 & 0 & -19.3 & 38.7 & -19.3 \\ 0 & 0 & 2.4 & 19.3 & 16.9 \end{pmatrix}
$$

(Columns numbered 1, 2, 3, 4, 5 above the matrix; rows numbered 1, 2, 3, 4, 5 at the right.)

Global load vector



$$
F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \\ P \\ 0 \\ R_5 \end{pmatrix}
$$

By the equilibrium equation KQ=F, solving the matrix we have Q2, Q3 and Q4 values

$$10^5 \begin{pmatrix} 18.6 & -21.3 & 2.6 & 0 & 0 \\ -21.3 & 42.6 & -21.3 & 0 & 0 \\ 2.6 & -21.3 & 35.5 & -19.3 & 2.4 \\ 0 & 0 & -19.3 & 38.1 & -19.3 \\ 0 & 0 & 2.4 & 19.3 & -16.9 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \\ P \\ 0 \\ R_5 \end{pmatrix}$$

$Q_2 = 1.25 \times 10^{-7}$ mm

$Q_3 = 2.14 \times 10^{-3}$ mm

$Q_5 = 5.13 \times 10^{-3}$ mm

Stress components in each element

For element 1 @ node 1

$$\sigma_{1/1} = \frac{2}{l_1} \begin{bmatrix} \frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi \end{bmatrix} \begin{bmatrix} Q1 \\ Q2 \\ Q3 \end{bmatrix} E$$

$$\sigma_{1/1} = \frac{2}{150} \begin{bmatrix} -3/2 & -½ & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0.01 \\ 0.02 \end{bmatrix} 2 \times 10^5$$

$= 93.1\ N/mm^2$

For element 1 @ node 2

$$\sigma_{1/2} = \frac{2}{l_1} \begin{bmatrix} \frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi \end{bmatrix} \begin{bmatrix} Q1 \\ Q2 \\ Q3 \end{bmatrix} E$$

$$\sigma_{1/2} = \frac{2}{150} \begin{bmatrix} -½ & ½ & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.01 \\ 0.02 \end{bmatrix} 2 \times 10^5$$

$= 13.33\ N/mm^2$

For element 1 @ node 3

$$\sigma_{1/3} = \frac{2}{l_1}\left(\frac{(2\xi-1)}{2} \quad \frac{(2\xi+1)}{2} \quad -2\xi\right)\begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} E$$

$$\sigma_{1/3} = \frac{2}{150}\left(½ \quad 3/2 \quad -2\right)\begin{pmatrix} 0 \\ 0.01 \\ 0.02 \end{pmatrix} 2 \times 10^5$$

$= -66.5 \text{ N/mm}^2$

For element 2 @ node 3

$$\sigma_{2/3} = \frac{2}{l_2}\left(\frac{(2\xi-1)}{2} \quad \frac{(2\xi+1)}{2} \quad -2\xi\right)\begin{pmatrix} Q3 \\ Q4 \\ Q5 \end{pmatrix} E$$

$$\sigma_{2/3} = \frac{2}{220}\left(-3/2 \quad -½ \quad 2\right)\begin{pmatrix} 0.02 \\ 0.01 \\ 0 \end{pmatrix} 2 \times 10^5$$

$= -63.63 \text{ N/mm}^2$

For element 2 @ node 4

$$\sigma_{2/4} = \frac{2}{l_2}\left(\frac{(2\xi-1)}{2} \quad \frac{(2\xi+1)}{2} \quad -2\xi\right)\begin{pmatrix} Q3 \\ Q4 \\ Q5 \end{pmatrix} E$$

$$\sigma_{2/4} = \frac{2}{220}\left(-½ \quad ½ \quad 0\right)\begin{pmatrix} 0.02 \\ 0.01 \\ 0 \end{pmatrix} 2 \times 10^5$$

$= -9.09 \text{ N/mm}^2$

For element 2 @ node 5

$$\sigma_{2/5} = \frac{2}{l_1}\begin{bmatrix} \frac{(2\xi-1)}{2} & \frac{(2\xi+1)}{2} & -2\xi \end{bmatrix} \begin{Bmatrix} Q3 \\ Q4 \\ Q5 \end{Bmatrix} E$$

$$\sigma_{2/5} = \frac{2}{150}\begin{bmatrix} ½ & 3/2 & -2 \end{bmatrix} \begin{Bmatrix} 0.02 \\ 0.01 \\ 0 \end{Bmatrix} 2 \times 10^5$$

$$= 45.45 \text{ N/mm}^2$$

Solution to Simultaneous Algebraic Equations – Gauss Elimination Method:

Consider n simultaneous equations ,

$a_{11} x_1 + a_{12} x_2 +, a_{13} x_3$ .... $+ a_{1n} x_n$ b_1

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3$ ....$+ a_{2n} x_n$ b_2

$a_{31} x_1 + a_{32} x_2 + a_{33} x_3$ .... $+ a_{3n} x_n$ b_3

$a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3$ .... $+ a_{nn} x_n$ · b_n

write the given set of equations in matrix form,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ . & . & . & \ldots & \\ . & . & . & \ldots & \\ . & . & . & \ldots & \\ . & . & . & \ldots & \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ b_n \end{bmatrix}$$

In Gauss elimination method the variables x_2, … ………… …… x_{n-1}, will be successively eliminated using **Row Operations**. This step is called **forward elimination** . The given matrix will be converted to into an upper triangular matrix, Lower triangular elements become zeros.

After forward elimination the n^{th} equation (last equation) become simple , it as an equation with one variable x_n , determine x_n. Now using $(n-1)^{th}$ equation x_{n-1} can be determined. Similarly using $(n-2)^{nd}$ equation x_{n-2} can be determined. Using $(n-3)^{rd}$ equation x_{n-3} can be determined. Continue up to first equation until all the unknowns are determined. This is called **backward substitution.**

Forward Elimination

Step 1 : a_{11} becomes pivot, eliminate x_1 from row2 , row3, row4, row n etc

Row2 $a_{21} = a_{21} - (a_{21} / a_{11})\, a_{11}$ a_{21} becomes 0

$a_{22} = a_{22} - (a_{21} / a_{11})\, a_{12}$ a_{22} changes

$a_{23} = a_{23} - (a_{21} / a_{11})\, a_{13}$ a_{23} changes

etc up to $a_{1n} = a_{2n} - (a_{21} / a_{11})\, a_{1n}$ a_{2n} changes

$b_2 = b_2 - (a_{21} / a_{11})\, b_1$ b_2 changes

whatever we did to make $a_{21} = 0$ applied the same to other elements of that row

Row3 $a_{31} = a_{31} - (a_{31} / a_{11})\, a_{11}$ a_{31} becomes 0

$a_{32} = a_{32} - (a_{31} / a_{11})\, a_{12}$ a_{32} changes

$a_{33} = a_{33} - (a_{31} / a_{11})\, a_{13}$ a_{33} changes

etc up to $a_{3n} = a_{3n} - (a_{31} / a_{11})\, a_{1n}$ a_{3n} changes

$b_3 = b_3 - (a_{31} / a_{11})\, b_1$ b_3 changes

whatever we did to make $a_{31} = 0$ applied the same to other elements of that row

……………………………………………………………………………
……………………………………………………………………………
……………………………………………………………………………

Row n $a_{n1} = a_{n1} - (a_{n1} / a_{11})\, a_{11}$ a_{n1} becomes 0

$a_{n2} = a_{n2} - (a_{n1} / a_{11})\, a_{12}$ a_{n2} changes

$a_{n3} = a_{n3} - (a_{n1} / a_{11})\, a_{13}$ a_{n3} changes

etc up to $a_{nn} = a_{nn} - (a_{n1} / a_{11})\, a_{1n}$ a_{nn} changes

$b_3 = b_3 - (a_{31} / a_{11})\, b_1$ b_3 changes

whatever we did to make $a_{n1} = 0$ applied the same to other elements of that row.

Now, re- write the whole matrix equation. First row remains same, elements of other rows will be different.

Step2 : : a_{22} becomes pivot, eliminate x_2 from row3 , row4, row5, etc., row n following the same method

Now, re-write the whole matrix equation. First row , Second row remains same, elements of other rows will be different

Step3 : : a_{33} becomes pivot, eliminate x_3 from row4 , row5, row6, etc., row n following the same method

Now, re-write the whole matrix equation. First row , Second row , Third row remains same, elements of other rows will be different

Continue until the variables x_2, x_3, x_4 … ………………………… x_{n-1} will be successively eliminated and all the lower triangular elements becomes zero.

Backward substitution.
After forward elimination the n^{th} equation (last equation) become simple , it as an equation with one variable x_n , determine x_n. Now , using $(n-1)^{th}$ equation x_{n-1} can be determined. Similarly using $(n-2)^{nd}$ equation x_{n-2} can be determined. Using $(n-3)^{rd}$ equation x_{n-3} can be determined. Continue up to first equation until all the unknowns are determined. The method is best understood by solving problems.

Different Methods used to Solve Set of Simultaneous Equations in FEM.

Method of Matrix Inversion
Gauss elimination method
Cholesky Decomposition Technique
Gauss-Seidal Iteration Technique
Relaxation Method

Numerical examples illustrating Gauss elimination method :

Problem 1. Solve the following set of equation by Gaussian elimination technique.

5x1 + 3x2 + 2x3 + x4 = 4
4x1 + 3x2 - 3x3 - 2x4 = 5
x1 + 2x2 - 2x3 + 3x4 = 6 -
4x1 + 3x2 – 5x3 + 2x4 = 7

Solution : Write the given equations in Matrix Form

5	3	2	1		x1		4
4	3	-3	-2		x2		5
1	2	-2	3	*	x3	=	6
-4	3	-5	2		x4		7

[CO] [X] = [CONS]
[a] [x] = [b]

Step 1 : $a_{ij} = a_{ij} - (a_{i1} / a_{11})\, a_{1j}$ $b_i = b_i - (a_{i1} / a_{11})\, b_1$

i = 2, j = 1,2,3,4

Row 2 i = 2 j = 1,2,3,4

4 – (4/5) 5 = 4 – (4) = 0.0 3 – (4/5) 3 = 3 – 2.4 = 0.6

-3 – (4/5)2 = -3 –1.6 = -4.6 -2 – (4/5) 1 = -2 – 0.8 = -2.8

Row 3 i = 3 j = 1,2,3,4

1 – (1/5)5 = 1 – 1 = 0.0 2 – (1/5)3 = 2 -0 .6 = 1.4

-2 – (1/5) 2 = -2– 0.4= -2.4 3 – (1/5) 1 = 3 – 0.2 = 2.8

Row 4 : i = 4 j = 1,2,3,4

-4 – (-4 / 5)5 = 4 - 4 = 0 3 – (-4 / 5)3 = 3+2.4 = 5.4

-5 – (-4 /5) 2 = -5 + 1.6 = -3 .4 2 – (-4 /5) 1 = 3 – 0.2 = 2.8

b **i** = b **i** – (a **i1**/ a**11**) b **1**

i = 2 b2 = 5 – (4/5)4 = 1.8

i = 3 b3 = 6 – (1/5)4 = 5.2

i = 4 b4 = 7 – (-4/5)4 = 10.2

The modified matrix equation , after eliminating x1 from 2nd , 3rd and 4th equations.

5	3	2	1		x1		4
0	0.6	-4.6	-2.8		x2		1.8
0	1.4	-2.4	2.8	*	x3	=	5.2
0	5.4	-3.4	2.8		x4		10.2

Step 2 : To eliminate x2 from Row 3 and Row 4

$a_{ij} = a_{ij} - (a_{i2} / a_{22})\, a_{2j}$ $b_i = b_i - (a_{i2} / a_{22})\, b_2$ $i = 3, j = 2,3,4$

Row 3   i =3   j = 2,3,4
1.4 – (1.4/0.6) 0.6 =1.4 – 1.4 = 0.0   j = 2
-2.4 – (1.4/0.6) (-4.6) = – 2.4+10.73 =8.33 j = 3
2.8 – (1.4/0.6)(-2.8) = -2.8 – 6.53 = 9.33   j = 4

Row 4   i = 4   j = 2,3,4

5.4– (5.4/0.6)0.6 = 5.4-5.4 = 0.0   j = 2
-3.4 – (5.4/0.6)(-4.6) = -3.4+41.4 = 38 j = 3
2.8 – (5.4/0.6)2.8 = 2.8 +25.2 = 28   j = 4

b i = b i – (a i2/ a22) b 2

i = 3 b3 = b3 – (a32/a22)b2
5.2 - (1.4/0.6) 1.8 = 5.2 – 4.2 = 1

i = 4 b4 = b4 – (a42/a22)b2
10.2 - (5.4/0.6)1.8 = 10.2 -16.2 = -6

The modified matrix after step 2 eliminating x2 from 3rd and 4th equations.

5	3	2	1		x1		4
0	0.6	-4.6	-2.8		x2		1.8
0	0	8.33	9.32	*	x3	=	1
0	0	38	28		x4		-6

Step 3 : To eliminate x3 from Row 4

$a_{ij} = a_{ij} - (a_{i3} / a_{33})\, a_{3j}$ $b_i = b_i - (a_{i3} / a_{33})\, b_3$ $i = 4, j = 3,4$

a43 = a43 – (a43/ a**33)** a**33** = 38 – (38 / 8.33) 8.33 = 38 – 38 = 0.0

a44 = a44- (a43/ a**33**) a34 = 28 - (38 / 8.33) 9.32
= 28 – 42.52 = -1 4.52

b i = b i – (a **i3**/ a**33**) b **3** i = 4,
b4 = b4 - (a43/ a**33**) b3 b4 = -6 - (38 / 8.33) 1 = - 6 - 4.56 = - 10.56
The modified matrix, after step 3 , eliminating x3 from 4th equation.

5	3	2	1		x1		4
0	0.6	-4.6	-2.8		x2		1.8
0	0	8.33	9.32	*	x3	=	1
0	0	0	-14.52		x4		-10.56

Back Substitution:

The modified equations are

5 x1 + 3 x2 + 2 x3 + x4 = 4
0.6 x2 – 4.6 x3 - 2.8 x4 = 1.8
8.33 x3 + 9.32 x4 = 1
-14.52 x4 = -10.56
x4 = (-1 0.56/ -1 4.52) = **0.727**

8.33 x3 + 9.32 (0.727) = 1
x3 = (1 - 6.776) / 8.33 = **- 0.693**

0.6 x2 – 4.6(- 0.693) -2.8 (0.727) =
1.8 0.6 x2 = (1.8 - 3.1878 + 2.0356)
x2 = (1.8 - 3.1878 + 2.0356) / 0.6 = **1.079**

5x1 + 3 (1.079) + 2 (- 0.693) + 0.727 = 4
x1 = (4 - 3 (1.079) + 2 (0.693) - 0.727) / 5 = **0.155**

x1= 0.155 x2=1.079 x3= - 0.693 x4= 0.727

Prob2 : Solve using gauss elimination method

x1- 2x2 + 6 x3 = 0 2x1 +2x2 + 3x3 = 3 -x 1 + 3x2 = 2

1	-2	6		x1		0
2	2	3		x2	=	3
-1	3	0		x3		2

Step 1 : $a_{ij} = a_{ij} - (a_{i1} / a_{11})\, a_{1j}$ I = 2,3 j = 2,3

2- (2/1)* 1 = 0 2 – (2/1) (-2) = 6 3 – (2/1) (6) = -9
-1- (-1/1) 1 = 0 3 – (-1/1) (-2) = 1 0 – (-1/1) 6 = 6

$$b_i = b_i - (a_{i1} / a_{11})\, b_1$$

3 – (2/1) *0 = 3 2 – (-1/1) *0 = 2

Modified Matrix After step1

1	-2	6		x1		0
0	6	-9		x2	=	3
0	1	6		x3		2

Step 2 : $a_{ij} = a_{ij} - (a_{i2} / a_{22})\, a_{2j}$ i = 3 j = 2,3

1 – (1/6)* (6) = 0 6 – (1/6) (-9) = 7.5

$$b_i = b_i - (a_{i2} / a_{22})\, b_2$$

2 – (1/6) 3 = 1.5

Modified Matrix After step2

1	-2	6	x1	=	0
0	6	-9	x2		3
0	0	7.5	x3		1.5

Back Substitution :

7.5 x3 = 1.5 x3 = 1.5 / 7.5 = 0.2 x3 = 0.2

6 x2 – 9x3 = 3 x2=(3 + 9(0.2))/6 =0.8 x2 = 0.8

x1 – 2x2 +6x3 = 0 x1 = 2(0.8)–6(0.2) = 0.4 x1 = 0.4

Prob 3 :Solve using gauss elimination method

4x1 +6x2 +8x3 = 2 8x1 +4x2 + 6x3 = 4 6x1 + 2x2+4x3= 6

Solution : writing the equations in matrix form

4	6	8	=	x1	2
8	4	6		x2	4
6	2	4		x3	6

Step 1 : $a_{ij} = a_{ij} - (a_{i1} / a_{11})\, a_{1j}$ i = 2,3 j = 2,3

8 - (8/4)* 4 = 0 4– (8/4) (6) = -8 6 -(8/4)8 = -10

6 -(6/4) 4 = 0 2– (6/4 (6) = -7 4 – (6 /4) 8 = -8

$b_i = b_i - (a_{i1} / a_{11})\, b_1$

4 – (8/4) 2 = 0 6 – (6/4) 2 = 3

Modified matrix equation after step 1

4	6	8	=	X1		2
0	-8	-10		X2		0
0	-7	-8		X3		3

Step 2 : $a_{ij} = a_{ij} - (a_{i2} / a_{22})\, a_{2j}$ $i = 3$ $j = 2,3$

$-7 - (-7/-8)(-8) = 0$

$-8 \;\; -(-7 / -8)(-10) = -8 \quad +70/8 = -8 + 8.75 = 0.75$

$b_i = b_i - (a_{i2} / a_{22})\, b_2$

$3 - (-7/-8)\, 0 = 3$

Modified matrix equation after step 2 :

4	6	8	X1		2
0	-8	-10	X2		0
0	0	0.75	x3		3

$0.75\, x3 = 3 \quad x3 = 3/0.75 = 4 \qquad\qquad x3 = 4$

$-8\, x2 - 10\, x3 = 0 \qquad -8x2 - 10\,(4) = 0 \qquad -8x2 = 40 \quad x2 = -5$

$4x1 + 6x2 + 8x3 = 2 \qquad 4x1 + 6(-5) + 8(4) = 2$

$x1 = (2 + 30 - 32) / 4 = 0$

$x1 = 0 \quad x2 = -5 \qquad x3 = 4$

Prob 4 : Solve using gauss elimination method

$3x1 - 3x2 - 2x3 = 5 \qquad 2x1 + 2x2 + 3x3 = 6 \qquad 3x1 - 5x2 + 2x3 = 7$

3	-3	-2		5
2	2	3		6
3	-5	2		7

Step 1 :

$a_{ij} = a_{ij} – (a_{i1} / a_{11}) a_{1j}$ i = 2,3 j = 2,3

2 - (2/3)3 = 0   2 - (2/3)(-3) = 4 3   -(2/3)(-2) = 4.33

3 -(3/3) 3 = 0   -5 – (3/3 (-3) = -2   2 – (3/3)(-2) = 4

$b_i = b_i - (a_{i1} / a_{11}) b_1$

6 – (2/3) 5 = 6-10/3 = 2.667 7 – (3/3) 5 = 2

Modified matrix is

3	-3	-2		5
0	4	4.33		2.667
0	-2	4		2

Step 2 :

$a_{ij} = a_{ij} – (a_{i2} / a_{22}) a_{2j}$ i = 3 j = 2,3

-2 - (-2/4)4 = 0   4 - (-2/4)(4.333) = 4+ 2.166 = 6.166

$b_i = b_i - (a_{i2} / a_{22}) b_2$

2 – (-2/4) 2.667 = 2+(2.667/2) = 2+1.333 = 3.333

3	-3	-2		5
0	4	4.33		2.667
0	0	6.166		3.333

6.166 x3 = 3.333 x3 = (3.333/6.166) = 0.504

4x2 + 4.333x3 = 2.667 4x2 = 2.667 - 4.333 (0.504) x2 = 0.483 /4 = 0.120

3x1 -3x2 – 2x3 = 5   3x1 -3(0.120) -2(0.504) = 5

x1   = (5 + 0.360 +1.008) / 3 = 2.122

x1   = 2.122 x2 = 0.120   x3 = 0.504

HIGHER ORDER ELEMENTS

Many engineering structures and mechanical components are subjected to loading in two directions. Shafts, gears, couplings, mechanical joints, plates, bearings, are few examples. Analysis of many three dimensional systems reduces to two dimensional, based on whether the loading is plane stress or plane strain type. Triangular elements or Quadrilateral elements are used in the analysis of such components and systems. The various load vectors, displacement vectors, stress vectors and strain vectors used in the analysis are as written below,

the displacement vector **u** = $[u, v]^T$,
u is the displacement along x direction, v is the displacement along y direction,

the body force vector **f** = $[f_x , f_y]^T$
f_x , is the component of body force along x direction, f_y is the component of body force along y direction

the traction force vector **T** = $[T_x , T_y]^T$
T_x , is the component of body force along x direction, T_y is the component of body force along y direction

Two dimensional stress strain equations

From theory of elasticity for a two dimensional body subjected to general loading the equations of equilibrium are given by

$$[\ _x / x] + [\ _{yx} / \ y] + F_x = 0$$

$$[\ _{xy} / x] + [\ _y / y] + F_y = 0$$

$$\text{Also} \quad _{xy} = \ _{yx}$$

The strain displacement relations are given by

$$_x = u / x, \ _y = v / y, \ _{xy} = u / y + v / x = [u / x, v / y, (u / y + v / x)]^T$$

The stress strain relationship for plane stress and plane strain conditions are given by the matrices shown in the next page. $_x$ $_y$ $_{xyx}$ $_y$ $_{xy}$ are usual stress strain components, v is the poisons ratio. E is young's modulus. Please note the differences in [D] matrix .

Two dimensional elements

Triangular elements and **Quadrilateral elements** are called two dimensional elements. A simple triangular element has straight edges and corner nodes. This is also a linear element. It can have constant thickness or variable thickness.

The stress strain relationship for plane stress loading is given by

x			1	V	0		z
y	=	$E / (1-v^2)$	V	1	0	*	y
xy			0	0	1-v / 2		yz

[ ] = [**D**] [ ]

The stress strain relationship for plane strain loading is give by

x			1-v	V	0		z
y			V	1-v	0		y
xy	=	E / (1+v)(1-2v)	0	0	½ -v	*	yz

[ ] = [**D**] [ ]

The element having mid side nodes along with corner nodes is a higher order element. Element having curved sides is also a higher order element.

A simple quadrilateral element has straight edges and corner nodes. This is also a linear element. It can have constant thickness or variable thickness. The quadrilateral having mid side nodes along with corner nodes is a higher order element. Element having curved sides is also a higher order element.

The given two dimensional component is divided in to number of triangular elements or quadrilateral elements. If the component has curved boundaries certain small region at the boundary is left uncovered by the elements. This leads to some error in the solution.

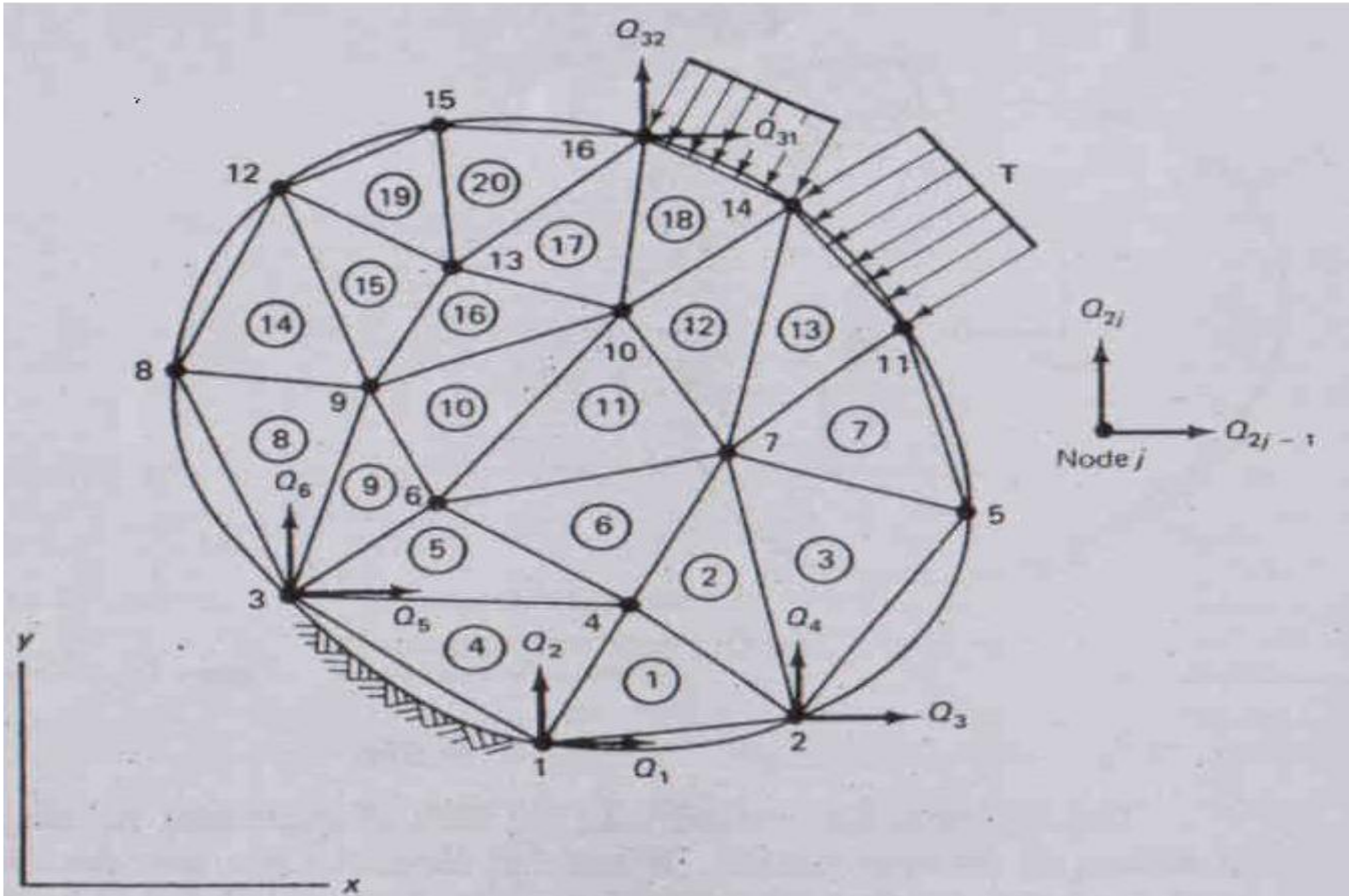
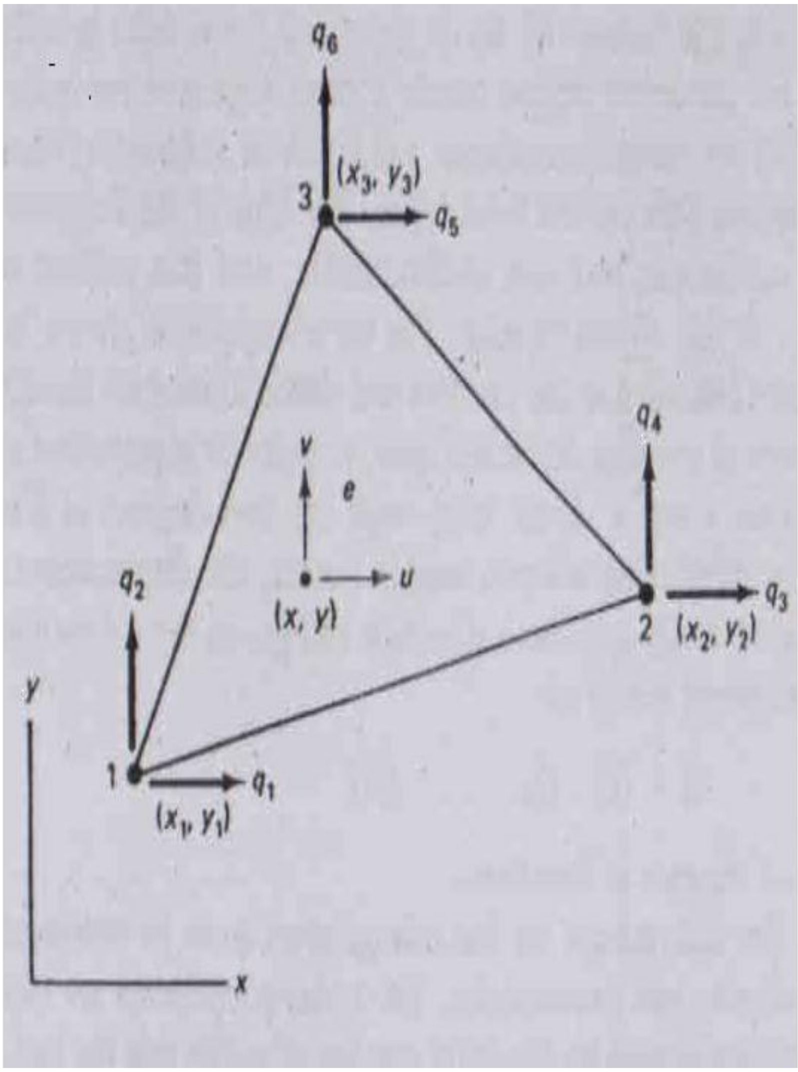
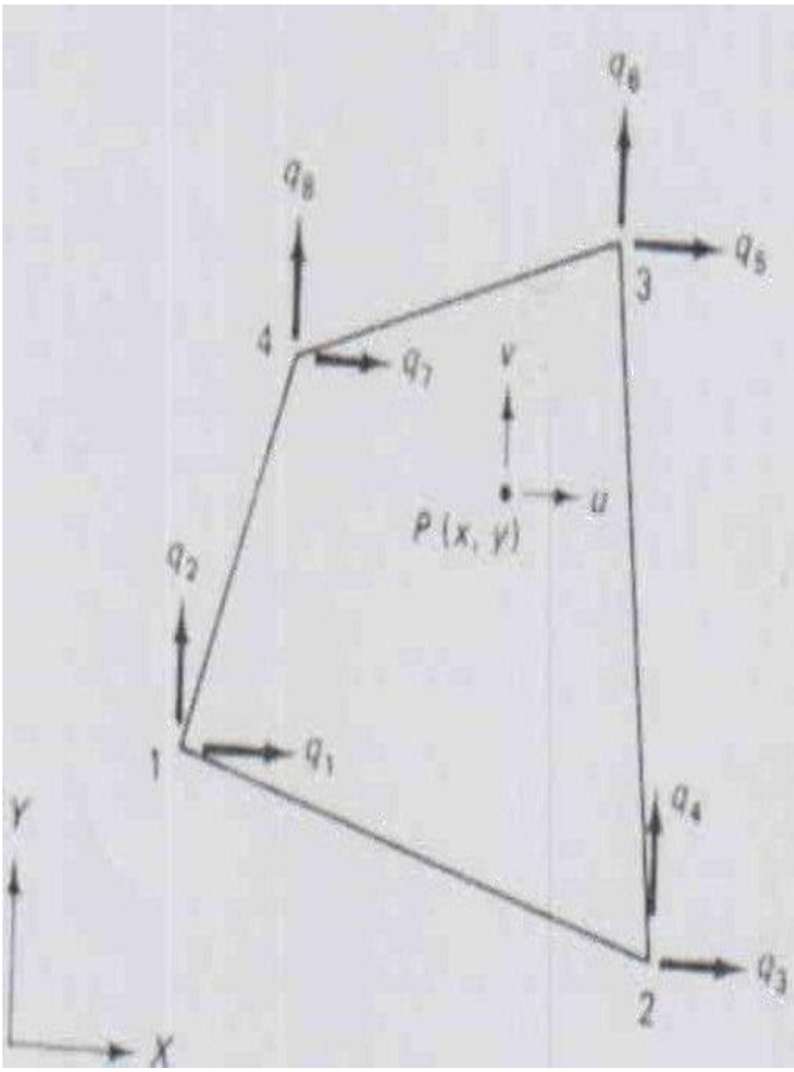


Figure 5.2 Finite element discretization.



Constant Strain Triangle



Quadrilateral

Constant Strain Triangle

It is a triangular element having three straight sides joined at three corners. and imagined to have a node at each corner. Thus it has three nodes, and each node is permitted to displace in the two directions, along x and y of the Cartesian coordinate system. The loads are applied at nodes. Direction of load will also be along x direction and y direction, +ve or –ve etc. Each node is said to have two degrees of freedom. The nodal displacement vector for each element is given by,

$\mathbf{q} = [q_1, q_2, q_3, q_4, q_5, q_6]$

q_1, q_3, q_5 are nodal displacements along x direction of node1, node2 and node3 simply called horizontal displacement components.

q_2, q_4, q_6 are nodal displacements along y direction of node1, node2 and node3 simply called vertical displacement components. q_{2j-1} is the displacement component in x direction and q_{2j} is the displacement component in y direction.

Similarly the nodal load vector has to be considered for each element. Point loads will be acting at various nodes along x and y …………………………

(x_1, y_1), (x_2, y_2), (x_3, y_3) are cartesian coordinates.of node 1 node 2 and node 3.

In the discretized model of the continuum the node numbers are progressive, like 1,2,3,4,5,6,7,8……….etc and the corresponding displacements are Q_1, Q_2, Q_3, Q_4, Q_5 Q_6, Q_7, Q_8, Q_9, Q_{10}….. Q_{16}, two displacement components at each node.

Q_{2j-1} is the displacement component in x direction and Q_{2j} is the displacement component in y direction. Let j = 10, ie 10th node, $Q_{2j-1} = Q_{19}$ $Q_{2j} = Q_{20}$

The element connectivity table shown establishes correspondence of local and global node numbers and the corresponding degrees of freedom. Also the (x_1, y_1), (x_2, y_2) and (x_3, y_3) have the global correspondence established through the table.

Element Connectivity Table Showing
Local – Global Node Numbers

Element Number	Local Nodes Numbers 1	2	3	
1	1	2	4	Corres-
2 3	4	2	7	-ponding-
..	..	..	..	Global-
11	6	7	10	Node-
..	..	..	..	
20	13	16	15	Numbers

Nodal Shape Functions: u nder the action of the given load the nodes are assumed to deform linearly. element has to deform elastically and the deformation has to become zero as soon as the loads are zero. It is required to define the magnitude of deformation

and nature of deformation for the element Shape functions or Interpolation functions are used to model the magnitude of displacement and nature of displacement.

The Triangular element has three nodes. Three shape functions N1 , N2 , N3 are used at nodes 1,2 and 3 to define the displacements. Any linear combination of these shape functions also represents a plane surface.

N1 = , N2 = , N3 = 1 – – (1.8)

The value of N1 is unity at node 1 and linearly reduces to 0 at node 2 and 3. It defines a plane surface as shown in the shaded fig. N2 and N3 are represented by similar surfaces having values of unity at nodes 2 and 3 respectively and dropping to 0 at the opposite edges. In particular N1 + N2 + N3 represents a plane at a height of 1 at nodes 1 , 2 and 3 The plane is thus parallel to triangle 1 2 3.

Shape Functions N_1 , N_2 , N_3

For every N1 , N2 and N3 , N1 + N2 + N3 = 1 N1 N2 and N3 are therefore not linearly independent.

N1 =N2 =N3 = 1 – – , where and are natural coordinates
The displacements inside the element are given by,
u = N1 q1 + N2 q3 + N3 q5
v = N1 q2 + N2 q4 + N3 q6 writing these in the matrix form

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} * \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

[u] = [N] [q]

Iso Paramatric Formulation :

The shape functions N1, N2, N3 are also used to define the geometry of the element apart from variations of displacement. This is called Iso-Parametric formulation

- u = N1 q1 + N2 q3 + N3 q5
  v = N1 q2 + N2 q4 + N3 q6 , defining variation of displacement.

- x = N1 x1 + N2 x2 + N3 x3
- y = N1 y1 + N2 y2 + N3 y3 , defining geometry.

Potential Energy :

Total Potential Energy of an Elastic body subjected to general loading is given by

= Elastic Strain Energy + Work Potential

$= ½ {}^T dv - u^T f dv - u^T T ds - u^T i Pi$

For the 2- D body under consideration P.E. is given by

$= ½ {}^T D \; te \, dA - u^T f \, t \, dA - u^T T \, t \, dl - u^T i \, Pi$

This expression is utilised in deriving the elemental properties such as Element stiffness matrix **[K]** , load vetors f^e , T^e , etc .

Derivation of Strain Displacement Equation and Stiffness Matrix for CST (derivation of [B] and [K]) :

Consider the equations

u = N1 q1 + N2 q3 + N3 q5 v = N1 q2 + N2 q4 + N3 q6

x = N1 x1 + N2 x2 + N3 x3 y = N1 y1 + N2 y2 + N3 y3 Eq (1)

We Know that u and v are functions of x and y and they in turn are functions ofand .

u = u (x (,) , y (,)) v = v (x (,) , y (,))

taking partial derivatives for u , using chain rule, we have equation (A) given by

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

Eq (A)

Similarly, taking partial derivatives for v using chain rule, we have equation (B) given by

$$\frac{\partial v}{\partial \xi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta}$$

Eq (B)

now consider equation (A), writing it in matrix form

$$\begin{matrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{matrix} = \begin{matrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{matrix} \qquad \begin{matrix} +u \\ -x \\ +u \\ -y \end{matrix}$$

$$\begin{matrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{matrix}$$

Is called **JACOBIAN [J]**

Jacobian is used in determining the strain components, now we can get

$$\begin{matrix} \bullet u \\ x \\ \bullet u \\ y \end{matrix} = [\mathbf{J}]^{-1} \begin{matrix} \partial u \\ \partial \xi \\ \\ \partial u \\ \partial \eta \end{matrix}$$

In the Left vector u / x = x , is the strain component along x-dirction.

Similarly writing equation (B) in matrix form and considering [J] we get ,

$$\begin{matrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{matrix} = [\mathbf{J}]^{-1} \begin{matrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{matrix}$$

In the left vector v / y = y , is the strain component along y-direction..

u/ x = x , v/ y = y , xy = u/ y + v/ x

We have to determine [J] , $[J]^{-1}$ which is same for both the equations.

First we will take up the determination u/ x = x and u/ y using J and J^{-1} , Consider the equations

u = N1 q1 + N2 q3 + N3 q5   v = N1 q2 + N2 q4 + N3 q6

Substituting for N1 , N2 and N3 , in the above equations we get

u = q1 + q3 + (1 – –) q5   = (q1 - q5) + (q3 - q5) + q5

= q 15 + q 35 + q 5

u / = q15   u / = q 35

v = q2 + q4 + (1 – –) q6   = (q2 - q6) + (q4 - q6) + q6

= q 26 + q 46 + q 6

v / = q 26   v / = q 46

Consider x = N1 x1 + N2 x2 + N3 x3

y = N1 y1 + N2 y2 + N3 y3

Substituting for N1 , N2 and N3 , in the above equations we get

x = x1 + x2 + (1 – –) x3

x = (x1 - x3) + (x2 - x3) + x3   = x 13 + x 23 + x3

x / = x13   x / = q 23

y = y1 + y2 + (1 – –) y3

y = (y1 - y3) + (y2 - y3) + y3   = y 13 + y 23 + y3

y / = y13   y / = y23

To determine [J] , $[J]^{-1}$

u / = q15   u / = q 35   v / = q 26   v / = q 46 y

x / = x13   x / = y23   y / = y13   / = y23

[J] =x /   y /   [J] = x13 , y13   x1 - x 3 , y1 - y 3

x /   y /   x23 , y23   x2 - x 3 , y2 - y3

To determine $[J]^{-1}$: find out co factors [J]

co-factors of x ij = $(-1)^{i+j}$ | |

co-factors [co] =   (y2 - y3), -(x2 - x 3)   y23 , x32

-(y1 - y3), (x1 - x 3)   y31, x13

Adj [J] = [co]^T = y23 y31
x32 x13

[J] $^{-1}$ = Adj [J] / | J |

[J] $^{-1}$ = (1/ |J|) y23 y31
x32 x13

Also we have
u / = q15 = q1- q5 u / = q 35 = q3 –q5

u / x = [J] $^{-1}$ u /
u / y u /

u / x = (1/ |J|) y23 y31 q1- q5
u / y x32 x13 q3 –q5

u / x = (1/ |J|) y23 q1- q5 + y31 q3 –q5
u / y x32 q1- q5 + x13 q3 –q5

u / x = (1/ |J|) y23 q1- y23 q5 + y31 q3 – y31q5
u / y x32 q1- x32 q5 + x13 q3 – x13q5

u / x = (1/ |J|) y23q1 + y31 q3 - y23 q5– y31q5
u / y x32 q1 + x13 q3 - x32 q5– x13q5

u / x = (1/ |J|) y23q1 + y31 q3 - q5 (y2 - y3+ y3 -y1)
u / y x32 q1 + x13 q3 - q5 (x3 - x2+x1 - x3)

u / x = (1/ |J|) y23q1 + y31 q3 - q5 (y2 -y1)
u / y x32 q1 + x13 q3 - q5 (- x2+x1)

u / x = (1/ |J|) y23q1 + y31 q3 + q5 (y1- y2)
u / y x32 q1 + x13 q3 + q5 (x2 -x1)

u / x = (1/ |J|) y23q1 + y31 q3 + y12 q5
u / y x32 q1 + x13 q3 + x21 q5

Writing the R.H.S of above equation in Matrix form

u / x = 1/ |J| y23 0 y31 0 y12 0 q1
u / y x32 0 x13 0 x21 0 q2
q3
q4
q5
q6
…… eq (6)

Similarly Considering equation (B) we get

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = [\mathbf{J}]^{-1} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

[J] = x / y / =x13 , y13 x1 - x 3 , y1 - y 3
x /y / x23 , y23 x2 - x 3 , y2 - y3

[J] -1 = 1/ |J| y23 y31
x32 x13

consider v = N1 q2 + N2 q4 + N3 q6

v = q2 + q4 + (1 – –) q6
v = (q2 - q6) + (q4 - q6) + q6
= q26 + q46 + q6

v / = q26 v
/ = q46

v / x = [J] $^{-1}$ v / v
v / y /

$$\begin{array}{lllll} v / x & = (1/|J|) & y23 & y31 & q2 - q6 \\ v / y & & x32 & x13 & q4 - q6 \end{array}$$

$$\begin{array}{lll} v / x & = (1/|J|) & y23 \ (q2 - q6) + y31 \ (q4 - q6) \\ v / y & & x32 \ (q2 - q6) + x13 \ (q4 - q6) \end{array}$$

$$\begin{array}{lll} v / x & = (1/|J|) & y23 \ q2 - y23 \ q6 + y31 \ q4 - y31q6 \\ v / y & & x32 \ q2 - x32q6 + x13 \ q4 - x13q6 \end{array}$$

$$\begin{array}{lll} v / x & = (1/|J|) & y23q2 + y31 \ q4 - y23q6 - y31q6 \\ v / y & & x32 \ q2 + x13 \ q4 - x32q6 - x13q6 \end{array}$$

$$\begin{array}{lll} v / x & = (1/|J|) & y23q2 + y31q4 - q6(y2 - y3 + y3 - y1) \\ v / y & & x32 \ q2 + x13 \ q4 - q6(x3 - x2 + x1 - x3) \end{array}$$

canceling y3 and x3 , we get

$$\begin{array}{lll} v / x & = (1/|J|) & y23q2 + y31q4 - q6(y2 - y1) \\ v / y & & x32 \ q2 + x13 \ q4 - q6(- x2 + x1) \end{array}$$

$$\begin{array}{lll} v / x & = (1/|J|) & y23q2 + y31q4 + q6(y1 + y2) \\ v / y & & x32 \ q2 + x13 \ q4 + q6(x2 + x1) \end{array}$$

$$\begin{array}{lll} v / x & = (1/|J|) & y23q2 + y31q4 + y12 \ q6 \\ v / y & & x32 \ q2 + x13 \ q4 + x21q \end{array}$$

Writing in matrix form

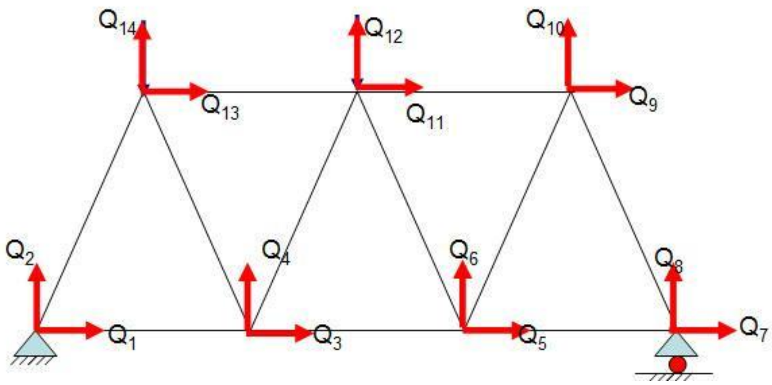
$$\begin{array}{llcccccccc} v / x & = 1/|J| & 0 & y23 & 0 & y31 & 0 & y12 & & q1 \\ v / y & & 0 & x32 & 0 & x13 & 0 & x21 & & q2 \\ & & & & & & & & & q3 \\ & & & & & & & & & q4 \\ & & & & & & & & & q5 \\ & & & & & & & & & q6 \end{array}$$

ANALYSIS OF TRUSSES

A Truss is a two force members made up of bars that are connected at the ends by joints. Every stress element is in either tension or compression. Trusses can be classified as plane truss and space truss.

- Plane truss is one where the plane of the structure remain in plane even after the application of loads
- While space truss plane will not be in a same plane

Fig shows 2d truss structure and each node has two degrees of freedom. The only difference between bar element and truss element is that in bars both local and global coordinate systems are same where in truss these are different.



There are always assumptions associated with every finite element analysis. If all the assumptions below are all valid for a given situation, then truss element will yield an exact solution. Some of the assumptions are:

Truss element is only a prismatic member ie cross sectional area is uniform along its length
It should be a isotropic material
Constant load ie load is independent of time Homogenous material

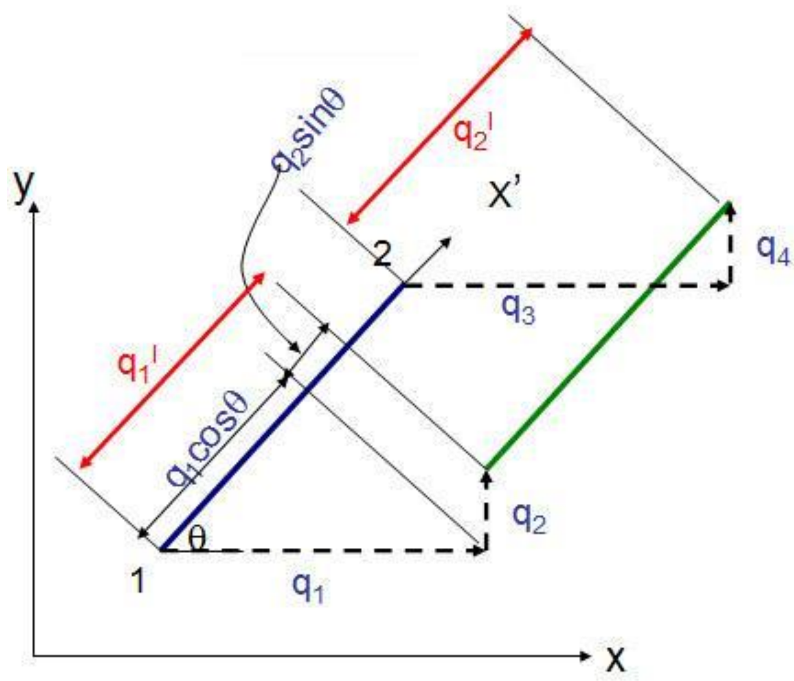
A load on a truss can only be applied at the joints (nodes)

Due to the load applied each bar of a truss is either induced with tensile/compressive forces

The joints in a truss are assumed to be frictionless pin joints

Self weight of the bars are neglected

Consider one truss element as shown that has nodes 1 and 2 .The coordinate system that passes along the element (x^1 axis) is called local coordinate and X-Y system is called as global coordinate system. After the loads applied let the element takes new position say locally node 1 has displaced by an amount q_1^1 and node2 has moved by an amount equal to q_2^1.As each node has 2 dof in global coordinate system .let node 1 has displacements q_1 and q_2 along x and y axis respectively similarly q_3 and q_4 at node 2.



Resolving the components q_1, q_2, q_3 and q_4 along the bar we get two equations as

$$q_1^l = q_1 \cos\theta + q_2 \sin\theta$$

$$q_2^l = q_3 \cos\theta + q_4 \sin\theta$$

Or

$$q_1^l = q_1 \ell + q_2 m$$

$$q_2^l = q_3 \ell + q_4 m$$

Writing the same equation into the matrix form

$$\begin{Bmatrix} q_1^l \\ q_2^l \end{Bmatrix} = \begin{pmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{pmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$q^l = L\, q$$

Where L is called transformation matrix that is used for local –global correspondence.

Strain energy for a bar element we have

$$U = \tfrac{1}{2}\, q^T K q$$

For a truss element we can write

$$U = \tfrac{1}{2}\, q^{lT}\, K\, q^l$$

Where $q^l = L\, q$ and $q^{lT} = L^T\, q^T$

Therefore

$$U = \frac{1}{2} q^{lT} K q^{l}$$

$$= \frac{1}{2} L^T q^T K L q$$

$$= \frac{1}{2} q^T (L^T K L) q$$

$$= \frac{1}{2} q^T K_T q$$

Where K_T is the stiffness matrix of truss element

$$K_T = L^T K L$$

$$L = \begin{pmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{pmatrix} \qquad L^T = \begin{pmatrix} \ell & 0 \\ m & 0 \\ 0 & \ell \\ 0 & m \end{pmatrix}$$

$$K = \frac{AE}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Taking the product of all these matrix we have stiffness matrix for truss element which is given as

$$K_T = \frac{AE}{L} \begin{pmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{pmatrix}$$

Stress component for truss element

The stress σ in a truss element is given by

$$\sigma = \varepsilon E$$

But strain $\varepsilon = B\, q^{l}$ and $q^{l} = T\, q$

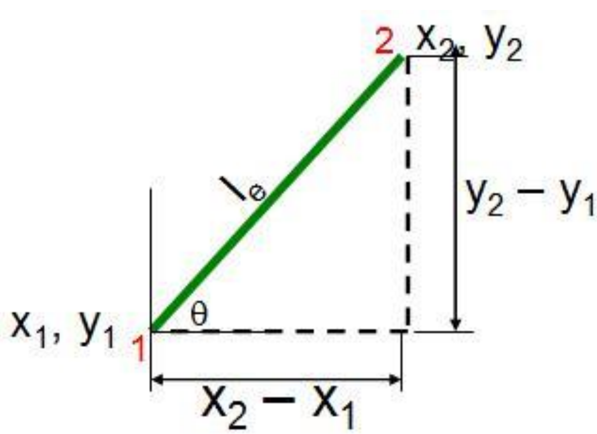
$$\text{where } B = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

Therefore

$$\sigma = \frac{E}{L_e}\begin{pmatrix} -\ell & -m & \ell & m \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

How to calculate direction cosines

Consider a element that has node 1 and node 2 inclined by an angle θ as shown .let (x1, y1) be the coordinate of node 1 and (x2,y2) be the coordinates at node 2.



When orientation of an element is know we use this angle to calculate and m as:

cosθ m = cos (90 - θ sinθ

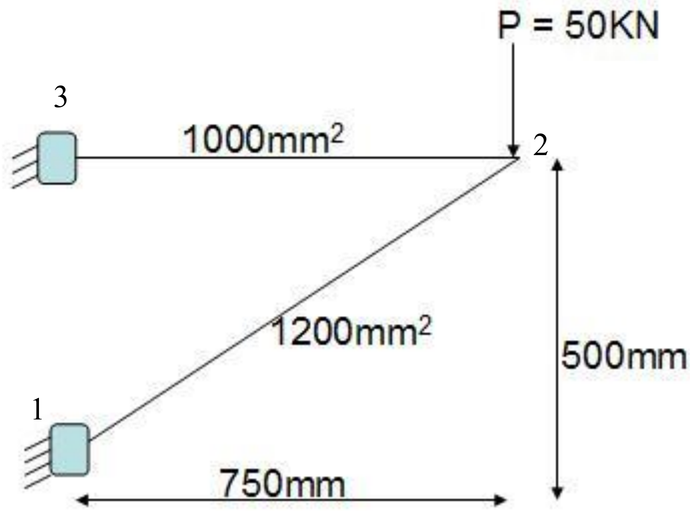
and by using nodal coordinates we can calculate using the relation

$$\ell = \frac{x_2 - x_1}{l_e} \quad m = \frac{y_2 - y_1}{l_e}$$

We can calculate length of the element as

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 6



Solution: For given structure if node numbering is not given we have to number them which depend on user. Each node has 2 dof say q1 q2 be the displacement at node 1, q3 & q4 be displacement at node 2, q5 &q6 at node 3.

Tabulate the following parameters as shown

Element	θ	L	$\ell = \cos\theta$	$m = \sin\theta$
1	33.6	901.3	0.832	0.554
2	0	750	1	0

For element 1 θ can be calculate by using $\tan\theta = 500/700$ ie $\theta = 33.6$, length of the element is

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$= 901.3$ mm

Similarly calculate all the parameters for element 2 and tabulate

Calculate stiffness matrix for both the elements

$$K_T = \frac{AE}{L}\begin{pmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{pmatrix}$$

$$K_1 = 10^5 \begin{pmatrix} 1.84 & 1.22 & -1.84 & -1.22 \\ 1.22 & 0.816 & -1.22 & -0.816 \\ -1.84 & -1.22 & 1.84 & 1.22 \\ -1.22 & -0.816 & 1.22 & 0.816 \end{pmatrix} \quad \text{(rows/columns 1, 2, 3, 4)}$$

$$K_2 = 10^5 \begin{pmatrix} 2.66 & 0 & -2.66 & 0 \\ 0 & 0 & 0 & 0 \\ -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(rows/columns 3, 4, 5, 6)}$$

Element 1 has displacements q1, q2, q3, q4. Hence numbering scheme for the first stiffness matrix (K1) as 1 2 3 4 similarly for K_2 3 4 5 & 6 as shown above.

Global stiffness matrix: the structure has 3 nodes at each node 3 dof hence size of global stiffness matrix will be 3 X 2 = 6

ie 6 X 6

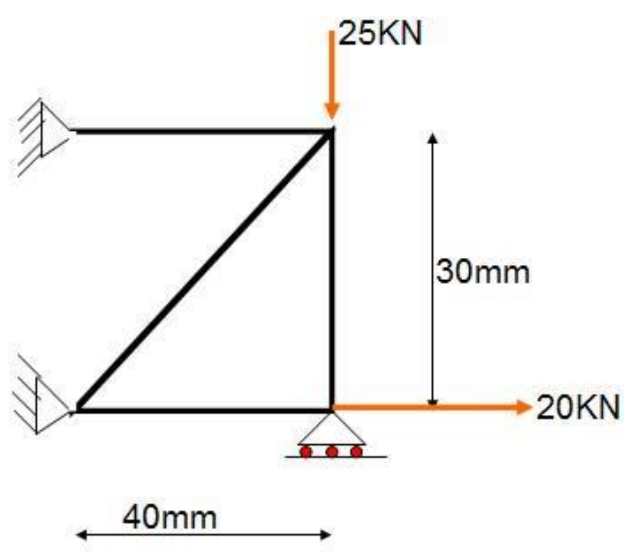
$$K = 10^5 \begin{pmatrix} 1.84 & 1.22 & -1.84 & -1.22 & 0 & 0 \\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0 \\ -1.84 & -1.22 & 4.5 & 1.22 & -2.66 & 0 \\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(rows/columns 1–6)}$$

From the equation KQ = F we have the following matrix. Since node 1 is fixed q1=q2=0 and also at node 3 q5 = q6 = 0 .At node 2 q3 & q4 are free hence has displacements.
In the load vector applied force is at node 2 ie F4 = 50KN rest other forces zero.

$$10^5 \begin{bmatrix} 1.84 & 1.22 & -1.84 & -1.22 & 0 & 0 \\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0 \\ -1.84 & -1.22 & 4.5 & 1.22 & -2.66 & 0 \\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -50 \times 10^3 \\ 0 \\ 0 \end{Bmatrix}$$

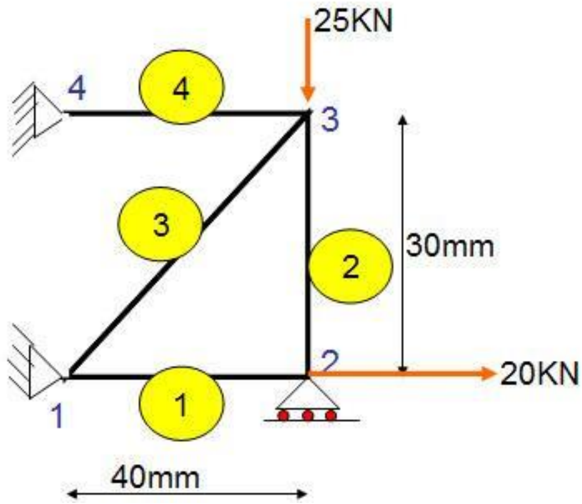
By elimination method the matrix reduces to 2 X 2 and solving we get Q3= 0.28mm and Q4 = -1.03mm. With these displacements we calculate stresses in each element.

Example 7



$E = 29.5 \times 10^6\ N/mm^2 \qquad A = 1mm^2$

Solution: Node numbering and element numbering is followed for the given structure if not specified, as shown below



Let Q1, Q2 …..Q8 be displacements from node 1 to node 4
and F1, F2……F8 be load vector from node 1 to node 4.

Tabulate the following parameters

Element	θ	L	ℓ =cosθ	m=sin θ
1	0	40	1	0
2	90	30	0	1
3	36.8	50	0.8	0.6
4	0	40	1	0

Determine the stiffness matrix for all the elements

$$K_1 = 10^5 \begin{bmatrix} 5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \quad (\text{columns } 1, 2, 3, 4)$$

$$K_2 = 10^5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 6.66 & 0 & -6.66 \\ 0 & 0 & 0 & 0 \\ 0 & -6.66 & 0 & 6.66 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \quad (\text{columns } 3, 4, 5, 6)$$

$$K_3 = 10^5 \begin{array}{c} \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} \\ \begin{bmatrix} 2.56 & 1.92 & -2.56 & -1.92 \\ 1.92 & 1.44 & -1.92 & -1.44 \\ -2.56 & -1.92 & 2.56 & 1.92 \\ -1.92 & -1.44 & 1.92 & 1.44 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 2 \end{array} \qquad K_4 = 10^5 \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 5 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Global stiffness matrix: the structure has 4 nodes at each node 3 dof hence size of global stiffness matrix will be 4 X 2 = 8

ie 8 X 8

$$\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 7.56 & 1.92 & -5 & 0 & -2.56 & -1.92 & 0 & 0 \\ 1.92 & 1.44 & 0 & 0 & -1.92 & -1.44 & 0 & 0 \\ -5 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.66 & 0 & -6.66 & 0 & 0 \\ -2.56 & -1.92 & 0 & 0 & 7.56 & 1.92 & -5 & 0 \\ -1.92 & -1.44 & 0 & -6.66 & 1.92 & 8.11 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

From the equation KQ = F we have the following matrix. Since node 1 is fixed q1=q2=0 and also at node 4 q7 = q8 = 0 .At node 2 because of roller support q3=0 & q4 is free hence has displacements. q5 and q6 also have displacement as they are free to move.

In the load vector applied force is at node 2 ie F3 = 20KN and at node 3 F6 = 25KN, rest other forces zero.

$$
10^5 \begin{pmatrix}
7.56 & 1.92 & -5 & 0 & -2.56 & -1.92 & 0 & 0 \\
1.92 & 1.44 & 0 & 0 & -1.92 & -1.44 & 0 & 0 \\
-5 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6.66 & 0 & -6.66 & 0 & 0 \\
-2.56 & -1.92 & 0 & 0 & 7.56 & 1.92 & -5 & 0 \\
-1.92 & -1.44 & 0 & -6.66 & 1.92 & 8.11 & 0 & 0 \\
0 & 0 & 0 & 0 & -5 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \\ Q7 \\ Q8 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 20 \times 10^3 \\ 0 \\ 0 \\ -25 \times 10^3 \\ 0 \\ 0 \end{pmatrix}
$$

Solving the matrix gives the value of q3, q5 and q6.