Filter Design Techniques

- Filter
  - Filter is a system that passes certain frequency components and totally rejects all others
- Stages of the design filter
  - Specification of the desired properties of the system
  - Approximation of the specification using a causal discrete-time system
  - Realization of the system

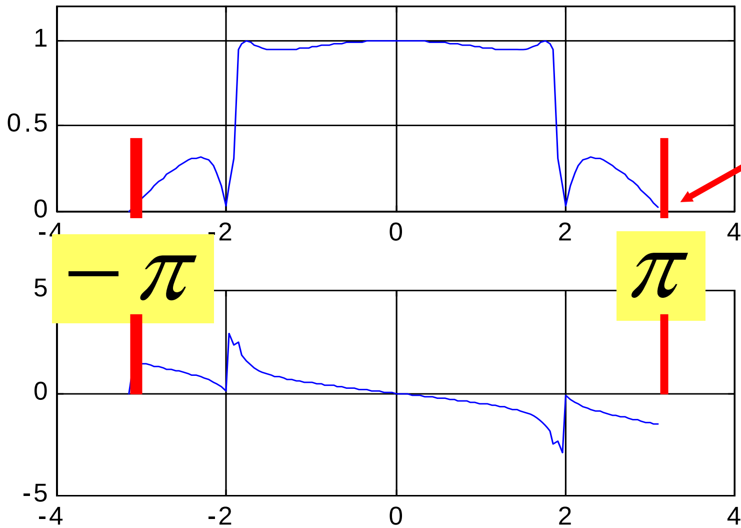
Review of discrete-time systems

Frequency response :

- periodic : period = 2π
- for a real impulse response h[k]

  Magnitude response $\left|H(e^{j\omega})\right|$ is even function

  Phase response $\angle H(e^{j\omega})$ is odd function
- example :

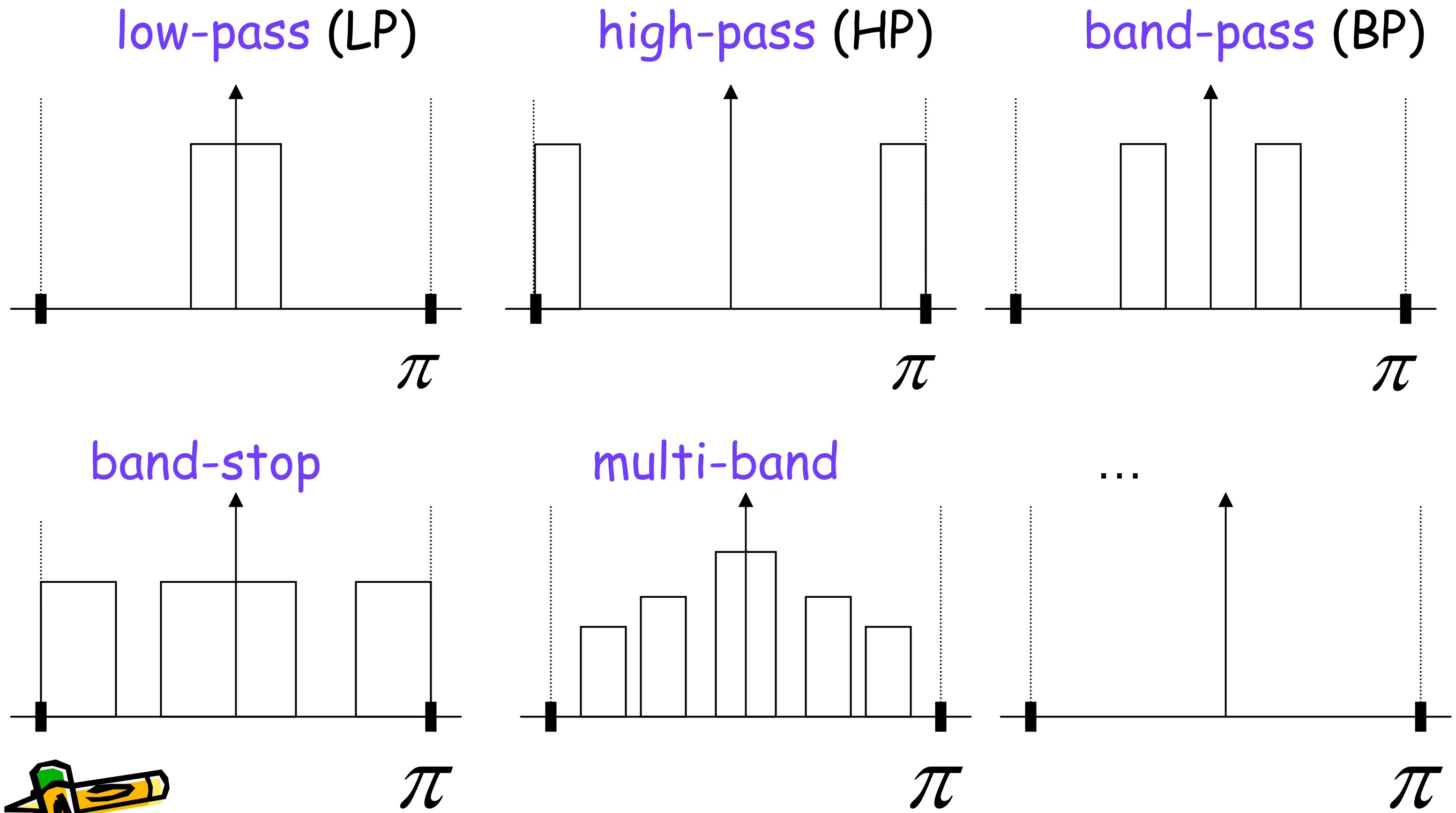


Nyquist frequency

$$e^{j\pi k} = \ldots,1,-1,1,-1,1,\ldots$$

Review of discrete-time systems

'Popular' frequency responses for filter design :



Review of discrete-time systems

"FIR filters" (finite impulse response):

$$H(z) = \frac{B(z)}{z^N} = b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}$$

- "Moving average filters" (MA filters)
- N poles at the origin z=0 (hence guaranteed stability)
- N zeros (zeros of B(z)), "all zero" filters
- corresponds to difference equation

$$y[k] = b_0 u[k] + b_1 u[k-1] + \ldots + b_N u[k-N]$$

- impulse response

$$h[0] = b_0, h[1] = b_1, \ldots, h[N] = b_N, h[N+1] = 0, \ldots$$

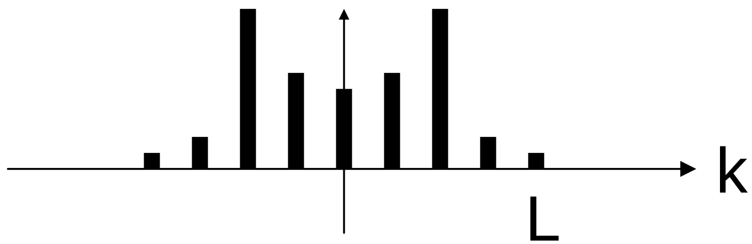
Linear Phase FIR Filters

Non-causal zero-phase filters :

example: symmetric impulse response

h[-L],....h[-1],h[0],h[1],...,h[L]

h[k]=h[-k], k=1..L



frequency response is

$$H(e^{j\omega}) = \sum_{k=-L}^{+L} h[k] e^{-j\omega.k} = ... = \sum_{k=0}^{L} a_k \cos(\omega k)$$

- i.e. real-valued (=zero-phase) transfer function
- causal implementation by introducing (group) delay

Linear Phase FIR Filters

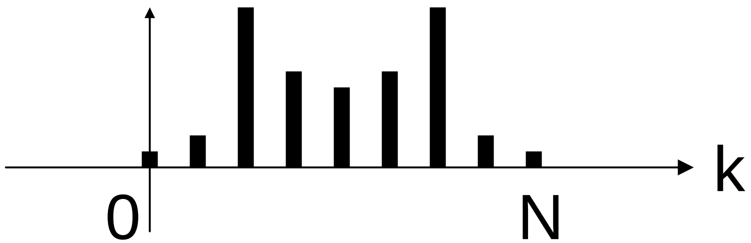
- Causal linear-phase filters = non-causal zero-phase + delay

example: symmetric impulse response & N even

$h[0], h[1], \ldots, h[N]$

$N=2L$ (even)

$h[k]=h[N-k]$, $k=0..L$



frequency response is

$$H(e^{j\omega}) = \sum_{k=0}^{N} h[k] e^{-j\omega.k} = \ldots = e^{-j\omega L} \sum_{k=0}^{L} a_k \cos(\omega k)$$

= i.e. causal implementation of zero-phase filter, by introducing (group) delay $\left. z^{-L} \right|_{z=e^{j\omega}} = e^{-j\omega L}$

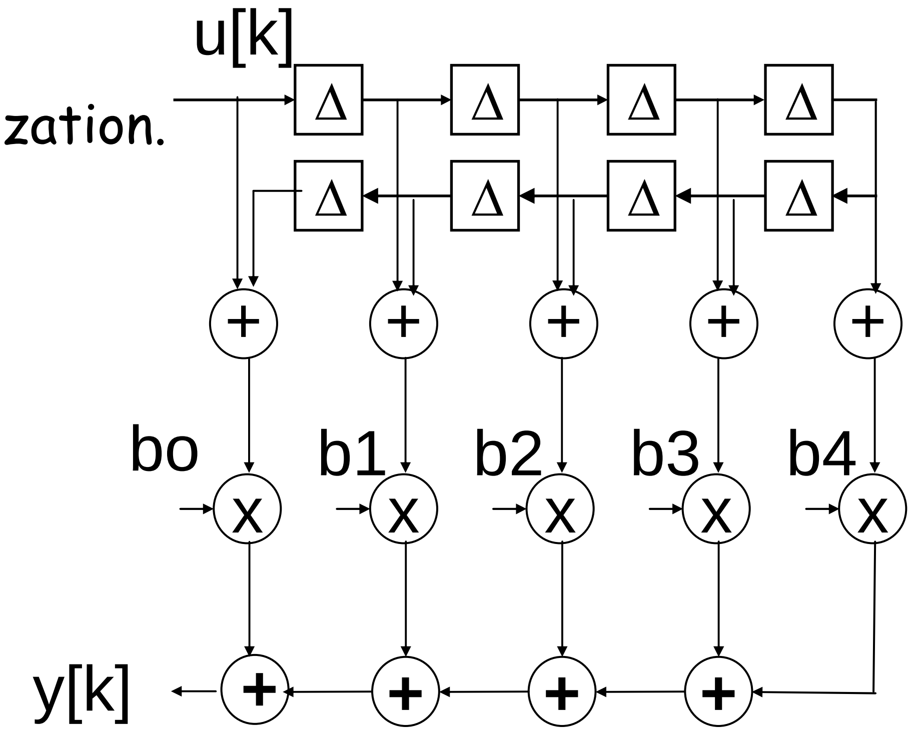
Linear Phase FIR Filters

Type-1	Type-2	Type-3	Type-4
N=2L=even	N=2L+1=odd	N=2L=even	N=2L+1=odd
symmetric	symmetric	anti-symmetric	anti-symmetric
h[k]=h[N-k]	h[k]=h[N-k]	h[k]=-h[N-k]	h[k]=-h[N-k]
$e^{-j\omega N/2}\sum_{k=0}^{L} a_k \cos(\omega k)$	$e^{-j\omega N/2}\cos(\frac{\omega}{2})\sum_{k=0}^{L} a_k \cos(\omega k)$	$je^{-j\omega N/2}\sin(\omega)\sum_{k=0}^{L-1} a_k \cos(\omega k)$	$j.e^{-j\omega N/2}\sin(\frac{\omega}{2})\sum_{k=0}^{L} a_k \cos(\omega k)$
	zero at $\omega = \pi$	zero at $\omega = 0, \pi$	zero at $\omega = 0$
LP/HP/BP	LP/BP		HP

Linear Phase FIR Filters

- efficient direct-form realization.

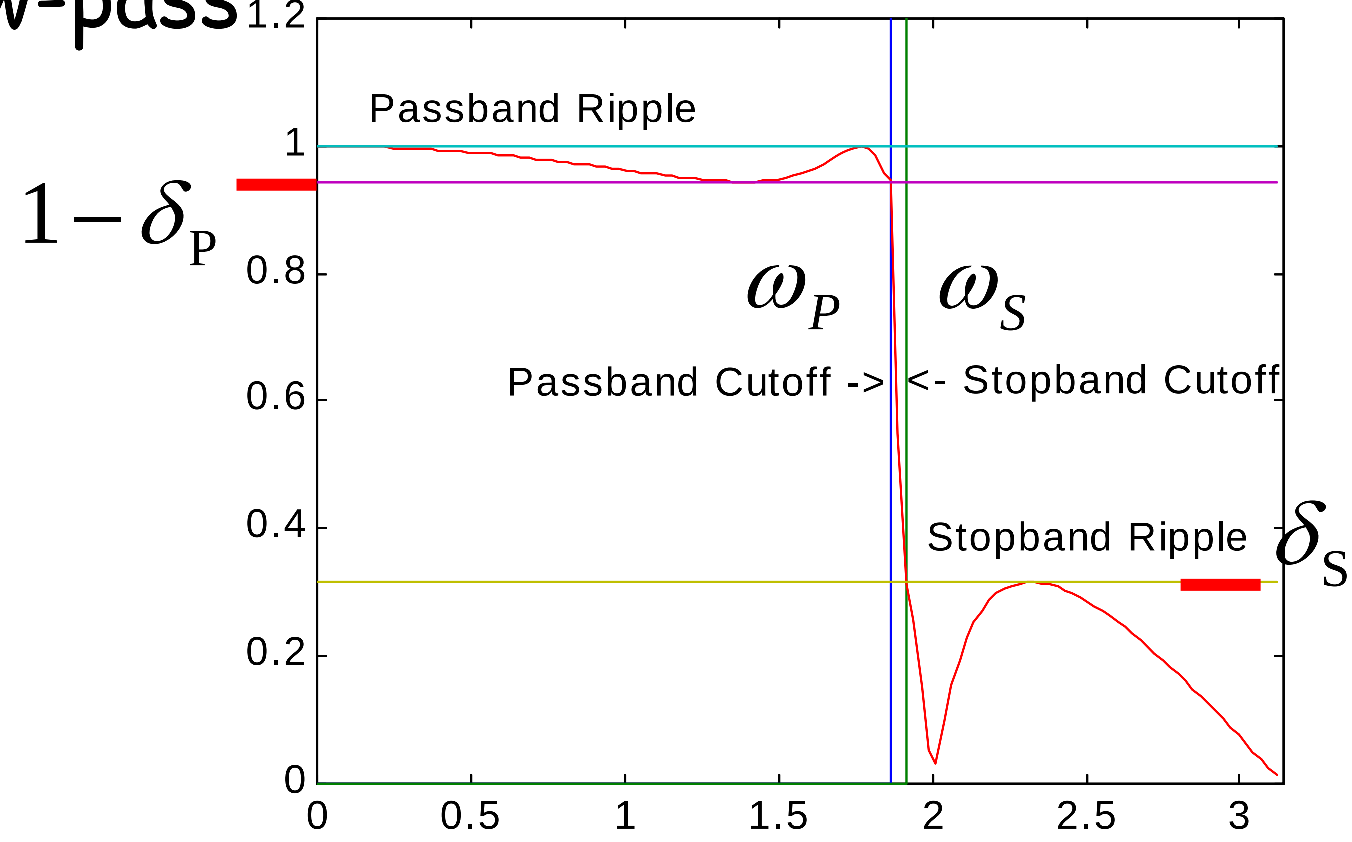
  <u>*example:*</u>



- PS: IIR filters can NEVER have linear-phase property !

Filter Specification

Ex: Low-pass



Filter Design Problem

- Design of filters is a problem of function approximation

- For FIR filter, it implies polynomial approximation
- For IIR filter, it implies approximation by a rational function of z

Filter Design by Optimization

(I) Weighted Least Squares Design :

- select one of the basic forms that yield linear phase

  e.g. Type-1 $$H(e^{j\omega}) = e^{-j\omega N/2} \sum_{k=0}^{L} a_k \cos(\omega k) = e^{-j\omega N/2} A(\omega)$$

- specify desired frequency response (LP,HP,BP,…)

$$H_d(\omega) = e^{-j\omega N/2} A_d(\omega)$$

- optimization criterion is

$$\min_{a_0,\ldots,a_L} \int_{-\pi}^{+\pi} W(\omega) \left| H(e^{j\omega}) - H_d(\omega) \right|^2 d\omega = \min_{a_0,\ldots,a_L} \underbrace{\int_{-\pi}^{+\pi} W(\omega) \left| A(\omega) - A_d(\omega) \right|^2 d\omega}_{F(a_0,\ldots,a_L)}$$

where $W(\omega) \geq 0$ is a weighting function

Filter Design by Optimization

- …this is equivalent to

$$\min_x \{\overbrace{x^T Q x - 2x^T p + \mu}^{F(a_0,\dots,a_L)}\}$$

$$x^T = \begin{bmatrix} a_0 & a_1 & \dots & a_L \end{bmatrix}$$

$$Q = \int_0^\pi W(\omega) c(\omega) c^T(\omega) d\omega$$

$$p = \int_0^\pi W(\omega) A_d(\omega) c(\omega) d\omega$$

$$c^T(\omega) = \begin{bmatrix} 1 & \cos(\omega) & \dots & \cos(L\omega) \end{bmatrix}$$

$$\mu = \dots$$

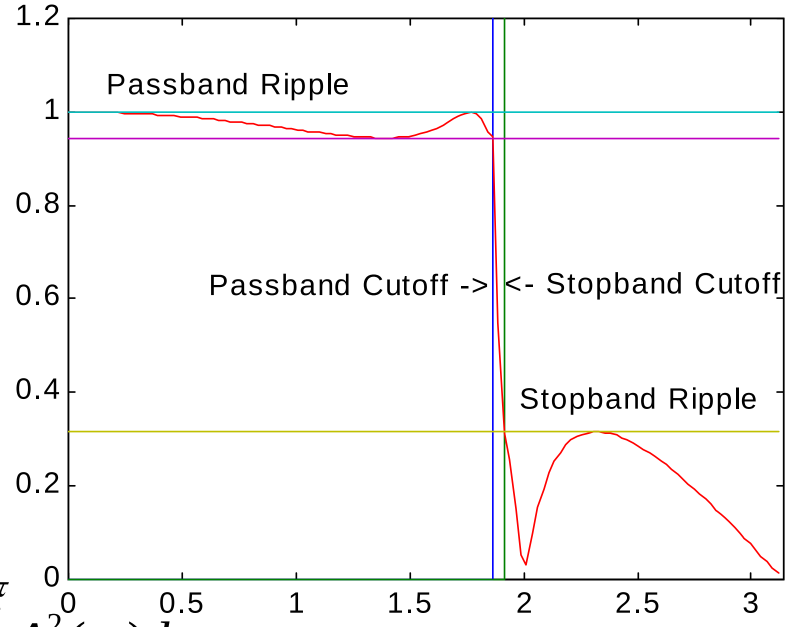
=standard 'Quadratic Optimization' problem

$$x_{OPT} = Q^{-1} p$$

Filter Design by Optimization

- Example: Low-pass design

$$A_d(\omega) = 1, |\omega| < \omega_P \qquad \text{(pass-band)}$$
$$A_d(\omega) = 0, \omega_S \le |\omega| \le \pi \qquad \text{(stop-band)}$$



optimization function is

$$F(a_0,\ldots,a_L) = \underbrace{\int_0^{\omega_P} |A(\omega)-1|^2 d\omega}_{\text{pass-band}} + \gamma.\underbrace{\int_{\omega_S}^{+\pi} A^2(\omega) d\omega}_{\text{stop-band}} = \ldots$$

i.e.

$$W(\omega) = \ldots$$

Filter Design by Optimization

- a simpler problem is obtained by replacing the F(..) by…

$$\underline{F}(a_0,\ldots,a_L) = \sum_i W(\omega_i)\left|A(\omega_i) - A_d(\omega_i)\right|^2 = \sum_i W(\omega_i)\left\{c^T(\omega_i)\begin{bmatrix} a_0 \\ : \\ a_L \end{bmatrix} - A_d(\omega_i)\right\}^2$$

where the w_i's are a set of n sample frequencies

The quadratic optimization problem is then equivalent to a least-squares problem

$$\min_x \left\|\underline{A}x - \underline{b}\right\|_2^2 = \min_x \{x^T \underbrace{\underline{A}^T\underline{A}}_{\sum_i W(\omega_i)c(\omega_i)c^T(\omega_i)} x - 2x^T \underbrace{\underline{A}^T\underline{b}}_{\sum_i \ldots} + \underbrace{\underline{b}^T\underline{b}}_{\sum_i \ldots}\}$$

$$x_{LS} = (\underline{A}^T\underline{A})^{-1}\underline{A}^T\underline{b}$$

Compare to p.12

+++ : simple

--- : unpredictable behavior in between sample frequencies.

Filter Design by Optimization

- ...then all this is often supplemented with
  additional constraints

Example: Low-pass (LP) design (continued)

pass-band ripple control :

$$\left|A(\omega)-1\right| \le \delta_{\mathrm{P}}, \left|\omega\right| < \omega_P \quad (\delta_{\mathrm{P}} \text{ is pass-band ripple})$$

stop-band ripple control :

$$\left|A(\omega)\right| \le \delta_{\mathrm{S}}, \omega_S \le \left|\omega\right| \le \pi \quad (\delta_{\mathrm{S}} \text{ is stop-band ripple})$$

Filter Design by Optimization

Example: Low-pass (LP) design (continued)

a realistic way to implement these constraints, is to impose the constraints (only) on a set of sample frequencies

$\omega_{P1}, \omega_{P2}, \ldots, \omega_{Pm}$ in the pass-band

and $\omega_{S1}, \omega_{S2}, \ldots, \omega_{Sn}$ in the stop-band

The resulting optimization problem is :

minimize : $F(a_0, \ldots, a_L) = \ldots$

$$x^T = \begin{bmatrix} a_0 & a_1 & \ldots & a_L \end{bmatrix}$$

subject to $A_P x \leq b_P$ (pass-band constraints)

$A_S x \leq b_S$ (stop-band constraints)

= \`Quadratic Linear Programming' problem

Filter Design by Optimization

(II) `Minimax' Design :

- select one of the basic forms that yield linear phase

  e.g. Type-1 $$H(e^{j\omega}) = e^{-j\omega N/2}\sum_{k=0}^{L} a[k]\cos(\omega k) = e^{-j\omega N/2}A(\omega)$$

- specify desired frequency response (LP,HP,BP,…)

$$H_d(\omega) = e^{-j\omega N/2}A_d(\omega)$$

- optimization criterion is

$$\min_{a_0,\ldots,a_L}\ \max_{0\le\omega\le\pi} W(\omega)\left|H(e^{j\omega}) - H_d(\omega)\right| = \min_{a_0,\ldots,a_L}\ \max_{0\le\omega\le\pi} W(\omega)\left|A(\omega) - A_d(\omega)\right|$$

where $W(\omega)\ge 0$ is a weighting function

Filter Design by Optimization

- Conclusion:

  (I) weighted least squares design

  (II) minimax design

  provide general `framework', procedures to translate filter design problems into standard optimization problems

- In practice (and in textbooks):

  emphasis on specific (ad-hoc) procedures :

  - filter design based on 'windows'

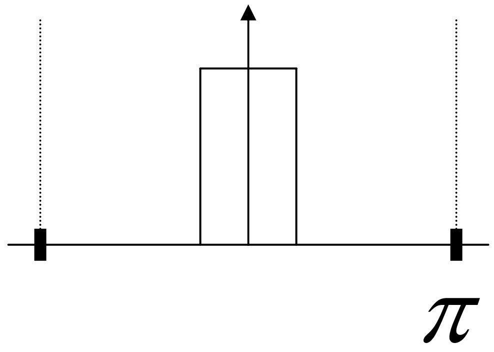
  - equi-ripple design

Filter Design using 'Windows'

Example : Low-pass filter design

- ideal low-pass filter is

$$H_d(\omega) = \begin{cases} 1 & |\omega| < \omega_C \\ 0 & \omega_C < |\omega| < \pi \end{cases}$$



- hence ideal time-domain impulse response is

$$h_d[k] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H_d(e^{j\omega}) e^{j\omega.k} d\omega = \ldots = \alpha \frac{\sin(\omega_c k)}{\omega_c k}$$

Non-causal and infinitely long

- truncate $h_d[k]$ to N+1 samples :

$$h[k] = \begin{cases} h_d[k] & -N/2 < k < N/2 \\ 0 & \text{otherwise} \end{cases}$$

- add (group) delay to turn into causal filter

Filter Design using 'Windows'

Example : Low-pass filter design (continued)

- note : it can be shown that time-domain truncation corresponds to solving a weighted least-squares optimization problem with the given H_d, and weighting function $W(\omega)=1$

- truncation corresponds to applying a 'rectangular window' :

$$h[k]=h_d[k]w[k]$$

$$w[k]=\begin{cases}1 & -N/2<k<N/2\\ 0 & \text{otherwise}\end{cases}$$

- +++: simple procedure (also for HP,BP,…)
- --- : truncation in the time-domain results in 'Gibbs effect' in the frequency domain, i.e. large ripple in pass-band and stop-band, which cannot be reduced by increasing the filter order N.

Filter Design using 'Windows'

Remedy : apply windows other than rectangular window:

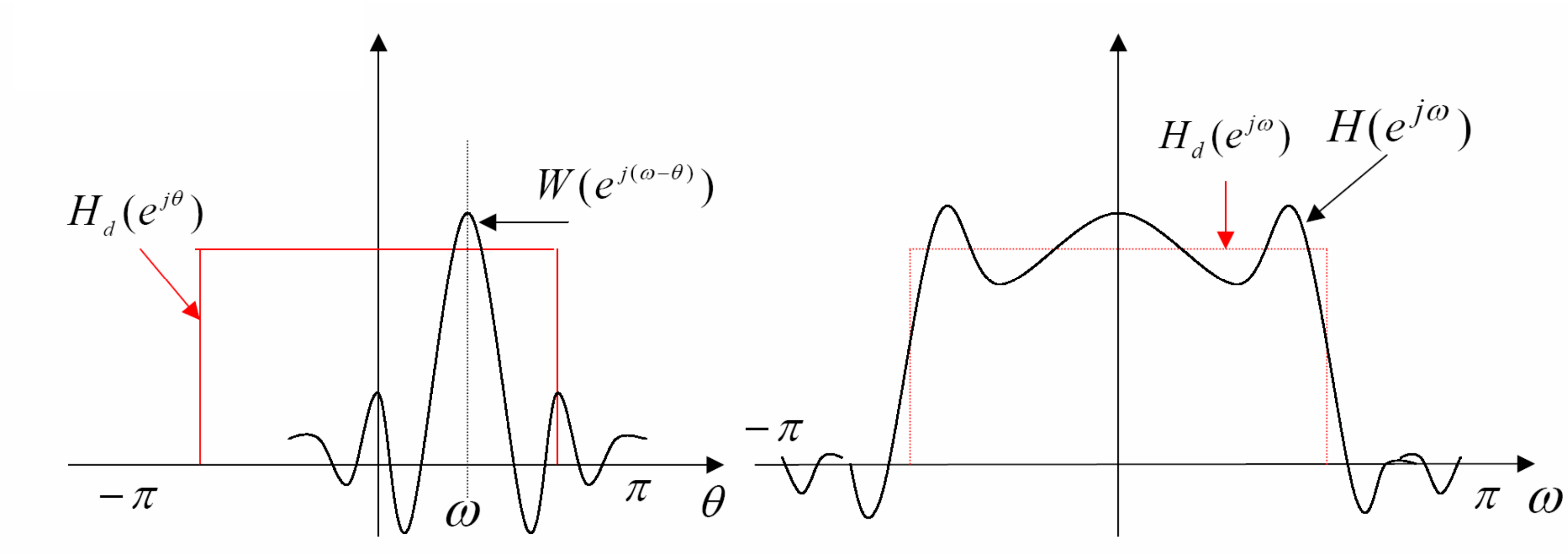
- time-domain multiplication with a window function w[k] corresponds to frequency domain convolution with W(z) :

$$h[k] = h_d[k]w[k]$$

$$H(z) = H_d(z) * W(z)$$

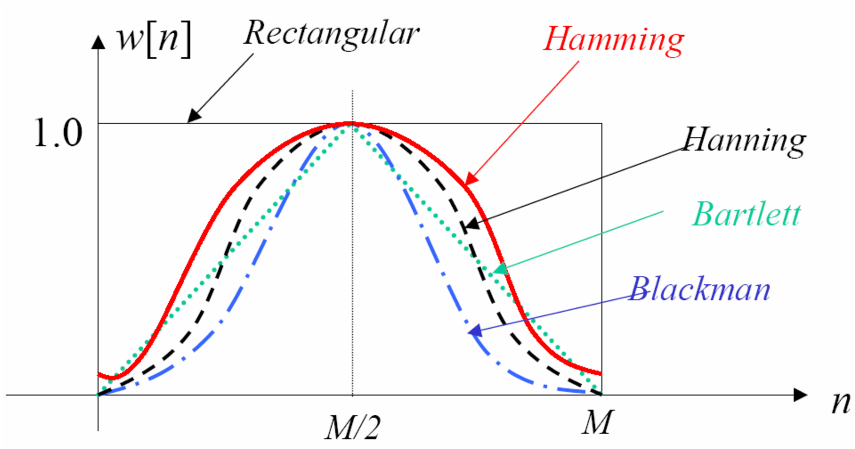
- candidate windows : Han, Hamming, Blackman, Kaiser,…. (see textbooks)
- window choice/design = trade-off between side-lobe levels (define peak pass-/stop-band ripple) and width main-lobe (defines transition bandwidth)

Windowing Effect

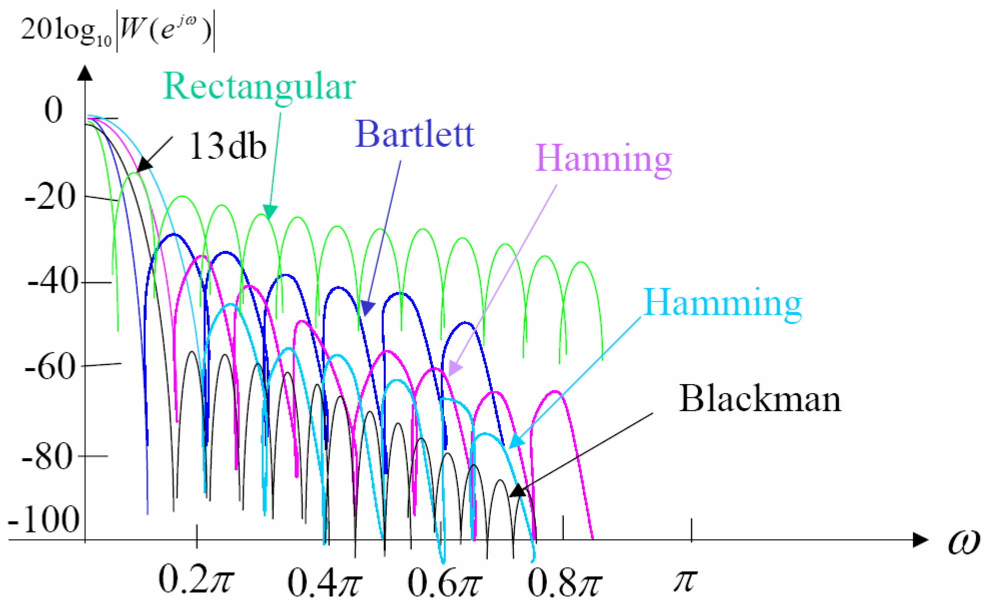


Gibbs phenomenon

Windowing



w[n]
Rectangular
Hamming
1.0
Hanning
Bartlett
Blackman
M/2
M
n



$20\log_{10}|W(e^{j\omega})|$
Rectangular
0
13db
Bartlett
Hanning
-20
-40
Hamming
-60
Blackman
-80
-100
0.2π
0.4π
0.6π
0.8π
π
ω

Equiripple Design

- Starting point is minimax criterion, e.g.

$$\min_{a_0,\ldots,a_L} \max_{0\le\omega\le\pi} W(\omega)\left|A(\omega)-A_d(\omega)\right| = \min_{a_0,\ldots,a_L} \max_{0\le\omega\le\pi} \left|E(\omega)\right|$$

- Based on theory of Chebyshev approximation and the 'alternation theorem', which (roughly) states that the optimal a_i's are such that the 'max' (maximum weighted approximation error) is obtained at L+2 extremal frequencies…

$$\max_{0\le\omega\le\pi} \left|E(\omega)\right| = \left|E(\omega_i)\right| \quad \text{for } i=1,..,L+2$$

  …that hence will exhibit the same maximum ripple ('equiripple')

- Iterative procedure for computing extremal frequencies, etc. (Remez exchange algorithm, Parks-McClellan algorithm)
- Very flexible, etc., available in many software packages
- Details omitted here (see textbooks)

Software

- FIR Filter design abundantly available in commercial software
- Matlab:

b=fir1(n,Wn,type,window), windowed linear-phase FIR design, n is filter order, Wn defines band-edges, type is `high',`stop',…

b=fir2(n,f,m,window), windowed FIR design based on inverse Fourier transform with frequency points f and corresponding magnitude response m

b=remez(n,f,m), equiripple linear-phase FIR design with Parks-McClellan (Remez exchange) algorithm