Module 2: ENERGY TRANSFER IN TURBOMACHINES

2.1 Introduction:

In this chapter, general analysis of kinematic and dynamic factors for different types of turbomachines is made. Kinematics relates to movement (velocities, accelerations, etc.), without paying attention to what brought about the motion. Dynamics is related to detailed examination of the forces that bring about the motion described by kinematics. The kinematic and dynamic factors depend on the velocities of fluid flow in the machine as well as the rotor velocity itself and the forces of interaction due to velocity changes.

2.2 Euler's Turbine Equation:

Question No 2.1: Derive Euler's turbine equation for power generating or power absorbing turbomachines and clearly state the assumptions made. ***(VTU, Jan/Feb-03, Dec-12, Jul-17)***

Answer: The figure 2.1 shows the rotor of a generalized turbomachine with axis of rotation 0-0, with an angular velocity ω. The fluid enters the rotor at radius r_1 with an absolute velocity V_1 and leaves the rotor at radius r_2 with an absolute velocity V_2.

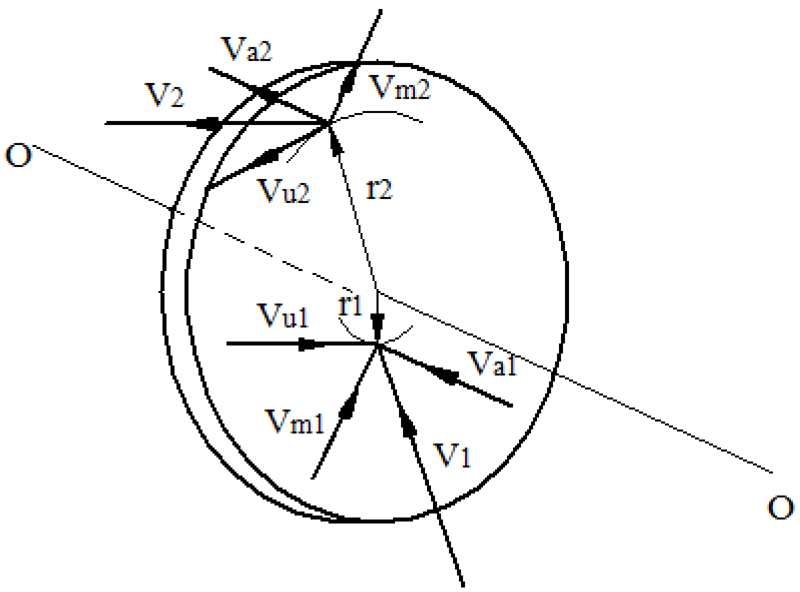


Fig. 2.1 Fluid flow through a rotor of a turbomachine.

Assumptions:

i. Fluid flow through the turbomachine is steady flow.
ii. Mass flow rate is constant and the state of the fluid doesn't vary with time.
iii. Rate of energy transfer at the rotor is constant.
iv. Losses due to leakage are neglected.

The absolute velocity of the fluid can be resolved in to three mutually perpendicular velocity components:

a. Axial component (V_a), which is parallel to the axis of rotation of the rotor.
b. Radial component (V_m), which is perpendicular to the axis of rotation of the rotor.
c. Tangential component (V_u), which is along the tangential direction of the rotor.

The only velocity component which changes the angular momentum of the rotor is the tangential component (V_u) and by Newton's second law of motion forces applied on the rotor is equal to rate of change of momentum of the fluid.

Force applied on the rotor = Rate of change of momentum

$$F = \Delta\left(\frac{mV_u}{t}\right) = \dot{m}(V_{u1} - V_{u2})$$

But, Torque = Force × Radius

$$\boldsymbol{\tau} = \boldsymbol{F} \times \boldsymbol{r}$$

Then, $\tau = \dot{m}(V_{u1}r_1 - V_{u2}r_2)$

But, Rate of energy transfer = Torque × Angular velocity

$$\dot{E} = \boldsymbol{\tau} \times \boldsymbol{\omega}$$

Then, $$\dot{E} = \dot{m}(V_{u1}r_1\omega_1 - V_{u2}r_2\omega_2)$$

But, tangential velocity of rotor $$U = r \times \omega$$

Then, $$\dot{E} = \dot{m}(U_1V_{u1} - U_2V_{u2})$$

Energy transfer per unit mass flow of fluid is

$$e = \frac{\dot{E}}{\dot{m}} = (U_1V_{u1} - U_2V_{u2}) \tag{2.1}$$

The equation (2.1) is the general Euler's equation for all kind of turbomachines.

For *power generating turbomachine energy transfer is positive* (i.e., $U_1V_{u1} > U_2V_{u2}$)

Therefore, $$\boldsymbol{e = (U_1V_{u1} - U_2V_{u2})} \tag{2.2}$$

For *power absorbing turbomachine energy transfer is negative* (i.e., $U_2V_{u2} > U_1V_{u1}$)

Therefore, $$\boldsymbol{e = (U_2V_{u2} - U_1V_{u1})} \tag{2.3}$$

Note: *(a) The change in magnitude of axial velocity components give rise to an axial thrust which must be taken up by the thrust bearings. The change in magnitude of radial velocity components give rise to a radial thrust which must be taken up by the journal bearing. Neither of these forces causes any angular rotation nor has any effect on the torque exerted on the rotor.*

(b) The Euler's turbine equation may be used for the flow of fluids like water, steam, air and combustion products, since their viscosities are reasonably small. For fluids of very large viscosity like heavy oils or petroleum products, errors in the calculated torque and power output may result due to: (i) non-uniformity of velocity profiles at the inlet and the exit and (ii) the boundary layers near the housing and the stator surfaces. Both these tend reduce the magnitude of the torque in comparison with the ideal torque predicted by Euler's turbine equation.

2.2.1 Procedure to Draw Velocity Diagram:

Question No 2.2: Explain the procedure to draw velocity triangles. Why velocity triangles are of utmost importance in the study of turbomachines? ***(VTU, Dec-10)***

Answer: In turbomachinery, a velocity triangle or a velocity diagram is a triangle representing the various components of velocities of the working fluid in a turbomachine. Velocity triangles may be drawn for both the inlet and outlet sections of any turbomachine. The vector nature of velocity is utilized in the triangles, and the most basic form of a velocity triangle consists of the tangential velocity, the absolute velocity and the relative velocity of the fluid making up three sides of the triangle.

Consider turbomachine consisting of a stator and a rotor. The three points that are very much important to draw the velocity triangles are entry to the stator, the gap between the stator and rotor and exit from the rotor. These points labelled 3, 1 and 2 respectively in figure 2.2 and combination of rotor and stator is called stage in turbomachines.

The fluid enters the stator at point 3 but as the stator is not moving there is no relative motion between the incoming flow and the stator so there is no velocity triangle to draw at this point. At point 1 the flow leaves the stator and enters the rotor. Here there are two flow velocities, the absolute velocity of the flow (V) viewed from the point of view of stationary stator and relative velocity of flow (V_r) viewed from the point of view of moving rotor. The rotor is moving with a tangential velocity of magnitude U. At point 2 the flow leaves the rotor and exits the stage. Again there are two flow velocities, one by viewing from the moving rotor and another by viewing from outside the rotor where there is no motion.

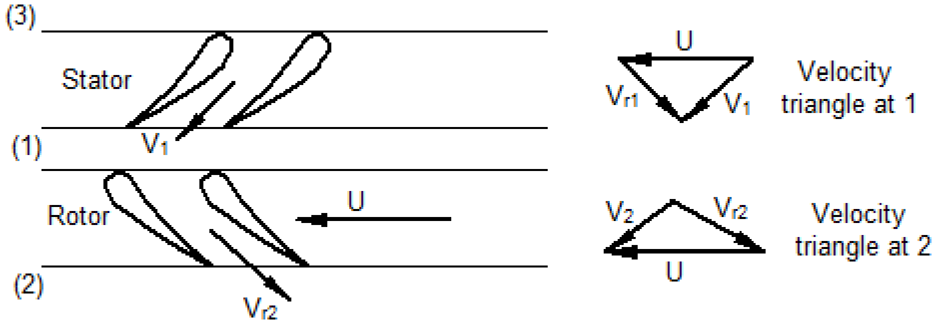


Fig.2.2 Velocity triangles for a turbomachine.

Therefore velocity triangles can be drawn for the point 1 and point 2 as shown in figure 2.2, the methodology for this is as follows:

1. Draw the flow that is known
2. Draw the blade speed

3. Close the triangle with the remaining vector
4. Check that the key rule applies: $\vec{V} = \vec{U} + \vec{V_r}$

The velocity triangles at inlet and outlet of the rotor are utmost important in deciding the size of the turbomachine for the given power output.

2.2.2 Energy components of Euler's Turbine Equation:

Question No 2.3: Derive an alternate (modified) form of Euler's turbine equation with usual notations and identify each component contained in the equation. ***(VTU, Jun/Jul-09, Dec-13/Jan-14, Jun/Jul-14)*** ***Or, Draw the velocity triangle at inlet and exit of a turbomachine in general and show that the energy transfer per unit mass is given by*** $e = \frac{1}{2}\left[\left(V_1^2 - V_2^2\right) + \left(U_1^2 - U_2^2\right) - \left(V_{r1}^2 - V_{r2}^2\right)\right]$ ***(VTU, Feb-06, Jul-13, Jun-12, Jan-14, Jul-14)***

Answer: Let us consider velocity diagram for generalised rotor as shown in figure 2.3.

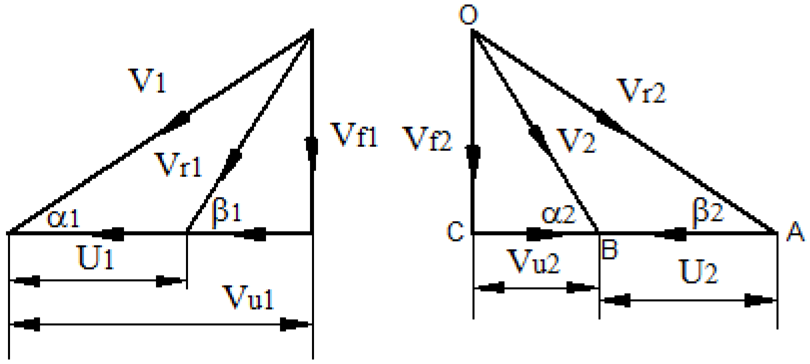


Fig. 2.3 Generalised velocity diagrams.

Let V= Absolute velocity of fluid

α = Angle made by V wrt tangential direction or nozzle angle or guide vane angle

V_r= Relative velocity of the fluid

β = Rotor angle or blade angle wrt tangential direction

U= Tangential velocity of the rotor

V_u= Tangential component of the absolute velocity or whirl velocity

V_f =V_m=V_a= Radial component or axial component of the absolute velocity or flow velocity.

Suffix 1 and 2 represents the values at inlet and outlet of the rotor.

Consider outlet velocity triangle, OBC

$$V_{f2}^2 = V_2^2 - V_{u2}^2 \tag{2.4}$$

From outlet velocity triangle, OAC

$$V_{f2}^2 = V_{r2}^2 - (U_2 - V_{u2})^2$$ (Because, U_2 and V_{u2} are in opposite direction)

$$V_{f2}^2 = V_{r2}^2 - U_2^2 - V_{u2}^2 + 2U_2V_{u2} \tag{2.5}$$

Compare equations (2.4) and (2.5)

$$V_2^2 - V_{u2}^2 = V_{r2}^2 - U_2^2 - V_{u2}^2 + 2U_2V_{u2}$$

$$2U_2V_{u2} = V_2^2 + U_2^2 - V_{r2}^2$$

Or

$$U_2V_{u2} = \frac{1}{2}(V_2^2 + U_2^2 - V_{r2}^2) \tag{2.6}$$

Similarly, for inlet velocity triangle

$$U_1V_{u1} = \frac{1}{2}(V_1^2 + U_1^2 - V_{r1}^2) \tag{2.7}$$

Substitute equations (2.6) and (2.7) in Euler's turbine equations (2.2) and (2.3)

For *power generating turbomachines*,

$$\boldsymbol{e = \frac{1}{2}\left[\left(V_1^2 - V_2^2\right) + \left(U_1^2 - U_2^2\right) - \left(V_{r1}^2 - V_{r2}^2\right)\right]} \tag{2.8}$$

For *power absorbing turbomachines*,

$$\boldsymbol{e = \frac{1}{2}\left[\left(V_2^2 - V_1^2\right) + \left(U_2^2 - U_1^2\right) - \left(V_{r2}^2 - V_{r1}^2\right)\right]} \tag{2.9}$$

First component: $\frac{(V_1^2 - V_2^2)}{2}$ or $\frac{(V_2^2 - V_1^2)}{2}$ change in the absolute kinetic energy and which causes a change in the dynamic head or dynamic pressure of the fluid through the machine.

Second component: $\frac{(U_1^2 - U_2^2)}{2}$ or $\frac{(U_2^2 - U_1^2)}{2}$ change in the centrifugal energy of the fluid in the motion. This is due to the change in the radius of rotation of the fluid. This causes a change in the static head or static pressure of the fluid through the rotor.

Third component: $\frac{(V_{r1}^2 - V_{r2}^2)}{2}$ or $\frac{(V_{r2}^2 - V_{r1}^2)}{2}$ change in the relative kinetic energy and which causes a change in the static head or static pressure of the fluid across the rotor.

Note: If directions of Vu_1 and Vu_2 are same then, $\boldsymbol{e} = (\boldsymbol{U_1V_{u1} - U_2V_{u2}})$ and if directions of Vu_1 and Vu_2 are opposite to each other then, $\boldsymbol{e} = (\boldsymbol{U_1V_{u1} + U_2V_{u2}})$.

Dynamic pressure is the kinetic energy per unit volume of a fluid particle. The dynamic pressure is equal to the difference between the stagnation pressure and static pressure. Dynamic pressure sometimes called velocity pressure. Static pressure is the actual pressure of the fluid, which is associated not with its motion but with its state. Stagnation or total pressure is sum of static pressure and dynamic pressure.

$$\boldsymbol{P_o = P + \frac{\rho V^2}{2}}$$

2.3 General Analysis of Turbomachines:

2.3.1 Impulse and Reaction Tubomachines: In general, turbomachines may be classified into impulse and reaction types, depending upon the type of energy exchange that occurs in the rotor blades. An impulse stage is one in which the static pressure at the rotor inlet is the same as that at the rotor outlet (i.e.$V_{r1} = V_{r2}$ and $U_1 = U_2$). In an impulse stage, the energy exchange is purely due to change in the direction of the fluid (i.e., change in dynamic pressure) and there is a negligible change in the magnitude of velocity as fluid flows over the rotor blades. The force exerted on the blades is due to change in the direction of the fluid during flow over the moving blade.

A reaction stage is one where a change in static pressure occurs during flow over each rotor stage. In a reaction stage, the direction and magnitude of the relative velocity are changed by shaping the blade passage as a nozzle (or as a diffuser, depending upon whether it is generating or absorbing power). The force exerted on the blades is due to both changes in magnitude and in direction of the fluid velocity.

2.3.2 Degree of Reaction (R): The degree of reaction is a parameter which describes the relation between the energy transfer due to static pressure change and the energy transfer due to dynamic pressure change. The *degree of reaction is the ratio of energy transfer due to the change in static pressure in the rotor to total energy transfer due to the change in total pressure in the rotor*.

Mathematically,

$$R = \frac{\frac{1}{2}[(U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]}{\frac{1}{2}[(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]}$$

Or

$$R = \frac{e - \frac{1}{2}(V_1^2 - V_2^2)}{e}$$

2.3.3 Utilization Factor (ε):

Question No 2.4: Define utilization factor and derive an expression for the same for a power developing turbomachines. ***(VTU, Jan/Feb-03)***

Answer: The *utilization factor is the ratio of the ideal (Euler) work output to the energy available for conversion into work*. Under ideal conditions, it should be possible to utilize all of the kinetic energy of the fluid at the rotor inlet and also the increase in kinetic energy obtained in the rotor due to static pressure drop (i.e. the reaction effect). Thus, the energy available for conversion into work in the turbine is: $e_a = \frac{1}{2}[V_1^2 + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]$

On the other hand, Euler work output is: $e = \frac{1}{2}[(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]$

Mathematically, utilization factor is:

$$\epsilon = \frac{\frac{1}{2}[(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]}{\frac{1}{2}[V_1^2 + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]}$$

Or

$$\epsilon = \frac{e}{e + \frac{V_2^2}{2}}$$

Question No 2.5: Define utilization factor for a turbine. If the isentropic efficiency of a turbine is 100% would its utilization factor also be 100%? Explain.

Answer: The *utilization factor is the ratio of the ideal (Euler) work output to the energy available for conversion into work.*

Yes, because the isentropic efficiency (adiabatic efficiency) is the product of two factors, the first called the utilization factor (diagram efficiency), the second due to non-isentropic flow conditions caused by friction, turbulence, eddies and other losses. Therefore if the isentropic efficiency has to be 100%, the utilization factor must be 100%.

Question No 2.6: Derive an expression relating utilization factor with the degree of reaction. Or, Show that utilization factor is given by $\epsilon = \frac{V_1^2 - V_2^2}{V_1^2 - RV_2^2}$, where R is the degree of reaction. For what value of R this relation is invalid? Why? (VTU, Jan/Feb-03, Jul/Aug-05, Dec-08/Jan-09, Dec-12, Jul-13, Jan-15, Jul-15, Jan-17)

Answer: Degree of reaction for generalised turbomachine is given by:

$$R = \frac{[(U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]}{[(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]}$$

$$R(V_1^2 - V_2^2) + R[(U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)] = (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)$$

$$R(V_1^2 - V_2^2) = (1 - R)[(U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]$$

Then,

$$(U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2) = \frac{R}{(1-R)}(V_1^2 - V_2^2) \qquad (2.10)$$

The utilization factor for any type of turbine is given by:

$$\epsilon = \frac{[(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]}{[V_1^2 + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)]}$$

From equation (2.10)

$$\epsilon = \frac{(V_1^2 - V_2^2) + \frac{R}{(1-R)}(V_1^2 - V_2^2)}{V_1^2 + \frac{R}{(1-R)}(V_1^2 - V_2^2)}$$

$$\epsilon = \frac{(1-R)(V_1^2 - V_2^2) + R(V_1^2 - V_2^2)}{(1-R)V_1^2 + R(V_1^2 - V_2^2)}$$

$$\boldsymbol{\epsilon = \frac{V_1^2 - V_2^2}{V_1^2 - RV_2^2}}$$

The above equation is the general utilization factor irrespective of any type of turbines whether it is axial or radial type. Clearly, it is invalid when R=1, since ϵ=1. Therefore the above equation is valid for all values of R in the range of 0≤R<1.

2.3.4 Condition for Maximum Utilization Factor:

Question No 2.7: In a turbomachine, prove that the maximum utilization factor is given by $\epsilon_{max} = \frac{2\varphi cos\alpha_1}{1+2R\varphi cos\alpha_1}$, where φ is speed ratio, R is degree of reaction and α_1 is nozzle angle.

(VTU, Jan/Feb-05, Dec-11)

Answer: For maximum utilization, the value of V_2 should be the minimum and from the velocity triangle, it is apparent that V_2 is having minimum value when it is axial or radial (i.e., $V_2=V_{f2}$). Then the velocity diagram of generalized turbomachine for maximum utilization is as shown in figure 2.4.

Energy transfer of a generalized turbomachine is given by:

$$e = \frac{1}{2}[(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)] = (U_1 V_{u1} - U_2 V_{u2})$$

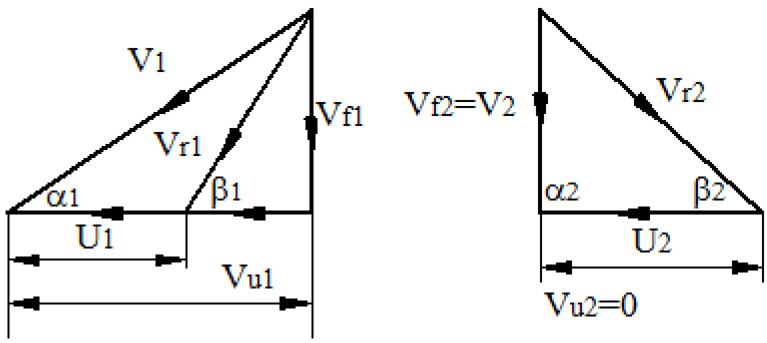


Fig. 2.4 Velocity diagram of generalized turbomachine for maximum utilization

For maximum utilization $V_{u2}=0$,

$$\frac{1}{2}[(V_1^2 - V_2^2) + (U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2)] = U_1 V_{u1}$$

From equation (2.10), $(U_1^2 - U_2^2) - (V_{r1}^2 - V_{r2}^2) = \frac{R}{(1-R)}(V_1^2 - V_2^2)$

Then,

$$\frac{1}{2}\left[(V_1^2 - V_2^2) + \frac{R}{(1-R)}(V_1^2 - V_2^2)\right] = U_1 V_{u1}$$

For maximum utilization $V_2=V_{f2}$ and from inlet velocity diagram $V_{u1}= V_1 cos\alpha_1$,

$$\frac{1}{2}\left[\left(V_1^2 - V_{f2}^2\right) + \frac{R}{(1-R)}\left(V_1^2 - V_{f2}^2\right)\right] = U_1 V_1 cos\alpha_1$$

$$\frac{\left(V_1^2 - V_{f2}^2\right)}{2(1-R)} = U_1 V_1 cos\alpha_1$$

$$\frac{\left(1 - \frac{V_{f2}^2}{V_1^2}\right)}{2(1-R)} = \frac{U_1}{V_1} cos\alpha_1$$

But blade speed ratio $\varphi = \frac{U}{V_1}$

$$\left(1 - \frac{V_{f2}^2}{V_1^2}\right) = 2(1-R)\varphi\ cos\alpha_1$$

Or,

$$\frac{V_{f2}^2}{V_1^2} = 1 - 2(1-R)\varphi\, cos\alpha_1 \qquad (2.11a)$$

Utilization factor is given by:

$$\epsilon = \frac{V_1^2 - V_2^2}{V_1^2 - RV_2^2}$$

For maximum utilization $V_2 = V_{f2}$,

$$\epsilon_{max} = \frac{V_1^2 - V_{f2}^2}{V_1^2 - RV_{f2}^2}$$

$$\epsilon_{max} = \frac{1 - \frac{V_{f2}^2}{V_1^2}}{1 - R\frac{V_{f2}^2}{V_1^2}}$$

From equation (2.11a)

$$\epsilon_{max} = \frac{1 - [1 - 2(1-R)\varphi\, cos\alpha_1]}{1 - R[1 - 2(1-R)\varphi\, cos\alpha_1]}$$

$$\epsilon_{max} = \frac{2(1-R)\varphi\, cos\alpha_1}{(1-R) + 2\varphi R(1-R)cos\alpha_1}$$

$$\boldsymbol{\epsilon_{max} = \frac{2\varphi\, cos\alpha_1}{1 + 2\varphi R cos\alpha_1}} \qquad (2.11b)$$

2.4 General Analysis of Turbines:

Power generating turbomachines are generally referred to as turbines. Turbines may run with compressible fluids like air or steam or with incompressible fluids like water. The quantity of interest in the power generating device is the work output. These machines are divided into axial, radial and mixed flow devices depending on the flow direction in the rotor blades.

2.4.1 Axial Flow Turbines: Axial flow machine are those in which the fluid enters and leaves the rotor at the same radius as shown in figure 2.5. Hence, for axial flow turbines $U_1=U_2$. In these kinds of machines, the flow velocity (V_f or V_a) is assumed to be constant from inlet to outlet. Axial flow turbines comprise the familiar steam turbines, gas turbines etc.

Energy transfer for axial flow turbine is:

$$e = \frac{1}{2}[(V_1^2 - V_2^2) - (V_{r1}^2 - V_{r2}^2)]$$

Degree of reaction for axial flow turbine is:

$$R = \frac{[(V_{r2}^2 - V_{r1}^2)]}{[(V_1^2 - V_2^2) + (V_{r2}^2 - V_{r1}^2)]}$$

Utilization factor for axial flow turbine is:

$$\epsilon = \frac{[(V_1^2 - V_2^2) - (V_{r1}^2 - V_{r2}^2)]}{[V_1^2 - (V_{r1}^2 - V_{r2}^2)]}$$

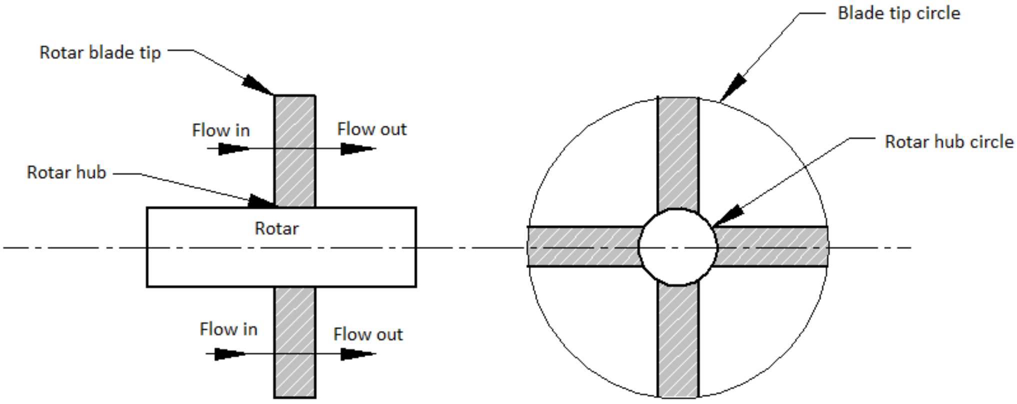


Fig. 2.5 Axial flow turbine

Question No 2.8(a): Explain why turbines with reaction R>1 and R<0 are not in practical use?

(VTU, Dec-10)

Answer: Degree of reaction can be given as:

$$R = \frac{Change\ in\ Static\ Pressure}{Change\ in\ Total\ Pressure} = \frac{Change\ in\ Static\ Pressure}{Change\ in\ Dynamic\ pressure + Change\ in\ Static\ Pressure}$$

$If\ R > 1,$

$$\frac{Change\ in\ Static\ Pressure}{Change\ in\ Total\ Pressure} > 1$$

Or, $Change\ in\ Static\ Pressure > Change\ in\ Total\ Pressure$, this is not practically possible. Therefore turbine with reaction R>1 is not in practical use.

Degree of reaction can also be given as:

$$R = \frac{Change\ in\ Total\ Pressure - Change\ in\ Dynamic\ Pressure}{Change\ in\ Total\ Pressure}$$

$If\ R < 0$

$$\frac{Change\ in\ Total\ Pressure - Change\ in\ Dynamic\ Pressure}{Change\ in\ Total\ Pressure} < 0$$

$$(Change\ in\ Total\ Pressure - Change\ in\ Dynamic\ Pressure) < 0$$

Or, $Change\ in\ Total\ Pressure < Change\ in\ Dynamic\ Pressure$, this is not practically possible. Therefore turbine with reaction R<0 is not in practical use.

2.4.1.1 Velocity Diagrams:

Question No 2.8(b): Sketch velocity diagrams for R=0, R=0.5 and R=1 and label. ***(VTU, Dec-12)***

Answer: For *impulse axial flow turbine*, R=0, thus V_{r1} should be equal to V_{r2} and if the blades are equiangular then, $\beta_1=\beta_2$ as shown in figure 2.6 (a). Here energy transfer is purely due to change in dynamic pressure.

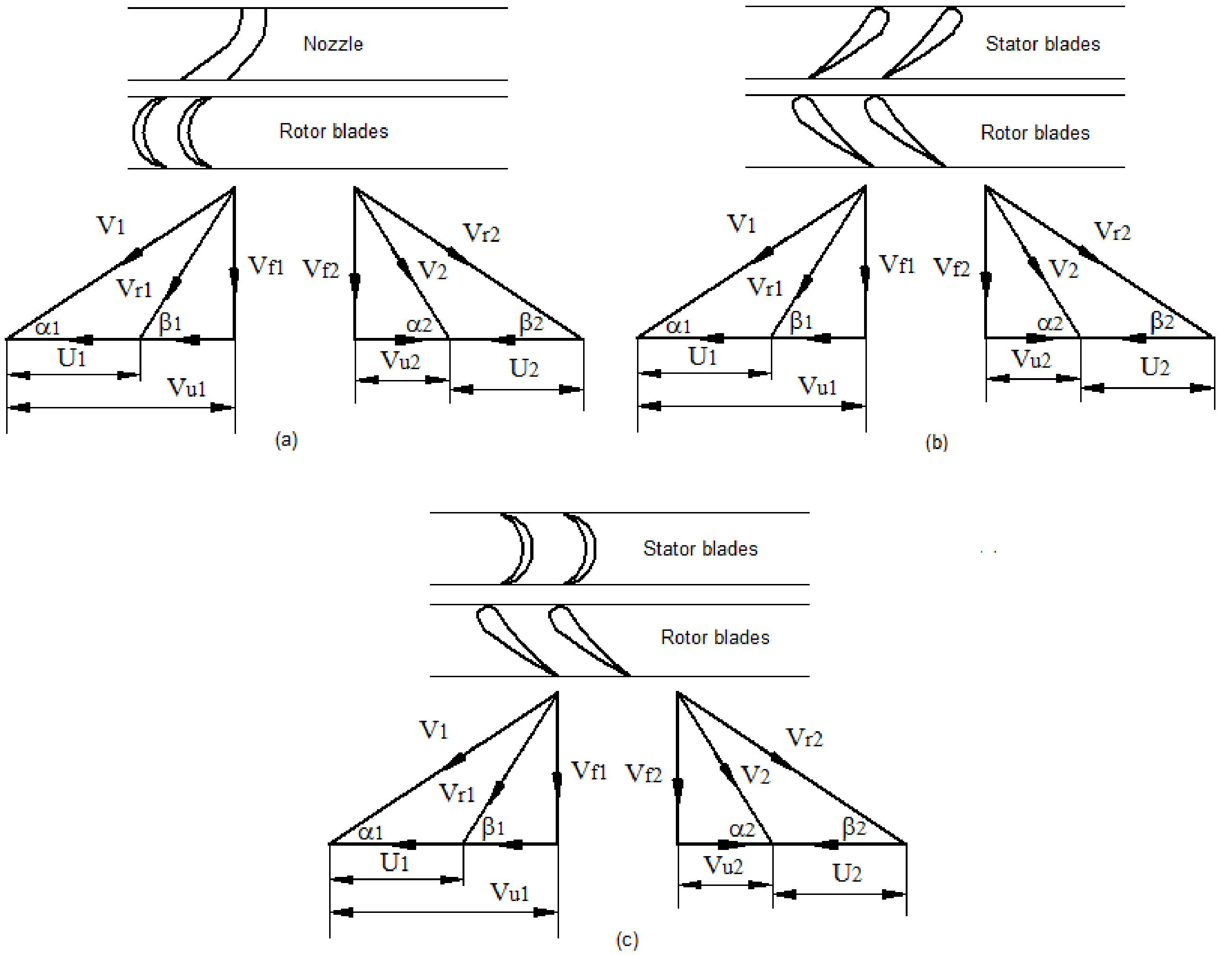


Fig. 2.6 Velocity triangles for axial flow turbine R = 0, R = 0.5 and R = 1

For *50% reaction axial flow turbine*, R=0.5, thus $(V_1^2 - V_2^2) = (V_{r2}^2 - V_{r1}^2)$ and if the stator and rotor blades are symmetric (two blades are identical but orientations are different) then, $\alpha_1=\beta_2$ and $\alpha_2=\beta_1$ and also $V_1=V_{r2}$ and $V_2=V_{r1}$ as shown in figure 2.6 (b). Here energy transfer due to change in dynamic pressure is equal to energy transfer due to change in static pressure.

For *fully (100%) reaction axial flow turbine*, R=1, thus V_1 should be equal to V_2 and also $\alpha_1=\alpha_2$ as shown in figure 2.6 (c). Here stator acts purely as a directional device and doesn't take part in the energy conversion process. The rotor acts both as the nozzle and as the energy transfer device, so energy transfer is purely due to change in static pressure.

2.4.1.2 Utilization Factor for R = 0 and R = 1:

Question No 2.9: Derive an expression for the utilization factor for an axial flow impulse turbine stage which has equiangular rotor blades, in terms of the fixed blade angle at inlet and speed ratio and show the variation of utilization factor and speed ratio in the form of a graph. ***(VTU, May/June-10)***

Answer: The velocity diagram for an axial flow impulse turbine stage with equiangular rotor blades is shown in figure 2.7.

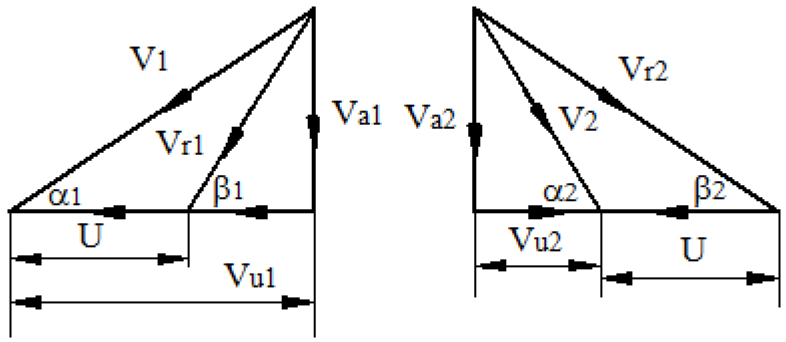


Fig. 2.7 Velocity diagram for axial flow impulse turbine.

For this machine, R=0 and $V_{r1}=V_{r2}$ and $\beta_1=\beta_2$ (equiangular blades).

Utilization factor is given by:

$$\epsilon = \frac{V_1^2 - V_2^2}{V_1^2 - RV_2^2}$$

But R=0,

$$\epsilon = \frac{V_1^2 - V_2^2}{V_1^2}$$

From outlet velocity diagram, $V_2^2 = V_{r2}^2 + U^2 - 2UV_{r2}cos\beta_2$

But $V_{r1}=V_{r2}$ and $\beta_1=\beta_2$, then $V_2^2 = V_{r1}^2 + U^2 - 2UV_{r1}cos\beta_1$

From inlet velocity diagram, $V_1^2 = V_{r1}^2 + U^2 - 2UV_{r1}cos(180-\beta_1)$

Or, $V_1^2 = V_{r1}^2 + U^2 + 2UV_{r1}cos\beta_1$

Then, $V_1^2 - V_2^2 = 4UV_{r1}cos\beta_1$

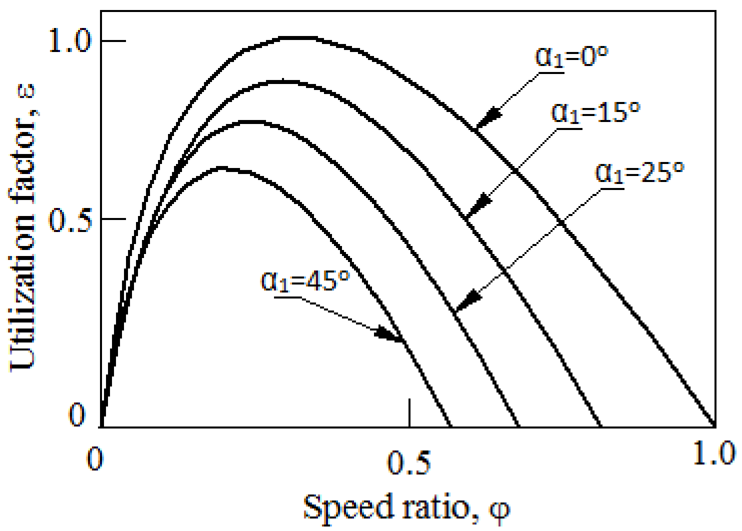


Fig.2.8 Variation of ε with φ for an impulse turbine

From inlet velocity diagram, $cos\beta_1 = \frac{(V_{u1}-U)}{V_{r1}}$ and $cos\alpha_1 = \frac{V_{u1}}{V_1}$

Then, $$V_1^2 - V_2^2 = 4U(V_{u1} - U)$$

Or, $$V_1^2 - V_2^2 = 4U(V_1 cos\alpha_1 - U)$$

Then,

$$\epsilon = \frac{4U(V_1 cos\alpha_1 - U)}{V_1^2}$$

But $\varphi = \frac{U}{V_1}$, then $$\boldsymbol{\epsilon = 4\varphi(cos\alpha_1 - \varphi)}$$

This means that utilization factor (ε) varies parabolically with the speed ratio (φ) and is zero both at φ=0 and at φ=$cos\alpha_1$. The variation of ε with φ is as shown in figure 2.8.

Question No 2.10: Derive an expression for the utilization factor for an fully reaction axial flow turbine stage, in terms of the fixed blade angle at inlet and speed ratio and show the variation of utilization factor and speed ratio in the form of a graph.

Answer: Figure 2.7 gives the velocity diagram for the axial flow turbine. For fully reaction axial flow turbine, R=1 and $V_1=V_2$ and also $\alpha_1=\alpha_2$.

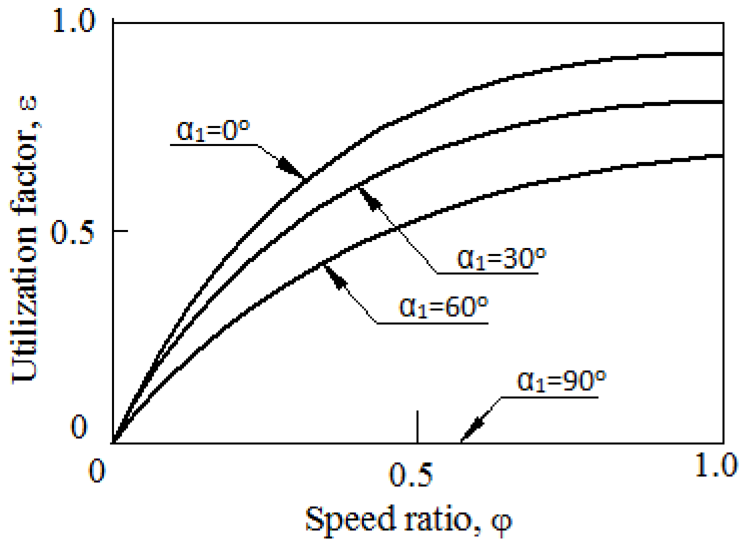


Fig.2.9 Variation of ε with φ for an fully reaction turbine

Utilization factor is given by:

$$\epsilon = \frac{e}{e + \frac{V_2^2}{2}} = \frac{U(V_{u1} + V_{u2})}{U(V_{u1} + V_{u2}) + \frac{V_1^2}{2}}$$

From inlet velocity diagram, $cos\alpha_1 = \frac{V_{u1}}{V_1} \Longrightarrow V_{u1} = V_1 cos\alpha_1$

And, from outlet velocity diagram, $cos\alpha_2 = \frac{V_{u2}}{V_2} \Longrightarrow V_{u2} = V_2 cos\alpha_2 = V_1 cos\alpha_1$

Then,

$$\epsilon = \frac{2UV_1 cos\alpha_1}{2UV_1 cos\alpha_1 + \frac{V_1^2}{2}} = \frac{1}{1 + \frac{V_1}{4Ucos\alpha_1}}$$

Or,

$$\boldsymbol{\epsilon = \frac{1}{1 + \frac{1}{4\varphi cos\alpha_1}}}$$

The variation of ε with φ for fully reaction axial flow turbine stage is as shown in figure 2.9. When α_1=90°, the utilization factor becomes zero irrespective of the speed ratio.

2.4.1.3 Variation of Maximum Utilization Factor with Nozzle (Stator) Angle: The general velocity diagram of axial flow turbine for maximum utilization is as shown if figure 2.10.

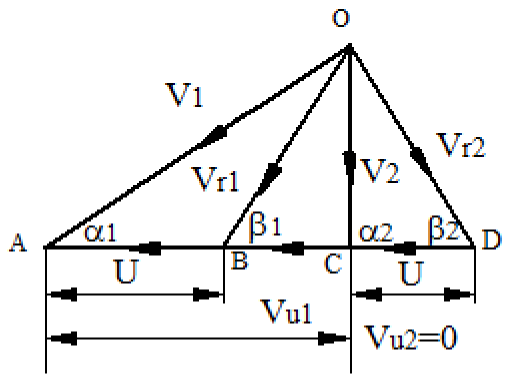


Fig. 2.10 General velocity diagram for maximum utilization (Common apex method)

Utilization factor is given by:

$$\epsilon = \frac{V_1^2 - V_2^2}{V_1^2 - RV_2^2}$$

From triangle OAC, $sin\alpha_1 = \frac{OC}{OA} = \frac{V_2}{V_1} \Longrightarrow V_2 = V_1 sin\alpha_1$

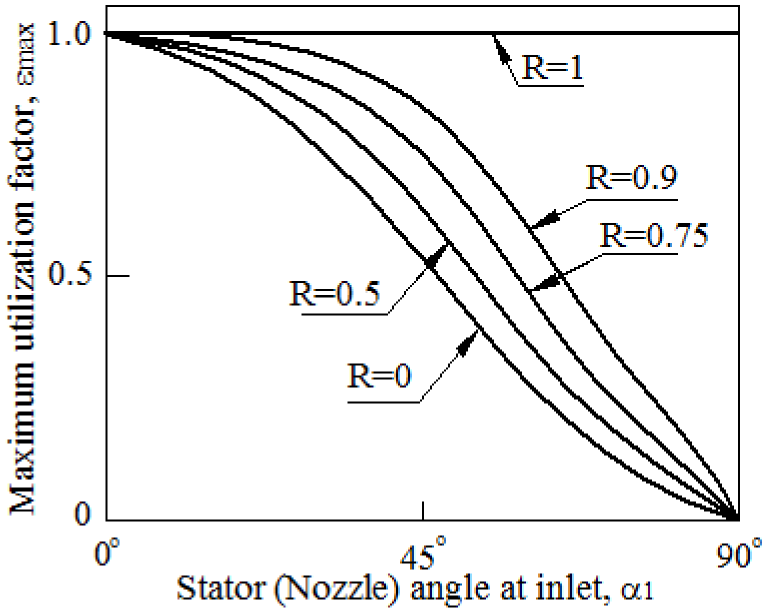


Fig. 2.11 Variation of ϵ_{max} with α_1 in an axial flow turbine stage

Then,

$$\epsilon_{max} = \frac{V_1^2 - V_1^2 sin^2\alpha_1}{V_1^2 - RV_1^2 sin^2\alpha_1} = \frac{V_1^2(1 - sin^2\alpha_1)}{V_1^2(1 - Rsin^2\alpha_1)}$$

$$\boldsymbol{\epsilon_{max} = \frac{cos^2\alpha_1}{1 - Rsin^2\alpha_1}}$$

For an axial flow impulse turbine R=0, then

$$\boldsymbol{\epsilon_{max} = cos^2\alpha_1} \tag{2.12}$$

For a 50% reaction axial flow turbine R=0.5, then

$$\boldsymbol{\epsilon_{max} = \frac{cos^2\alpha_1}{1-0.5\ sin^2\alpha_1}} \tag{2.13}$$

The variation of ϵ_{max} with α_1, using R as a parameter is exhibited in figure 2.11, for all values of R, ϵ_{max} is unity when α_1=0 and becomes zero when α_1=90°.

2.4.1.4 *Zero-angle Turbine:* When α_1=0 and if the requirements for maximum utilization are maintained ($V_2 = V_1 sin\alpha_1 = 0$), the velocity diagram collapse into a straight line, results in Zero-angle turbine. The shape of the rotor blade which theoretically achieves ϵ_{max}=1 is shown in figure 2.12. Evidently the blade is semi-cylindrical in shape, with a turning angle of 180°. This turbine cannot work in practice, since a finite velocity V_2 with an axial component is necessary to produce a steady flow at the wheel exit. However this shows that the nozzle angle should be as small as possible. The turbine that has a rotor-bucket with its shape approximately semi-cylindrical is the Pelton wheel, a hydraulic turbine. Even here, the bucket turns the water through 165° instead 180° so that the utilization factor is never unity in any real turbine.

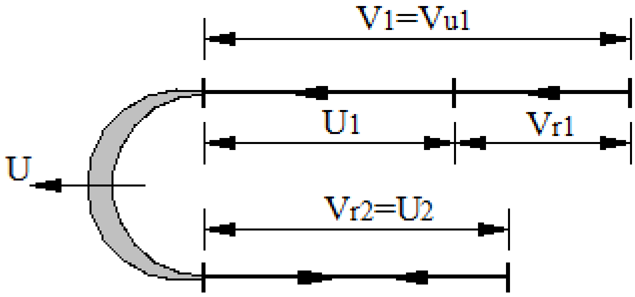


Fig. 2.12 Shape of blade needed to produce ϵ_{max}=1

2.4.1.5 Optimum Blade Speed Ratio for R = 0 and R = 0.5:

Question No 2.11: Draw the velocity diagram of an axial flow impulse turbine for maximum utilization and show that the optimum blade speed ratio for an axial flow impulse turbine is $\varphi_{opt} = \frac{cos\alpha_1}{2}$, where α_1 is the nozzle angle at inlet. Or

For an axial flow impulse turbine obtain the condition for maximum utilization factor.

(VTU, Jul/Aug-02)

Answer: The velocity diagram of an axial flow impulse turbine for maximum utilization is as shown in figure 2.13. For an axial flow impulse turbine V_{r1}=V_{r2} and β_1=β_2.

From triangle OAC,

$$cos\alpha_1 = \frac{AC}{OA} = \frac{AB + BC}{OA}$$

Velocity triangles OBC and OCD are congruent, hence BC=CD=U

Then,

$$cos\alpha_1 = \frac{U + U}{V_1} = \frac{2U}{V_1}$$

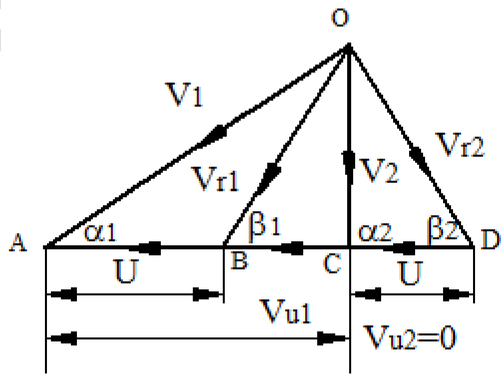


Fig. 2.13 Velocity diagram of an axial flow impulse turbine for maximum utilization

But blade speed ratio $\varphi = \frac{U}{V_1}$

Then, $$cos\alpha_1 = 2\varphi$$

Or, the optimum blade speed ratio

$$\boldsymbol{\varphi_{opt} = \frac{cos\alpha_1}{2}} \tag{2.14}$$

The optimum blade speed ratio is the blade speed ratio at which utilization factor will be the maximum.

From equation (2.11b),

$$\epsilon_{max} = \frac{2\varphi\, cos\alpha_1}{1 + 2\varphi R cos\alpha_1}$$

Substitute for axial flow impulse turbine R=0 and for maximum utilization condition $\varphi = \frac{cos\alpha_1}{2}$, then

$$\boldsymbol{\epsilon_{max} = cos^2\alpha_1}$$

The above equation same as equation (2.12)

Note: *Congruent triangles are exactly the same, that is their side lengths are the same and their interior/exterior angles are also same. We can create congruent triangles by rotating, translating or reflecting the original.*

Similar triangles look the same but their side lengths are proportional to each other and their interior/exterior angles are same. We can create similar triangles by dilating the original figure (in other words making it smaller or larger by a scale factor).

Question No 2.12: Draw the velocity diagram of an axial flow 50% reaction turbine for maximum utilization and show that the optimum blade speed ratio for an axial flow 50% reaction turbine is $\boldsymbol{\varphi_{opt} = cos\alpha_1}$, where α_1 is the nozzle angle at inlet.

Answer: For an axial flow 50% reaction turbine $V_1=V_{r2}$ and $V_2=V_{r1}$ and also $\alpha_1=\beta_2$ and $\alpha_2=\beta_1$. The velocity diagram of this turbine for maximum utilization is as shown in figure 2.14.

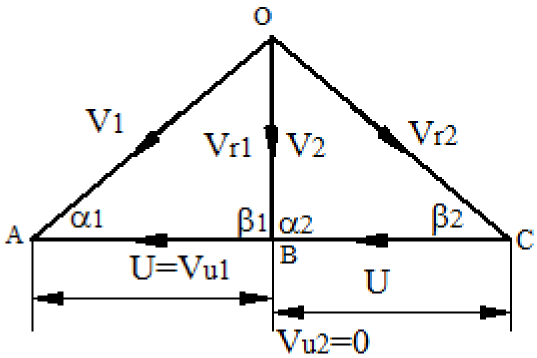


Fig. 2.14 Velocity diagram of an axial flow 50% reaction turbine for maximum utilization

From triangle OAB,

$$cos\alpha_1 = \frac{AB}{OA} = \frac{U}{V_1}$$

But blade speed ratio $\varphi = \frac{U}{V_1}$

Then, $$cos\alpha_1 = \varphi$$

Or, the optimum blade speed ratio

$$\boldsymbol{\varphi_{opt} = cos\alpha_1} \tag{2.15}$$

The optimum blade speed ratio is the blade speed ratio at which utilization factor will be the maximum.

From equation (2.11b),

$$\epsilon_{max} = \frac{2\varphi\ cos\alpha_1}{1 + 2\varphi R cos\alpha_1}$$

Substitute for a 50% reaction axial flow turbine R=0.5 and for maximum utilization condition $\varphi = Cos\alpha_1$, then

$$\epsilon_{max} = \frac{2cos^2\alpha_1}{1 + 2(0.5)\ cos^2\alpha_1}$$

$$\epsilon_{max} = \frac{cos^2\alpha_1}{\frac{1}{2}[1 + 2(0.5)cos^2\alpha_1]}$$

$$\epsilon_{max} = \frac{cos^2\alpha_1}{\frac{1}{2}[1 + 2(0.5)(1 - sin^2\alpha_1)]}$$

$$\boldsymbol{\epsilon_{max} = \frac{cos^2\alpha_1}{1 - 0.5\ sin^2\alpha_1}}$$

The above equation same as equation (2.13)

2.4.1.6 Comparison between Impulse Turbine and 50% Reaction Turbine:

Question No 2.13: Show that for maximum utilization the work output per stage of an axial flow impulse machine (with equiangular rotor blades) is double that of a 50% reaction stage which has the same blade speed. Assume that axial velocity remains constant for both machines.

(VTU, Dec-08/Jan-09)

Answer: Let U_I and U_R be the blade speed of an axial flow impulse turbine and 50% reaction turbine respectively.

Work output per stage or energy transfer per stage by impulse turbine is given by,

$$e_I = U_I(V_{u1} - V_{u2})$$

For maximum utilization factor, $V_{u2} = 0$

Then, $\quad e_I = U_I V_{u1}$

From impulse turbine velocity diagram for maximum utilization (Fig. 2.13)

$$AC = AB + BC \Longrightarrow V_{u1} = U_I + U_I = 2U_I$$

Then, $\quad \boldsymbol{e_I = 2U_I^2} \qquad (2.16)$

Work output per stage or energy transfer per stage by impulse turbine is given by,

$$e_R = U_R V_{u1}$$

From 50% reaction turbine velocity diagram for maximum utilization (Fig. 2.14)

$$AB = V_{u1} = U_R$$

Then, $$\boldsymbol{e_R = U_R^2} \tag{2.17}$$

From equations (2.16) and (2.17), for same blade speed $U_I = U_R$

$$\boldsymbol{e_I = 2e_R}$$

For the maximum utilization the energy transfer in axial flow impulse turbine is double than that of axial flow 50% reaction turbine for the same blade speed.

Question No 2.14: Show that for maximum utilization and for same amount of energy transfer in an axial flow impulse turbine and axial flow reaction turbine with 50% degree of reaction $U_R = \sqrt{2U_I^2}$, where U_R and U_I are blade speeds of reaction turbine and impulse turbine respectively.
(VTU, Feb-06)

Answer: From equation (2.16), for axial flow impulse turbine

$$e_I = 2U_I^2$$

From equation (2.17), for axial flow 50% reaction turbine

$$e_R = U_R^2$$

For same energy transfer, $e_R = e_I$

$$U_R^2 = 2U_I^2$$

Or, $$\boldsymbol{U_R = \sqrt{2U_I^2}}$$

Question No 2.15: Show that for maximum utilization and for same absolute velocity and inlet nozzle angle, the blade speed of axial flow 50% reaction turbine is double that of axial flow impulse turbine. (VTU, Jul-07)

Answer: From equation (2.13), optimum speed ratio for axial flow impulse turbine is

$$\varphi_{opt} = \frac{cos\alpha_1}{2} = \frac{U_I}{V_1}$$

Or, $$V_1 cos\alpha_1 = 2U_I$$

From equation (2.14), optimum speed ratio for axial flow 50% reaction turbine is

$$\varphi_{opt} = cos\alpha_1 = \frac{U_R}{V_1}$$

Or, $$V_1 cos\alpha_1 = U_R$$

For same absolute velocity (V_1) and inlet nozzle angle (α_1),

$$\boldsymbol{U_R = 2U_I}$$

For same absolute velocity and inlet nozzle angle, the blade speed of axial flow 50% reaction turbine is double that of axial flow impulse turbine.

Question No 2.16: Derive relations for maximum energy transfer and maximum utilization factor in case of axial flow impulse turbine and 50% reaction turbine. (VTU, May/Jun-10)

Answer: From equation (2.16), maximum energy transfer for axial flow impulse turbine is

$$e_I = 2U_I^2$$

From equation (2.17), maximum energy transfer for axial flow 50% reaction turbine is

$$e_R = U_R^2$$

From equation (2.12), maximum utilization factor for an axial flow impulse turbine is

$$\epsilon_{max} = cos^2\alpha_1$$

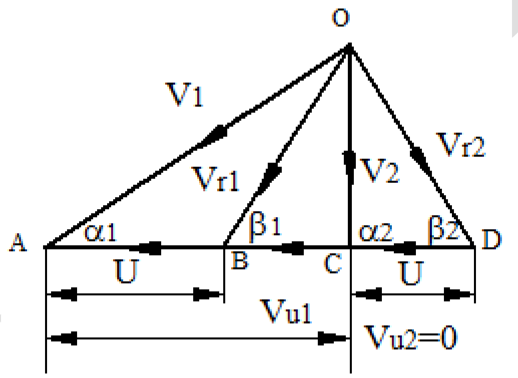
From equation (2.13), maximum utilization factor for an axial flow 50% reaction turbine is

$$\epsilon_{max} = \frac{cos^2\alpha_1}{1-0.5\ sin^2\alpha_1}$$

Question No 2.17: Show that maximum utilization factor of an axial flow turbine with degree of reaction ¼, the relationship of blade speed U to absolute velocity at rotor inlet V_1(speed ratio) is given by $\varphi = \frac{U}{V_1} = \frac{2}{3}cos\alpha_1$, where α_1 is the nozzle angle with respect to tangential direction at inlet.

(VTU, Jun/Jul-09, Jun/Jul-11, Jan-16)

Answer: The velocity diagram of axial flow turbine for maximum utilization is given in figure 2.10.



For axial flow turbine, degree of reaction is:

$$R = \frac{[(V_{r2}^2 - V_{r1}^2)]}{[(V_1^2 - V_2^2) + (V_{r2}^2 - V_{r1}^2)]} = \frac{1}{4}$$

$$(V_1^2 - V_2^2) + (V_{r2}^2 - V_{r1}^2) = 4(V_{r2}^2 - V_{r1}^2)$$

Or, $$(V_1^2 - V_2^2) = 3(V_{r2}^2 - V_{r1}^2) \qquad (2.18)$$

From triangle OAC, $sin\alpha_1 = \frac{OC}{OA} = \frac{V_2}{V_1} \Longrightarrow V_2 = V_1 sin\alpha_1$

Or, $$V_2^2 = V_1^2 sin^2\alpha_1$$

From triangle OCD, $$V_{r2}^2 = V_2^2 + U^2$$

Or, $$V_{r2}^2 = V_1^2 sin^2\alpha_1 + U^2$$

By applying cosine rule to triangle OAB,

$$V_{r1}^2 = V_1^2 + U^2 - 2UV_1 cos\alpha_1$$

Substitute the values of $V_2^2, V_{r2}^2\ and\ V_{r1}^2$ in equation (2.18),

$$V_1^2 - V_1^2 sin^2\alpha_1 = 3[V_1^2 sin^2\alpha_1 + U^2 - (V_1^2 + U^2 - 2UV_1 cos\alpha_1)]$$

$$4(V_1^2 - V_1^2 sin^2\alpha_1) = 6UV_1 cos\alpha_1$$

$$4V_1^2 cos^2\alpha_1 = 6UV_1 cos\alpha_1$$

Or,

$$\boldsymbol{\varphi = \frac{U}{V_1} = \frac{2}{3} cos\alpha_1}$$

2.4.2 Radial Flow Turbines: Radial flow turbines are radial inward flow turbomachines, here fluid flows across the rotor blades radially from outer radius (tip radius) to inner radius (hub radius) of the rotor as shown in figure 2.15. Therefore radial turbines are also known as *centripetal turbomachines*. Since the fluid enters and leaves the rotor at different radius $U_1 \neq U_2$.

Question No 2.18: A radial turbomachine has no inlet whirl. The blade speed at the exit is twice that of the inlet. Radial velocity is constant throughout. Taking the inlet blade angle as 45°, show that energy transfer per unit mass is given by $\boldsymbol{e = 2V_m^2(cot\beta_2 - 2)}$***, where*** $\boldsymbol{\beta_2}$ ***is the blade angle at exit with respect to tangential direction.*** ***(VTU, Jun/Jul-11)***

Answer: The data given in the problem are:

$$V_{u1} = 0, U_2 = 2U_1, V_{m1} = V_{m2} = V_m, \beta_1 = 45^o (\therefore U_1 = V_{m1})$$

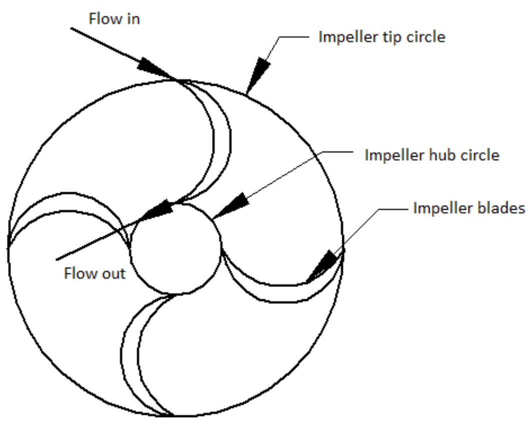
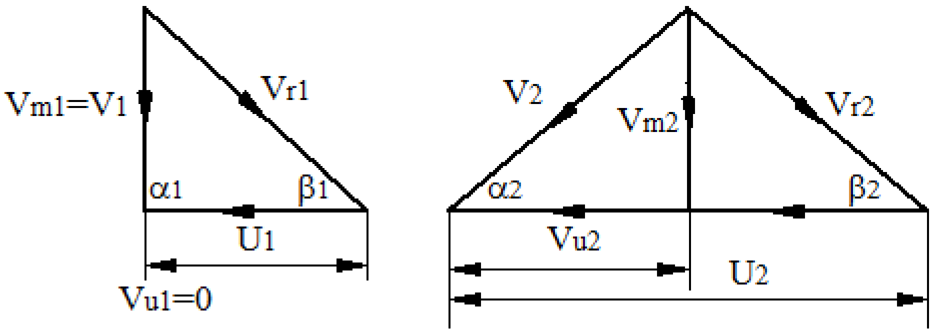


Fig. 2.15 Radial flow turbine

The velocity diagram for the above conditions is as follows



Energy transfer of general radial flow turbomachine is given by,

$$e = (U_1V_{u1} - U_2V_{u2})$$

But $V_{u1} = 0$,

$$e = -U_2V_{u2} \tag{2.19}$$

From outlet velocity triangle, $V_{u2} = U_2 - x_2$

But, $cot\beta_2 = \frac{x_2}{V_{m2}} \Longrightarrow x_2 = V_{m2}cot\beta_2$

Then, $V_{u2} = U_2 - V_{m2}cot\beta_2$

Substitute V_{u2} in equation (2.19)

$$e = -U_2(U_2 - V_{m2}cot\beta_2)$$

From given data, $U_2 = 2U_1 = 2V_{m1} = 2V_{m2} = 2V_m$

$$e = -2V_m(2V_m - V_mcot\beta_2)$$

$$\boldsymbol{e = 2V_m^2(cot\beta_2 - 2)} \tag{2.20}$$

Question No 2.19: A radial turbomachine has no inlet whirl. The blade speed at the exit is twice that of the inlet. Radial velocity is constant throughout. Taking the inlet blade angle as 45^o, show that degree of reaction is given by R = $\frac{(2+cot\beta_2)}{4}$, where β$_2$ is the blade angle at exit with respect to tangential direction. ***(VTU, Jun/Jul-11, Dec-12, Jun/Jul-13, Jul-16, Jul-17)***

Answer: The data given in the problem are:

$$V_{u1} = 0, U_2 = 2U_1, V_{m1} = V_{m2} = V_m, \beta_1 = 45^o (\because U_1 = V_{m1})$$

The velocity diagram for the above conditions is as same as the Question No 2.18.

Degree of reaction for general radial flow turbomachine is given by:

$$R = \frac{e - \frac{1}{2}(V_1^2 - V_2^2)}{e}$$

But, $e = 2V_m^2(cot\beta_2 - 2)$

From inlet velocity triangle, $V_1^2 = V_{m1}^2 = V_m^2$

By applying cosine rule to outlet velocity triangle, $V_2^2 = U_2^2 + V_{r2}^2 - 2U_2V_{r2}cos\beta_2$

But, $sin\beta_2 = \frac{V_{m2}}{V_{r2}} \Longrightarrow V_{r2} = \frac{V_{m2}}{sin\beta_2}$

Then, $V_2^2 = U_2^2 + \frac{V_{m2}^2}{sin^2\beta_2} - 2U_2\frac{V_{m2}}{sin\beta_2}cos\beta_2$

From given data, $U_2 = 2U_1 = 2V_{m1} = 2V_{m2} = 2V_m$

Then, $V_2^2 = 4V_m^2 + V_m^2cosec^2\beta_2 - 4V_m^2cot\beta_2$

Or, $V_2^2 = 4V_m^2 + V_m^2(1 + cot^2\beta_2) - 4V_m^2cot\beta_2$

$V_2^2 = 5V_m^2 + V_m^2 cot^2\beta_2 - 4V_m^2 cot\beta_2$

Then,

$$R = \frac{2V_m^2(cot\beta_2 - 2) - \frac{1}{2}[V_m^2 - (5V_m^2 + V_m^2 cot^2\beta_2 - 4V_m^2 cot\beta_2\ )]}{2V_m^2(cot\beta_2 - 2)}$$

$$R = \frac{4V_m^2(cot\beta_2 - 2) - [V_m^2 - (5V_m^2 + V_m^2 cot^2\beta_2 - 4V_m^2 cot\beta_2\ )]}{4V_m^2(cot\beta_2 - 2)}$$

$$R = \frac{V_m^2 cot^2\beta_2 - 4V_m^2}{4V_m^2(cot\beta_2 - 2)}$$

$$\boldsymbol{R = \frac{2 + cot\beta_2}{4}}$$

Question No 2.20: Why the discharge blade angle has considerable effect in the analysis of a turbomachine? Give reasons. ***(VTU, Dec-10,Jun/Jul-11)***

Answer: The energy transfer for radial flow turbomachines in terms of discharge blade angle is $e = 2V_m^2(cot\beta_2 - 2)$. This equation gives that, for $\beta_2 > 26.5^o$ 'e' is negative and continuously increases with β_2. As 'e' negative for these values of β_2, the machine will acts as pump or compressor. For $\beta_2 < 26.5^o$ 'e' is positive and machine will act as a turbine.

The degree of reaction for radial flow turbomachine in terms of discharge blade angle is $R = \frac{2+cot\beta_2}{4}$. This equation gives that, for β_2 in the range of 26.5^o to 153.5^o, the value of R decreases linearly from near unity to very small positive value. For $\beta_2 = 153.5^o$, R=0 and hence machine will act as impulse turbine.

The effect of discharge blade angle on energy transfer and degree of reaction of turbomachine is shown in figure 2.16.

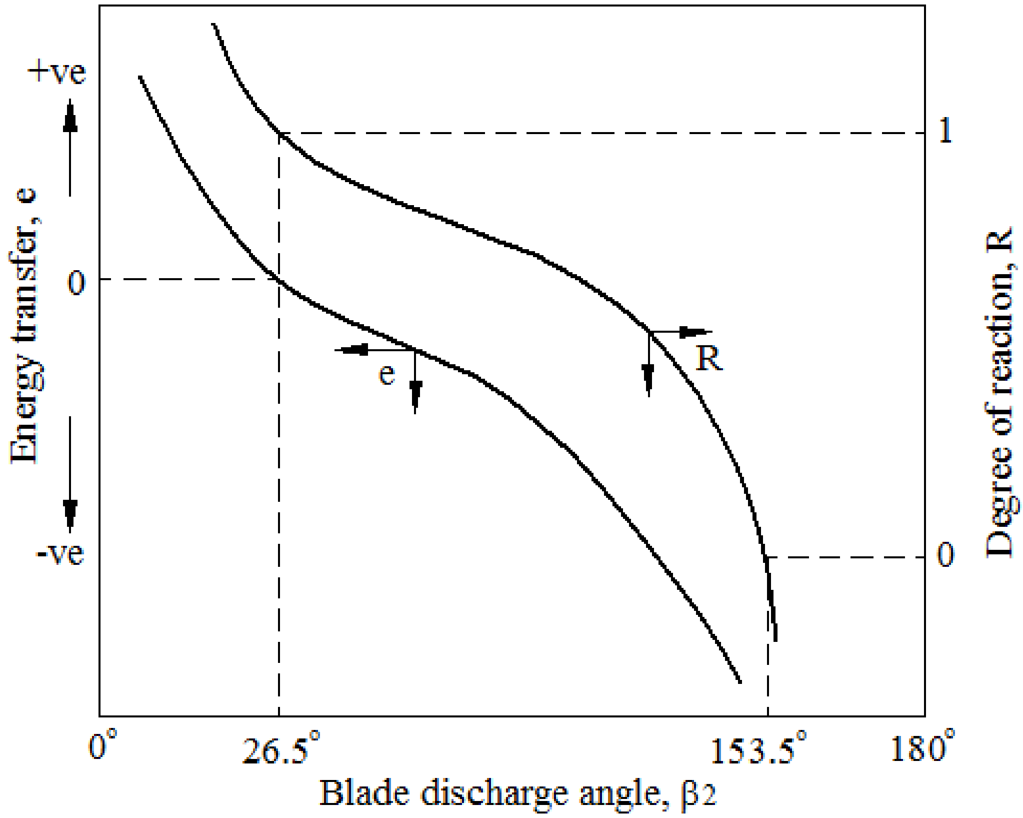


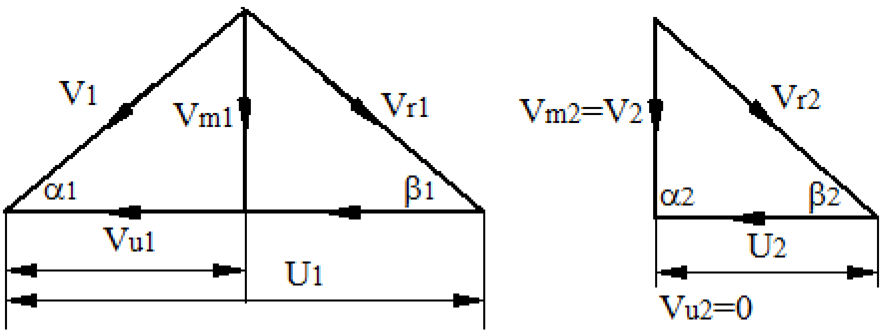
Fig. 2.16 Effect of discharge blade angle on energy transfer and degree of reaction

Question No 2.21: An inward flow radial reaction turbine has radial discharge at outlet with outlet blade angle is 45°. The radial component of absolute velocity remains constant throughout and equal to $\sqrt{2gH}$ where g is the acceleration due to gravity and H is the constant head. The blade speed at inlet is twice that at outlet. Express the energy transfer per unit mass and the degree of reaction in terms of α_1, where α_1 is the direction of the absolute velocity at inlet. At what value of α_1 will be the degree of reaction zero and unity? What are the corresponding values of energy transfer per unit mass? **(VTU, Jan/Feb-06)**

Answer: The data given in the problem are:

$$\alpha_2 = 90^o\ (\therefore V_{u2} = 0), \beta_2 = 45^o(\therefore U_2 = V_{m2}), V_{m1} = V_{m2} = \sqrt{2gH}, U_1 = 2U_2,$$

The velocity diagram for the above conditions is as follows



Energy transfer of inward radial flow reaction turbine is given by,

$$e = (U_1V_{u1} - U_2V_{u2})$$

But $V_{u2} = 0$,

$$e = U_1V_{u1}$$

From inlet velocity triangle, $cot\alpha_1 = \frac{V_{u1}}{V_{m1}} \Longrightarrow V_{u1} = V_{m1}cot\alpha_1$

Then, $$e = U_1V_{m1}cot\alpha_1$$

From given data, $U_1 = 2U_2 = 2V_{m2} = 2V_{m1}$

Then, $$e = 2V_{m1}^2cot\alpha_1$$

From given data, $V_{m1} = V_{m2} = \sqrt{2gH}$

Then, $$\boldsymbol{e = 4gHcot\alpha_1}$$

Degree of reaction for inward radial flow reaction turbine is given by:

$$R = \frac{e - \frac{1}{2}(V_1^2 - V_2^2)}{e}$$

But, $e = 4gHcot\alpha_1$

From inlet velocity triangle, $sin\alpha_1 = \frac{V_{m1}}{V_1} \Longrightarrow V_1 = \frac{\sqrt{2gH}}{sin\alpha_1} \Longrightarrow V_1^2 = \frac{2gH}{sin^2\alpha_1}$

From outlet velocity triangle, $V_2 = V_{m2} \Longrightarrow V_2^2 = V_{m2}^2 = 2gH$

Then,

$$R = \frac{4gHcot\alpha_1 - \frac{1}{2}\left[\frac{2gH}{sin^2\alpha_1} - 2gH\right]}{4gHcot\alpha_1}$$

$$R = \frac{4cot\alpha_1 - \left[\frac{1 - sin^2\alpha_1}{sin^2\alpha_1}\right]}{4cot\alpha_1} = \frac{4cot\alpha_1 - \left[\frac{cos^2\alpha_1}{sin^2\alpha_1}\right]}{4cot\alpha_1}$$

$$R = \frac{4cot\alpha_1 - cot^2\alpha_1}{4cot\alpha_1}$$

Or, $$\boldsymbol{R = \frac{4 - cot\alpha_1}{4}}$$

At $\alpha_1 = 14.04^o, R = 0$ then $e = 16gH\ J/kg$

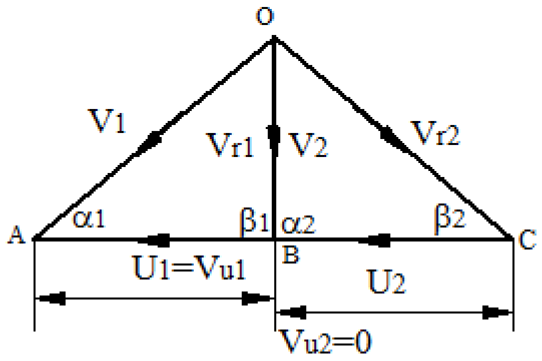
At $\alpha_1 = 90^o, R = 1$ then $e = 0$

Question No 2.22: For a centripetal turbine with guide blade angle α_1 and radial blades at the inlet. The radial velocity is constant and there is no whirl velocity at discharge. Show that the degree of reaction is 0.5. Also derive an expression for utilization factor in terms of α_1. Or Show that maximum blade efficiency $\eta_{blade\ max}$=$\frac{2cos^2\alpha_1}{1+cos^2\alpha_1}$ for a 50 % reaction Parson's. (VTU, Jun/Jul-08,Jan 15)

Answer: The data given in the problem are:

$$\beta_1 = 90^o, V_{m1} = V_{m2}, V_{u2} = 0(\therefore \alpha_2 = 90^o)$$

The velocity diagram for the above conditions is as follows



Degree of reaction for a centripetal turbine is given by:

$$R = \frac{e - \frac{1}{2}(V_1^2 - V_2^2)}{e}$$

Energy transfer of a centripetal turbine is given by, $e = (U_1 V_{u1} - U_2 V_{u2})$

But $V_{u2} = 0$, $\quad e = U_1 V_{u1}$

From inlet velocity triangle, $V_{u1} = U_1$

Then, $\quad e = U_1^2$

From inlet velocity triangle, $V_1^2 - V_{m1}^2 = U_1^2$

But $V_{m1} = V_{m2} = V_2$, then $V_1^2 - V_2^2 = U_1^2$

Thus,

$$R = \frac{U_1^2 - \frac{1}{2}(U_1^2)}{U_1^2}$$

$$\boldsymbol{R = 0.5}$$

Utilization factor for centripetal turbine is given by,

$$\epsilon = \frac{e}{e + \frac{V_2^2}{2}}$$

From inlet velocity triangle, $tan\alpha_1 = \frac{V_{r1}}{U_1} \Longrightarrow V_{r1} = U_1 tan\alpha_1$

But $V_2 = V_{m2} = V_{m1} = V_{r1} = U_1 tan\alpha_1$

Then,

$$\epsilon = \frac{U_1^2}{U_1^2 + \frac{U_1^2 tan^2\alpha_1}{2}}$$

$$\boldsymbol{\epsilon = \frac{2}{2 + tan^2\alpha_1}}$$

Or,

$$\epsilon = \frac{2}{2 + \frac{sin^2\alpha_1}{cos^2\alpha_1}} = \frac{2cos^2\alpha_1}{2cos^2\alpha_1 + sin^2\alpha_1}$$

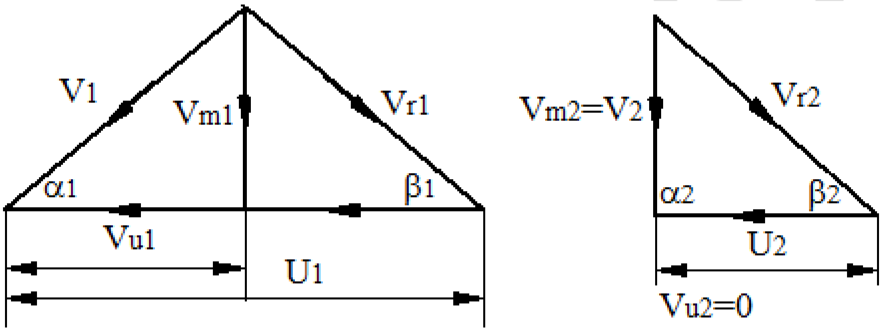
$$\boldsymbol{\epsilon = \frac{2cos^2\alpha_1}{1 + cos^2\alpha_1}}$$

Question No 2.23: In an inward flow radial hydraulic turbine, degree of reaction is R and utilization factor is ϵ. Assuming the radial velocity component is constant throughout and there is no tangential component of absolute velocity at outlet, show that the inlet nozzle angle is given by $\alpha_1 = cot^{-1}\sqrt{\frac{(1-R)\epsilon}{(1-\epsilon)}}$ (VTU, Jan-04, Dec-12)

Answer: The data given in the problem are:

$$V_{m1} = V_{m2} = V_m, V_{u2} = 0\ (\because \alpha_2 = 90^o)$$

The velocity diagram for the above conditions is as follows



Utilization factor is given by:

$$\epsilon = \frac{V_1^2 - V_2^2}{V_1^2 - RV_2^2}$$

From inlet velocity triangle, $sin\alpha_1 = \frac{V_{m1}}{V_1} \Longrightarrow V_{m1} = V_1 sin\alpha_1$

From outlet velocity triangle, $V_2 = V_{m2} = V_{m1} = V_1 sin\alpha_1$

Then,

$$\epsilon = \frac{V_1^2 - V_1^2 sin^2\alpha_1}{V_1^2 - RV_1^2 sin^2\alpha_1} = \frac{cos^2\alpha_1}{1 - Rsin^2\alpha_1}$$

$$\epsilon = \frac{\frac{cos^2\alpha_1}{sin^2\alpha_1}}{\frac{1}{sin^2\alpha_1} - R} = \frac{cot^2\alpha_1}{cosec^2\alpha_1 - R} = \frac{cot^2\alpha_1}{1 + cot^2\alpha_1 - R}$$

$$\epsilon(1 + cot^2\alpha_1 - R) = cot^2\alpha_1$$

$$\epsilon - \epsilon R = cot^2\alpha_1 - \epsilon cot^2\alpha_1$$

Or, $$cot^2\alpha_1(1 - \epsilon) = \epsilon(1 - R)$$

$$\alpha_1 = cot^{-1}\sqrt{\frac{(1-R)\epsilon}{(1-\epsilon)}}$$

2.5 General Analysis of Power-absorbing Turbomachines:

Compressors and pumps are power absorbing turbomachines, since they raise the stagnation pressure or enthalpy of a fluid through mechanical energy intake. The quantity of interest in the power absorbing device is the stagnation enthalpy or pressure rise of the flowing fluid due to the work. In power absorbing machines, the reference direction to define the various angles is often the axis than the tangent to the rotor-tip. Like turbines, these machines may be divided into axial, radial and mixed flow devices depending on the flow direction in the rotor blades.

2.5.1 Axial Flow Compressors and Pumps: In axial flow machines, the blade speed is the same at the rotor inlet and outlet. Each compressor stage consists usually of a stator and a rotor just as in a turbine. Further, there is diffuser at the exit to recover part of the exit kinetic energy of the fluid to produce an increase in static pressure. The pressure at the compressor exit will have risen due to the diffusive action in rotors and stators. If stator blades are present at the inlet they are called inlet guide-vanes. The blades at the exit section in the diffuser are called exit guide-vanes.

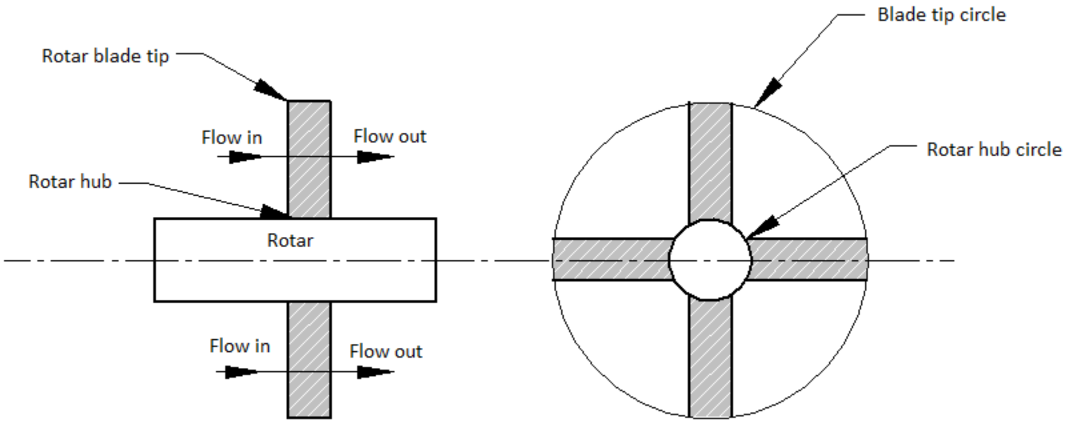


Fig. 2.17 Axial flow compressor

Energy transfer for axial flow compressor or pump is:

$$e = \frac{1}{2}[(V_2^2 - V_1^2) - (V_{r2}^2 - V_{r1}^2)]$$

Degree of reaction for axial flow compressor or pump is:

$$R = \frac{[(V_{r1}^2 - V_{r2}^2)]}{[(V_2^2 - V_1^2) - (V_{r2}^2 - V_{r1}^2)]} = \frac{e - \frac{1}{2}(V_2^2 - V_1^2)}{e}$$

Question No 2.24: Draw the set of velocity triangles for axial flow compressor stage and show that, $\Delta h_o = UV_a(tan\gamma_1 - tan\gamma_2)$, where V_a is axial velocity, U is blade speed and γ_1 and γ_2 are the inlet and outlet blade angles with respect to axial direction. Or,

Draw the set of velocity triangles for axial flow compressor stage and show that, $\Delta h_o = UV_a\left[\frac{tan\beta_2 - tan\beta_1}{tan\beta_1 tan\beta_2}\right]$, where V_a is axial velocity, U is blade speed and β_1 and β_2 are the inlet and outlet blade angles with respect to tangential direction.

Answer: The general velocity diagram for axial flow compressor stage is as shown in figure 2.18. For axial flow machines the blade speed and the axial velocity may assume to be constant. That is, $U_1 = U_2 = U\ and\ V_{a1} = V_{a2} = V_a$

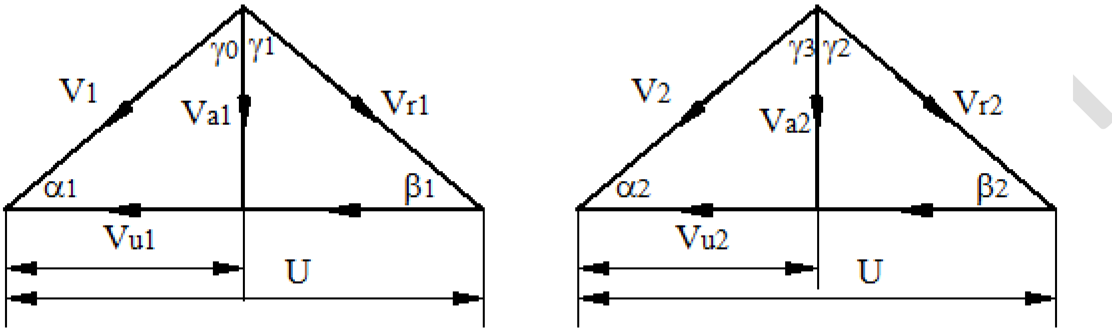


Fig. 2.18 General velocity diagram for axial flow compressor stage

Usually in an axial flow turbomachines the working fluid is either combustion gas or air. Whenever, the energy transfer occurs in these machines, then pressure energy or enthalpy of the working fluid changes. Therefore energy transfer of an axial flow compressor is given as:

$$e = \Delta h_o = U(V_{u2} - V_{u1})$$

From inlet velocity triangle,

$$tan\gamma_o = \frac{V_{u1}}{V_{a1}} \Longrightarrow V_{u1} = V_{a1}tan\gamma_o = V_a tan\gamma_o$$

$$tan\gamma_1 = \frac{AB}{V_{a1}} \Longrightarrow AB = V_{a1}tan\gamma_1 = V_a tan\gamma_1$$

$$U = AB + V_{u1} = V_a(tan\gamma_1 + tan\gamma_o)$$

From outlet velocity triangle,

$$tan\gamma_3 = \frac{V_{u2}}{V_{a2}} \Longrightarrow V_{u2} = V_{a2}tan\gamma_3 = V_a tan\gamma_3$$

$$tan\gamma_2 = \frac{AB}{V_{a2}} \Longrightarrow AB = V_{a2}tan\gamma_2 = V_a tan\gamma_2$$

$$U = AB + V_{u2} = V_a(tan\gamma_2 + tan\gamma_3)$$

Then, $$U = V_a(tan\gamma_1 + tan\gamma_o) = V_a(tan\gamma_2 + tan\gamma_3)$$

Or, $$\boldsymbol{(tan\gamma_1 - tan\gamma_2) = (tan\gamma_3 - tan\gamma_o)}$$

Then, $$\Delta h_o = U(V_{u2} - V_{u1}) = U(V_a tan\gamma_3 - V_a tan\gamma_o) = UV_a(tan\gamma_3 - tan\gamma_o)$$

Or, $$\boldsymbol{\Delta h_o = UV_a(tan\gamma_1 - tan\gamma_2)}$$

Where γ_1 and γ_2 are the inlet and outlet blade angles with respect to axial direction are also known as Air angles.

Or,
$$\Delta h_o = UV_a[\tan(90^o - \beta_1) - \tan(90^o - \beta_2)]$$
$$\boldsymbol{\Delta h_o = UV_a[\cot\beta_1 - \cot\beta_2]}$$
Or,
$$\Delta h_o = UV_a\left[\frac{1}{\tan\beta_1} - \frac{1}{\tan\beta_2}\right]$$
$$\boldsymbol{\Delta h_o = UV_a\left[\frac{tan\beta_2 - tan\beta_1}{tan\beta_1 tan\beta_2}\right]}$$

Where β_1 and β_2 are the inlet and outlet blade angles with respect to tangential direction.

Question No 2.25: With the help of inlet and outlet velocity triangles, show that the degree of reaction for axial flow compressor as $R = \frac{V_a}{U}tan\gamma_m$, where V_a is axial velocity, U is blade speed and $tan\gamma_m = \frac{tan\gamma_1 + tan\gamma_2}{2}$ γ_1 and γ_2 are the inlet and outlet blade angles with respect to axial direction. ***(VTU, Jun-12, Dec-06/Jan-07, Jun/Jul-13) Or,***

With the help of inlet and outlet velocity triangles, show that the degree of reaction for axial flow compressor as $R = \frac{V_a}{U}cot\beta_m$, where V_a is axial velocity, U is blade speed and $cot\beta_m = \frac{cot\beta_1 + cot\beta_2}{2}$ β_1 and β_2 are the inlet and outlet blade angles with respect to tangential direction. Or,

Draw the velocity triangles for an axial flow compressor and show that for an axial flow compressor having no axial thrust, the degree of reaction is given by: $R = \frac{V_a}{2U}\left[\frac{tan\beta_1 + tan\beta_2}{tan\beta_1 tan\beta_2}\right]$, where V_a is axial velocity, U is blade speed and β_1 and β_2 are the inlet and outlet blade angles with respect to tangential direction. ***(VTU, Feb-03, Jul-11, Jun-10, Jan-14)***

Answer: The general velocity diagram for axial flow compressor stage is as shown in figure 2.14. For axial flow machines the blade speed and the axial velocity may assume to be constant. That is, $U_1 = U_2 = U \; and \; V_{a1} = V_{a2} = V_a$

Degree of reaction for axial flow compressor is:
$$R = \frac{\frac{1}{2}[(V_{r1}^2 - V_{r2}^2)]}{\frac{1}{2}[(V_2^2 - V_1^2) - (V_{r2}^2 - V_{r1}^2)]} = \frac{\frac{1}{2}[(V_{r1}^2 - V_{r2}^2)]}{e}$$

But, $e = \Delta h_o = UV_a(tan\gamma_1 - tan\gamma_2)$

From inlet velocity triangle, $V_{r1}^2 = AB^2 + V_{a1}^2 = V_{a1}^2 tan^2\gamma_1 + V_{a1}^2$
$$V_{r1}^2 = V_a^2 + V_a^2 tan^2\gamma_1$$

Similarly from outlet velocity triangle,
$$V_{r2}^2 = V_a^2 + V_a^2 tan^2\gamma_2$$

Then,

$$R = \frac{V_a^2 + V_a^2 tan^2\gamma_1 - (V_a^2 + V_a^2 tan^2\gamma_2)}{2UV_a(tan\gamma_1 - tan\gamma_2)}$$

$$R = \frac{V_a(tan^2\gamma_1 - tan^2\gamma_2)}{2U(tan\gamma_1 - tan\gamma_2)}$$

$$R = \left(\frac{V_a}{U}\right)\frac{(tan\gamma_1 + tan\gamma_2)}{2}$$

$$\boldsymbol{R = \frac{V_a}{U} tan\gamma_m}$$

Where $tan\gamma_m = \frac{tan\gamma_1 + tan\gamma_2}{2}$ *γ_1 and γ_2 are the inlet and outlet blade angles with respect to axial direction.*

Or,

$$R = \left(\frac{V_a}{U}\right)\frac{[\tan(90^o - \beta_1) + \tan(90^o - \beta_2)]}{2}$$

$$R = \left(\frac{V_a}{U}\right)\frac{[\cot\beta_1 + cot\beta_2]}{2}$$

$$\boldsymbol{R = \frac{V_a}{U} cot\beta_m}$$

Where $cot\beta_m = \frac{cot\beta_1 + cot\beta_2}{2}$ *β_1 and β_2 are the inlet and outlet blade angles with respect to tangential direction.*

Or,

$$R = \left(\frac{V_a}{2U}\right)\left[\frac{1}{tan\beta_1} + \frac{1}{tan\beta_2}\right]$$

$$\boldsymbol{R = \frac{V_a}{2U}\left[\frac{tan\beta_1 + tan\beta_2}{tan\beta_1 tan\beta_2}\right]}$$

2.5.2 Radial Flow Compressors and Pumps: Radial flow compressors and pumps are radial outward flow turbomachines, here fluid flows across the rotor blades radially from inner radius (hub radius) to outer radius (tip radius) of the rotor as shown in figure 2.19. Therefore radial compressors and pumps are also known as *centrifugal turbomachines*. Since the fluid enters and leaves the rotor at different radius $U_1 \neq U_2$. In centrifugal compressor or pump usually the absolute velocity at the entry has no tangential component, i.e., $V_{u1} = 0$.

Question No 2.26: Derive a theoretical head capacity (H-Q) relationship for centrifugal pumps and compressors and explain the influence of outlet blade angle. ***(VTU, Jul/Aug-05, Dec-11, Jun/Jul-14)***

Answer: The velocity diagram for centrifugal pumps and compressor with $V_{u1} = 0$ is as shown in figure 2.20. Usually in a radial flow turbomachines the working fluid is either water or oil. Whenever, the energy transfer occurs in these machines, then pressure energy or potential energy of the working fluid changes.

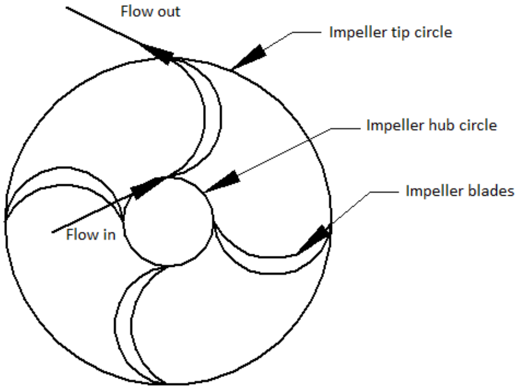


Fig. 2.19 Radial flow compressor or pump

The energy transfer of a centrifugal compressor and pump is given as:

$$e = gH = U_2V_{u2} - U_1V_{u1}$$

Or, $gH = U_2V_{u2}$ (Because, $V_{u1}= 0$)

From outlet velocity triangle, $V_{u2} = U_2 - x_2$

But, $cot\beta_2 = \frac{x_2}{V_{m2}} \Longrightarrow x_2 = V_{m2}cot\beta_2$

Then,

$$V_{u2} = U_2 - V_{m2}cot\beta_2$$

Therefore, $gH = U_2(U_2 - V_{m2}cot\beta_2)$

Or,

$$H = \frac{U_2^2}{g} - \frac{U_2V_{m2}}{g}cot\beta_2$$

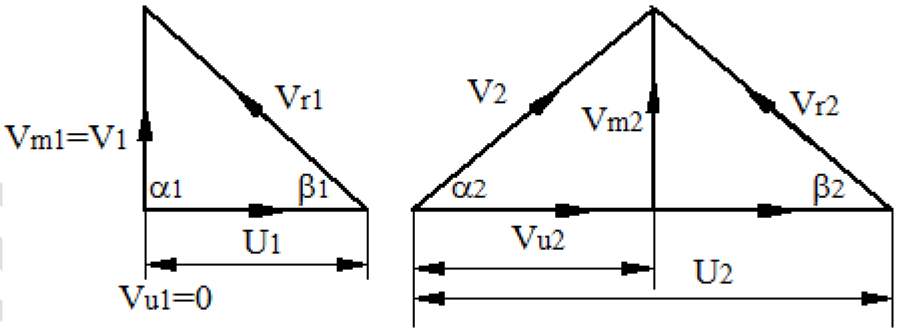


Fig. 2.20 Velocity diagram for centrifugal pumps and compressor with $V_{u1}= 0$

Discharge at outer radius of centrifugal machine = Area of flow × Flow velocity

$$Q = \pi D_2B_2 \times V_{m2}$$

$$V_{m2} = \frac{Q}{\pi D_2B_2}$$

Then,

$$\boldsymbol{H = \frac{U_2^2}{g} - \left(\frac{U_2}{g}\right)\left(\frac{Q}{\pi D_2B_2}\right)cot\beta_2}$$

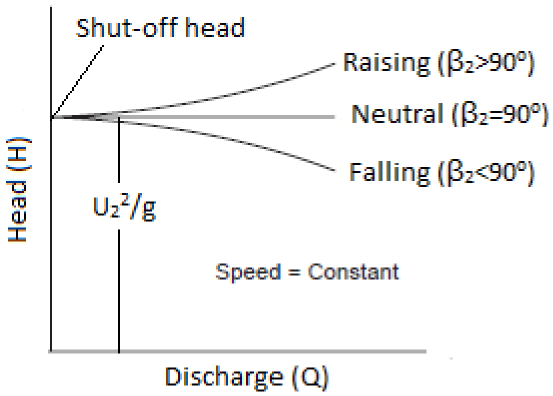


Fig. 2.21 H-Q characteristic curve for centrifugal machines

By using above equation, H-Q characteristic curve of a given impeller exit blade angle β_2 for different values of discharge is drawn in figure 2.21.

Question No 2.27: Draw the inlet and outlet velocity triangles for a radial flow power absorbing turbomachines with (i) Backward curved vane (ii) Radial vane (iii) Forward curved vane. Assume inlet whirl velocity to be zero. Draw and explain the head-capacity relations for the above 3 types of vanes. (VTU, Dec-08/Jan-09, Dec-12)

Answer: There are three types of vane shapes in centrifugal machines namely, (i) Backward curved vane (ii) Radial vane (iii) Forward curved vane.

The vane is said to be backward curved if the angle between the rotor blade-tip and the tangent to the rotor at the exit is acute ($\beta_2<90^o$). If it is a right angle ($\beta_2=90^o$) the blade said to be radial and if it is greater than 90^o, the blade is said to be forward curved. Here the blade angles measured with respect to direction of rotor (clockwise direction). The velocity triangles at the outlet of centrifugal machines are shown in figure 2.21.

The head-capacity characteristic curve for the above 3 types of vanes is given in figure 2.16, if β_2 lies between 0 to 90^o (backward curved vanes), $\cot\beta_2$ in H-Q relation is always positive. So for backward curved vanes the head developed by the machine falls with increasing discharge. For values of β_2 between 90^o to 180^o, $\cot\beta_2$ in H-Q relation is negative. So for forward curved vanes the head developed by the machine continuously rise with increasing discharge. For $\beta_2=90^o$ (radial vanes), the head is independent of flow rates and is remains constant. For centrifugal machines usually the absolute velocity at the entry has no tangential component (i.e., $V_{u1}= 0$), thus the inlet velocity triangle for all the 3 types of vanes is same.

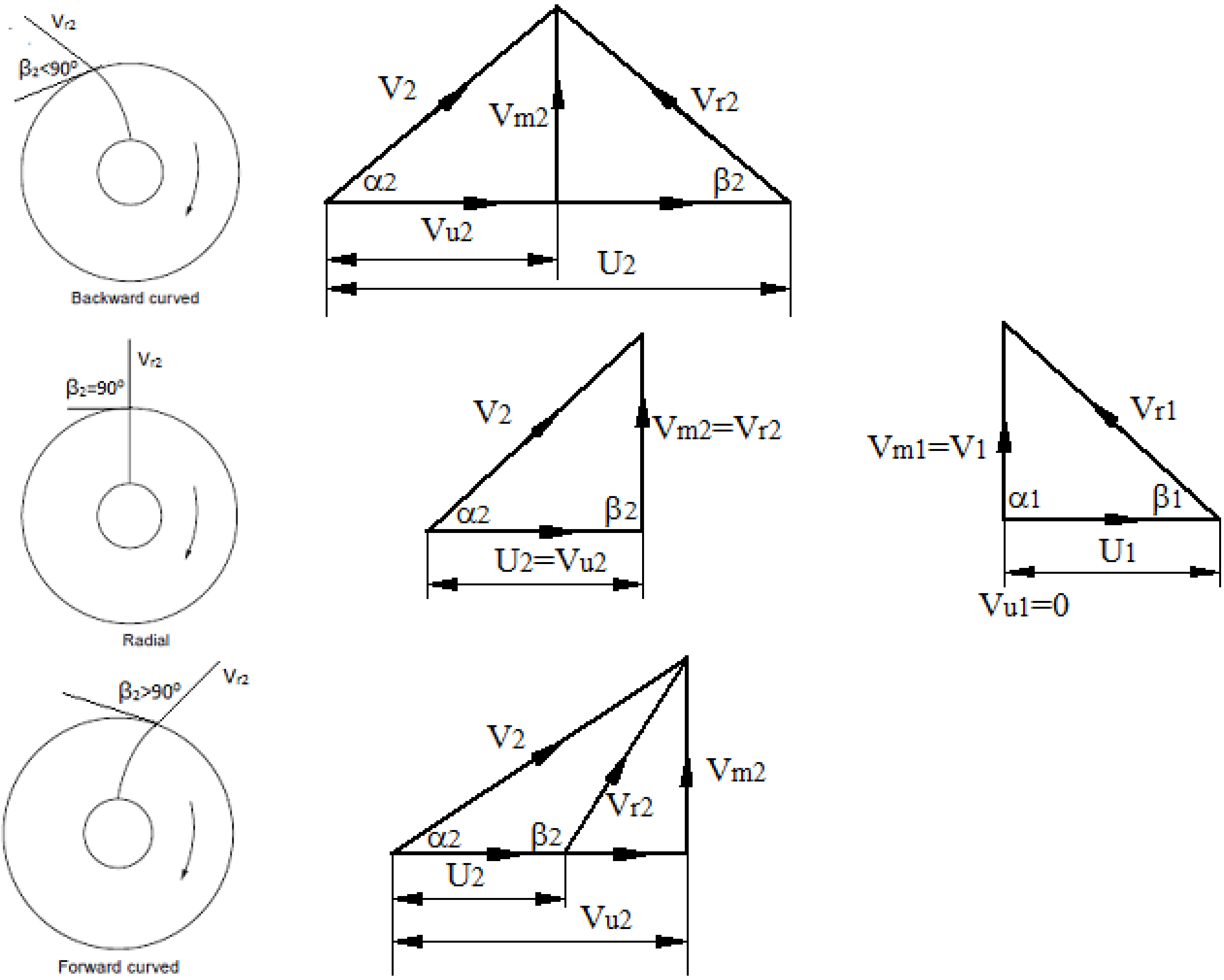


Fig. 2.21Types of centrifugal vanes

Question No 2.27: Draw the velocity diagram for a power absorbing radial flow turbomachine and show that $R = \frac{1}{2}\left[1 + \frac{V_{m2}\cot\beta_2}{U_2}\right]$***. (VTU, Dec-14/Jan-15)***