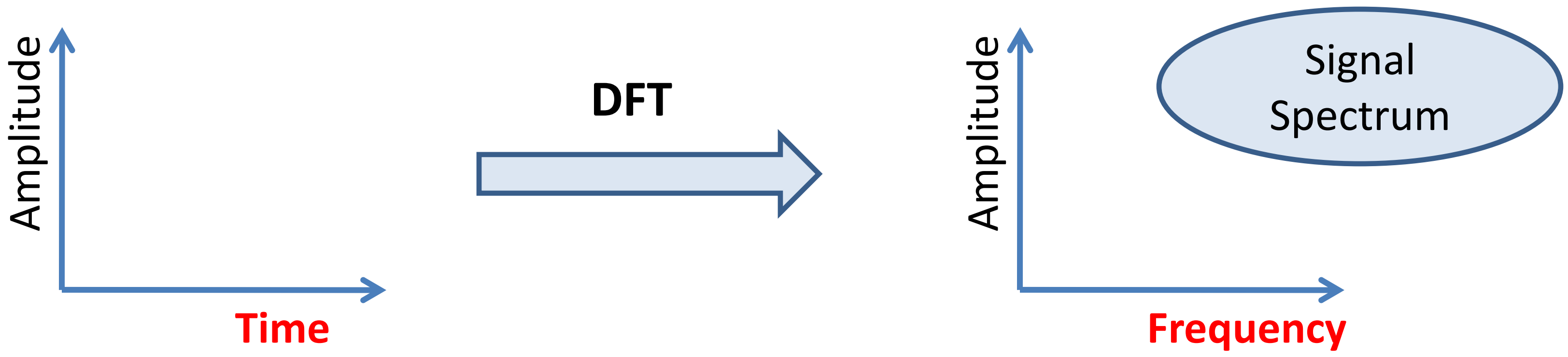
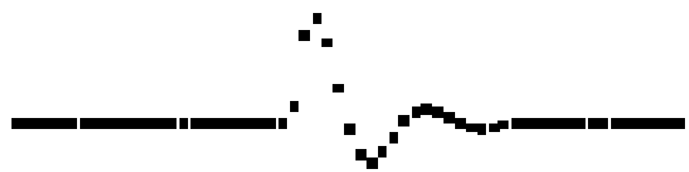
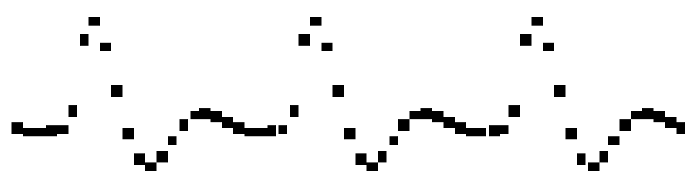
Discrete Fourier Transform (DFT)

DFT transforms the time domain signal samples to the frequency domain components.

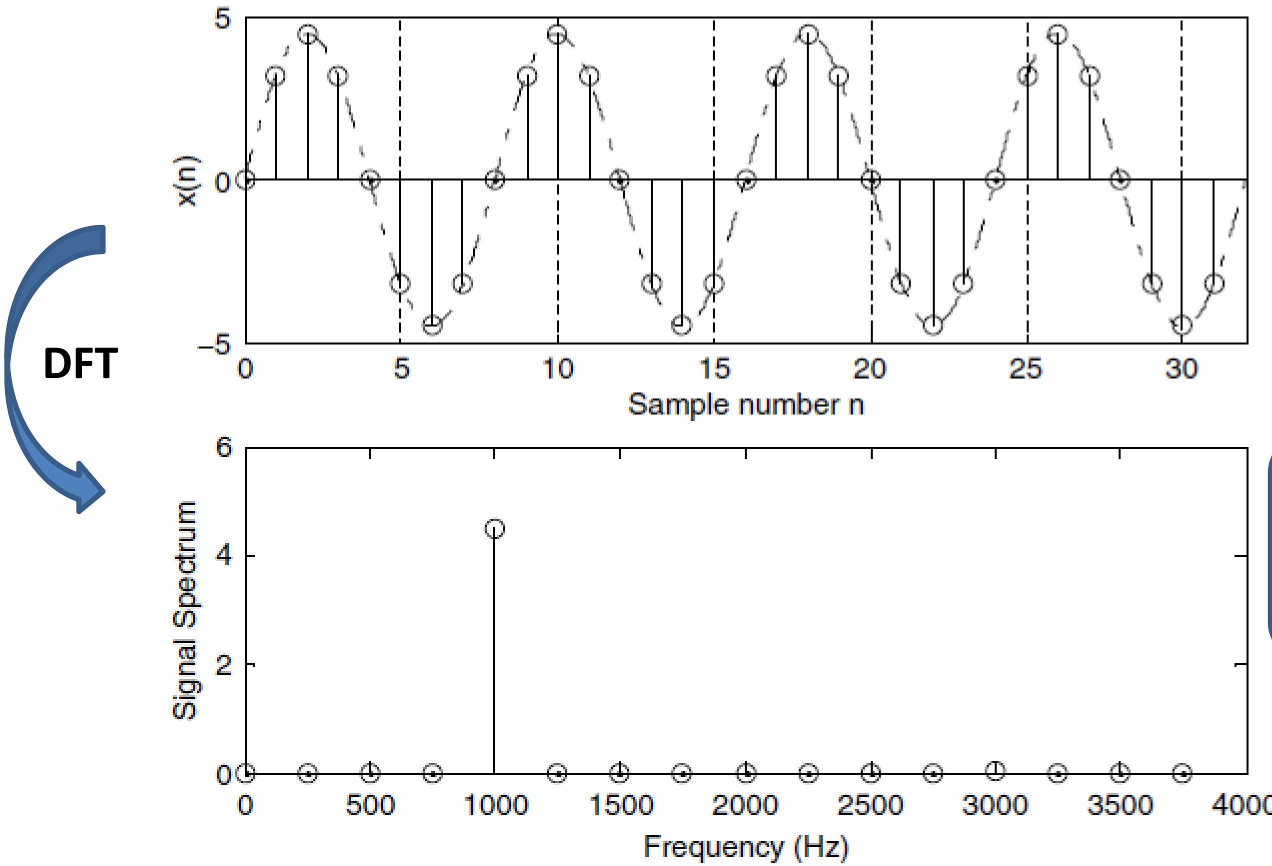


DFT is often used to do frequency analysis of a time domain signal.

Four Types of Fourier Transform

Type of Transform	Example Signal
Fourier Transform *signals that are continuous and aperiodic*	
Fourier Series *signals that are continuous and periodic*	
Discrete Time Fourier Transform *signals that are discrete and aperiodic*	
Discrete Fourier Transform *signals that are discrete and periodic*	

DFT: Graphical Example



1000 Hz sinusoid with 32 samples at 8000 Hz sampling rate.

Sampling rate

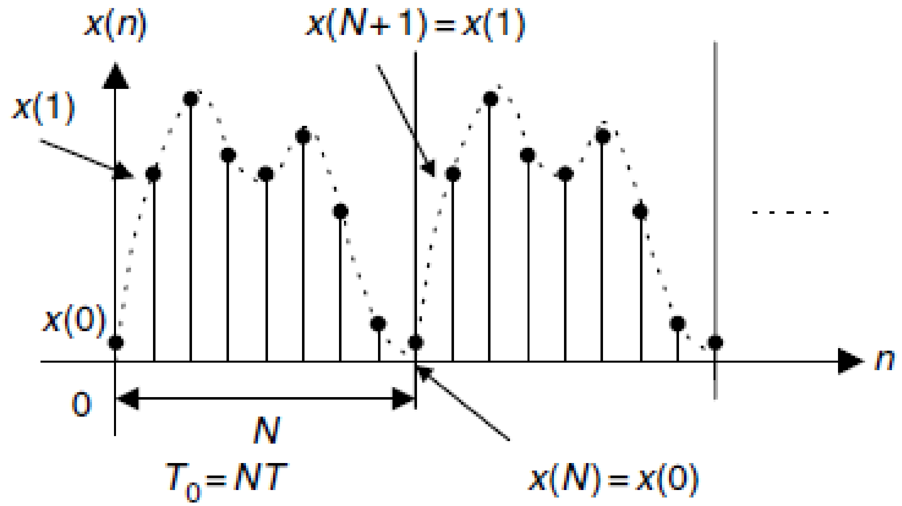
8000 samples = 1 second
32 samples = 32/8000 sec
= 4 millisecond

Frequency

1 second = 1000 cycles
32/8000 sec =
(1000*32/8000=) 4 cycles

DFT Coefficients of Periodic Signals

Periodic Digital Signal



Equation of DFT coefficients: $$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad -\infty < k < \infty$$

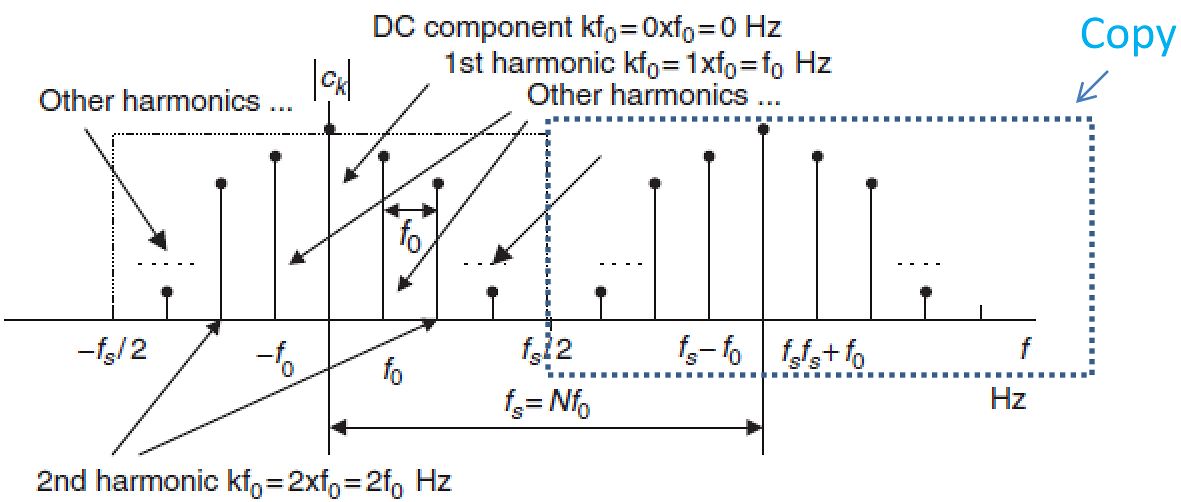
DFT Coefficients of Periodic Signals

Fourier series coefficient c_k is periodic of N

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k+N)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}e^{-j2\pi n}$$

$$\text{Since } e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1, \quad c_{k+N} = c_k.$$

Amplitude spectrum of the periodic digital signal



Example 1

The periodic signal: $\boxed{x(t) = \sin(2\pi t)}$ is sampled at $f_s = 4\,\text{Hz}$

a. Compute the spectrum c_k using the samples in one period.

b. Plot the two-sided amplitude spectrum $|c_k|$ over the range from -2 to 2 Hz.

Solution:

Fundamental frequency

a. We match $x(t) = \sin(2\pi t)$ with $x(t) = \sin(2\pi f t)$ and get ***f* = 1 Hz**.

Therefore the signal has 1 cycle or 1 period in 1 second.

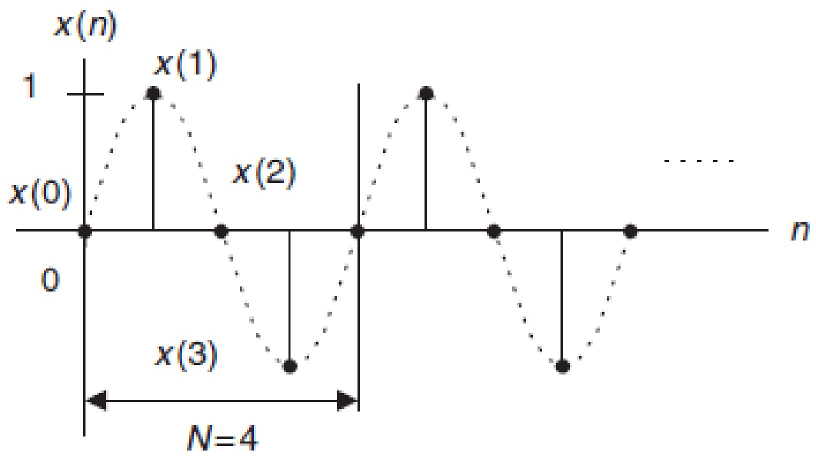
Sampling rate f_s = 4 Hz $\Rightarrow$ 1 second has 4 samples.

Hence, there are 4 samples in 1 period for this particular signal.

$T = 1/f_s = 0.25 \xrightarrow{\text{Sampled signal}} x(n) = x(nT) = \sin(2\pi nT) = \sin(0.5\pi n)$

Example 1 - contd. (1)

$x(0) = 0;\ x(1) = 1;\ x(2) = 0;\ \text{and}\ x(3) = -1$



b.

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad -\infty < k < \infty$$

$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x(n) = \frac{1}{4}(x(0) + x(1) + x(2) + x(3)) = \frac{1}{4}(0 + 1 + 0 - 1) = 0$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi\times 1n/4} = \frac{1}{4}\left(x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}\right)$$

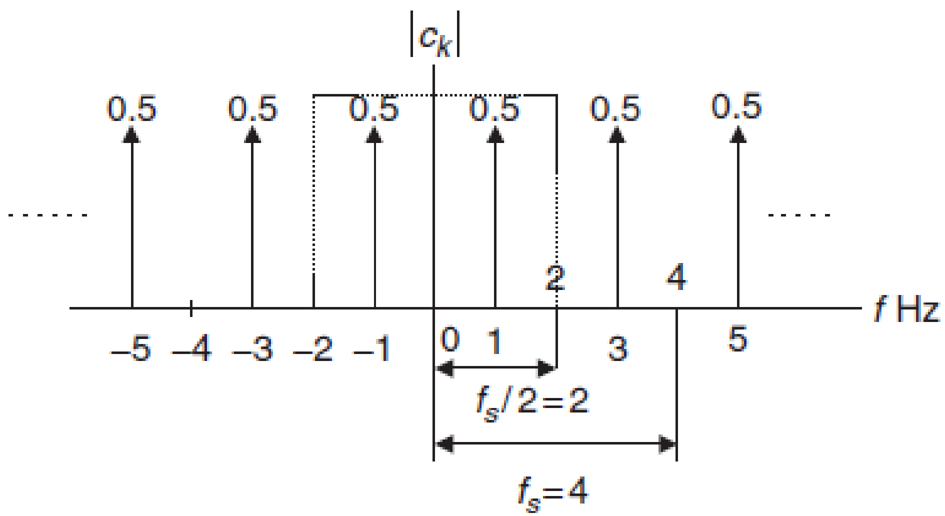
$$= \frac{1}{4}(x(0) - jx(1) - x(2) + jx(3) = 0 - j(1) - 0 + j(-1)) = -j0.5.$$

Example 1 - contd. (2)

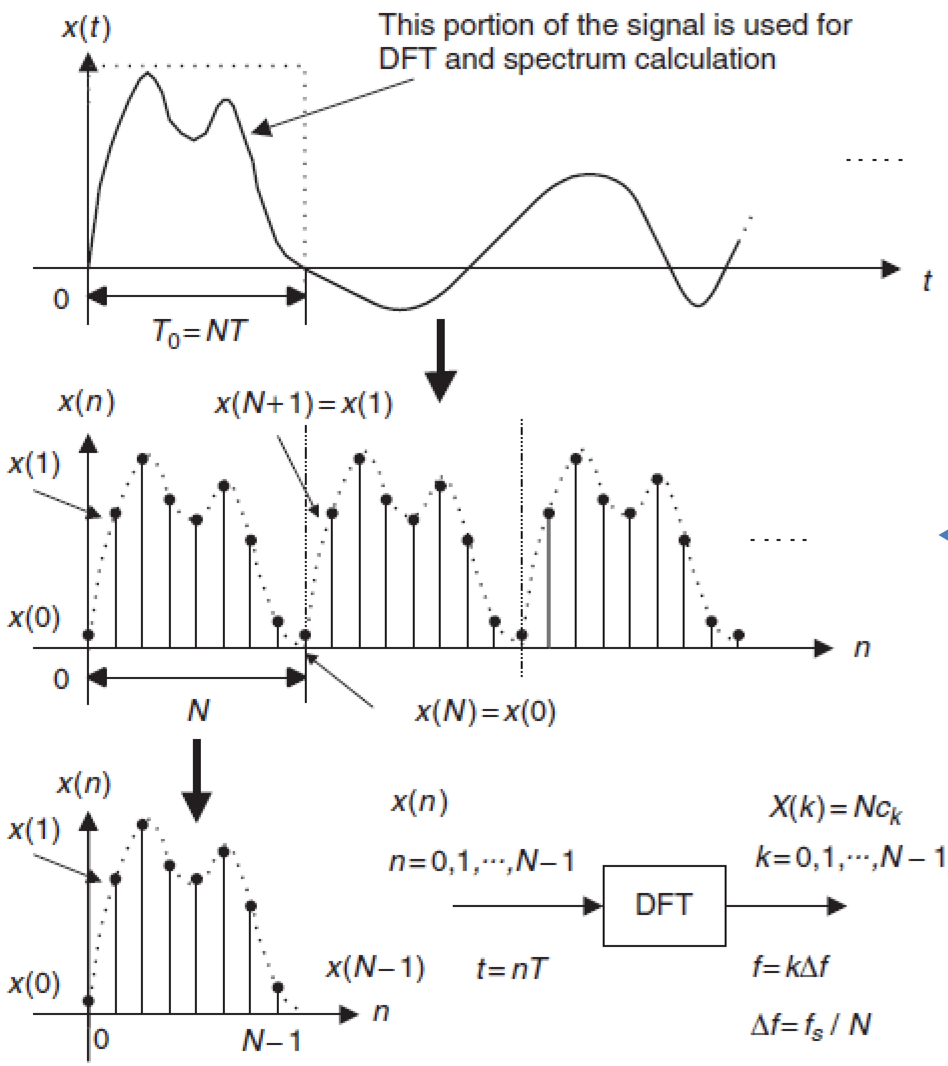
$$c_2 = \frac{1}{4}\sum_{k=0}^{3} x(n)e^{-j2\pi\times 2n/4} = 0, \text{ and } c_3 = \frac{1}{4}\sum_{n=0}^{3} x(k)e^{-j2\pi\times 3n/4} = j0.5.$$

Using periodicity, it follows that

$$c_{-1} = c_3 = j0.5, \text{ and } c_{-2} = c_2 = 0.$$



On the Way to DFT Formulas



Imagine periodicity of *N* samples.

Take first *N* samples (index 0 to *N* -1) as the input to DFT.

DFT Formulas

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{kn}, \text{ for } k = 0,\ 1, \ldots,\ N-1.$$

$$X(k) = x(0)\, W_N^{k0} + x(1)\, W_N^{k1} + x(2) W_N^{k2} + \ldots + x(N-1) W_N^{k(N-1)}$$

Where, $W_N = e^{-j2\pi/N} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right).$

Inverse DFT:

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}, \text{ for } n = 0{,}1, \ldots,\ N-1$$

MATLAB Functions

FFT: Fast Fourier Transform

MATLAB FFT functions.

X = fft(x)	% Calculate DFT coefficients
x = ifft(X)	% Inverse DFT
x = input vector	
X = DFT coefficient vector	

Example 2

Given a sequence $x(n)$ for $0 \le n \le 3$, where $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$,

a. Evaluate its DFT $X(k)$.

Solution:

$$N = 4 \text{ and } W_4 = e^{-j\frac{\pi}{2}} \quad \Rightarrow \quad X(k) = \sum_{n=0}^{3} x(n) W_4^{kn} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi kn}{2}}$$

Thus, for $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n) e^{-j0} = x(0)e^{-j0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0}$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 2 + 3 + 4 = 10$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}}$$

$$= x(0) - jx(1) - x(2) + jx(3)$$

$$= 1 - j2 - 3 + j4 = -2 + j2$$

Example 2 – contd.

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$
$$= x(0) - x(1) + x(2) - x(3)$$
$$= 1 - 2 + 3 - 4 = -2$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j\frac{3\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}}$$
$$= x(0) + jx(1) - x(2) - jx(3)$$
$$= 1 + j2 - 3 - j4 = -2 - j2$$

Using MATLAB,

$$\gg \mathrm{X} = \mathrm{fft}([1\ \ 2\ \ 3\ \ 4])$$
$$\mathrm{X} = 10.0000 \quad -2.0000 + 2.0000\mathrm{i} \quad -2.0000 \quad -2.0000 - 2.0000\mathrm{i}$$

Example 3

Inverse DFT of the previous example.

$N = 4$ and $W_4^{-1} = e^{j\frac{\pi}{2}}$ $\longrightarrow$ $x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)W_4^{-nk} = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{\pi kn}{2}}.$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j0} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j0} + X(2)e^{j0} + X(3)e^{j0}\right)$$
$$= \frac{1}{4}(10 + (-2 + j2) - 2 + (-2 - j2)) = 1$$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}}\right)$$
$$= \frac{1}{4}(X(0) + jX(1) - X(2) - jX(3))$$
$$= \frac{1}{4}(10 + j(-2 + j2) - (-2) - j(-2 - j2)) = 2$$

Example 3 – contd.

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{jk\pi} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right)$$

$$= \frac{1}{4}(X(0) - X(1) + X(2) - X(3))$$

$$= \frac{1}{4}(10 - (-2 + j2) + (-2) - (-2 - j2)) = 3$$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi 3}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}}\right)$$

$$= \frac{1}{4}(X(0) - jX(1) - X(2) + jX(3))$$

$$= \frac{1}{4}(10 - j(-2 + j2) - (-2) + j(-2 - j2)) = 4$$

Using MATLAB,

$$\gg \mathrm{x} = \mathrm{ifft}([10 \quad -2 + 2j \quad -2 \quad -2 - 2j])$$

$$\mathrm{x} = 1 \qquad 2 \qquad 3 \qquad 4.$$

Relationship Between Frequency Bin *k* and Its Associated Frequency in Hz

$$\boxed{f = \frac{kf_s}{N} \text{ (Hz)}}$$

Frequency step or frequency resolution: $\Delta f = \frac{f_s}{N} \text{ (Hz)}$

Example 4

In the previous example, if the sampling rate is 10 Hz,

a. Determine the sampling period, time index, and sampling time instant for a digital sample $x(3)$ in time domain.

b. Determine the frequency resolution, frequency bin number, and mapped frequency for each of the DFT coefficients $X(1)$ and $X(3)$ in frequency domain.

Example 4 - contd.

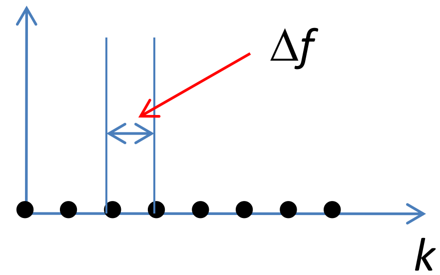
a.

Sampling period: $T = 1/f_s = 1/10 = 0.1$ second

For x(3), time index is n = 3, and sampling time instant is $t = nT = 3 \cdot 0.1 = 0.3$ second.

b.

Frequency resolution: $\Delta f = \frac{f_s}{N} = \frac{10}{4} = 2.5\,\text{Hz}.$



Frequency bin number for X(1) is k = 1, and its corresponding frequency is $f = \frac{kf_s}{N} = \frac{1 \times 10}{4} = 2.5\,\text{Hz}.$

Similarly, for X(3) is k = 3, and its corresponding frequency is $f = \frac{kf_s}{N} = \frac{3 \times 10}{4} = 7.5\,\text{Hz}.$

Amplitude and Power Spectrum

Since each calculated DFT coefficient is a complex number, it is not convenient to plot it versus its frequency index

Amplitude Spectrum:

$$A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{(\mathrm{Real}[X(k)])^2 + (\mathrm{Imag}[X(k)])^2},$$
$$k = 0, 1, 2, \ldots, N-1.$$

To find one-sided amplitude spectrum, we double the amplitude.

$$\bar{A}_k = \begin{cases} \frac{1}{N}|X(0)|, & k = 0 \\ \frac{2}{N}|X(k)|, & k = 1, \ldots, N/2 \end{cases}.$$

Amplitude and Power Spectrum –contd.

Power Spectrum:

$$P_k = \frac{1}{N^2}|X(k)|^2 = \frac{1}{N^2}\left\{(\text{Real}[X(k)])^2 + (\text{Imag}[X(k)])^2\right\},$$
$$k = 0,\ 1,\ 2, \ldots,\ N-1.$$

For, one-sided power spectrum:

$$\bar{P}_k = \begin{cases} \frac{1}{N^2}|X(0)|^2 & k = 0 \\ \frac{2}{N^2}|X(k)|^2 & k = 0,\ 1, \ldots,\ N/2 \end{cases}$$

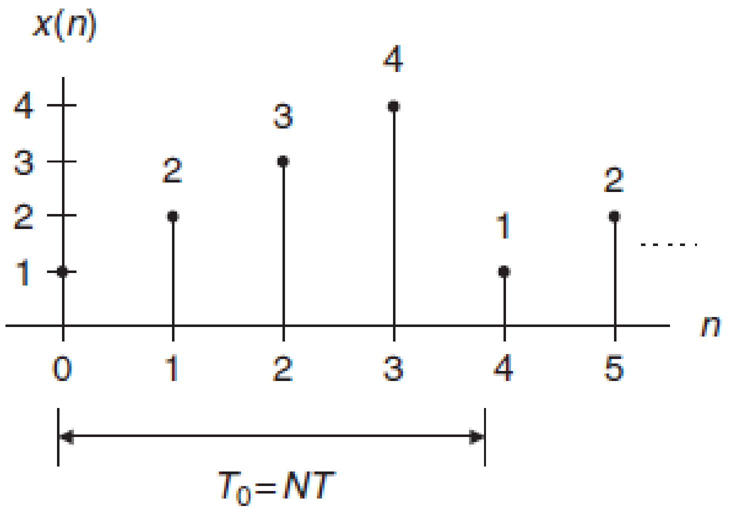
Phase Spectrum:

$$\varphi_k = \tan^{-1}\left(\frac{\text{Imag}[X(k)]}{\text{Real}[X(k)]}\right),\ k = 0,\ 1,\ 2, \ldots,\ N-1.$$

Example 5

Assuming that $f_s = 100\,\text{Hz}$,

a. Compute the amplitude spectrum, phase spectrum, and power spectrum.



Solution:

$$X(0) = 10$$
$$X(1) = -2 + j2$$
$$X(2) = -2$$
$$X(3) = -2 - j2.$$

See Example 2.

For $k = 0$, $f = k \cdot f_s / N = 0 \times 100/4 = 0\,\text{Hz}$,

$$A_0 = \frac{1}{4}|X(0)| = 2.5,\ \varphi_0 = \tan^{-1}\left(\frac{\text{Imag}[X(0)]}{\text{Real}([X(0)]}\right) = 0^0,$$

$$P_0 = \frac{1}{4^2}|X(0)|^2 = 6.25.$$

Example 5 - contd. (1)

For $k = 1$, $f = 1 \times 100/4 = 25\,\text{Hz}$,

$$A_1 = \frac{1}{4}|X(1)| = 0.7071,\ \varphi_1 = \tan^{-1}\left(\frac{\text{Imag}[X(1)]}{\text{Real}[X(1)]}\right) = 135^0,$$

$$P_1 = \frac{1}{4^2}|X(1)|^2 = 0.5000.$$

For $k = 2$, $f = 2 \times 100/4 = 50\,\text{Hz}$,

$$A_2 = \frac{1}{4}|X(2)| = 0.5,\ \varphi_2 = \tan^{-1}\left(\frac{\text{Imag}[X(2)]}{\text{Real}[X(2)]}\right) = 180^0,$$

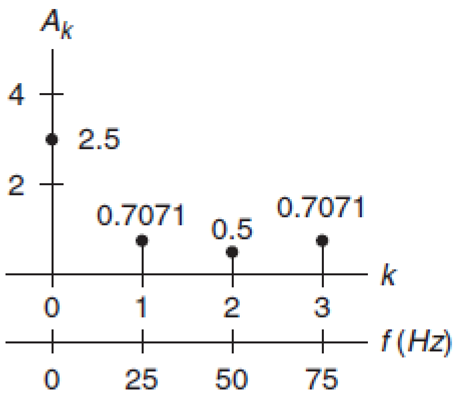
$$P_2 = \frac{1}{4^2}|X(2)|^2 = 0.2500.$$

Similarly, for $k = 3$, $f = 3 \times 100/4 = 75\,\text{Hz}$,

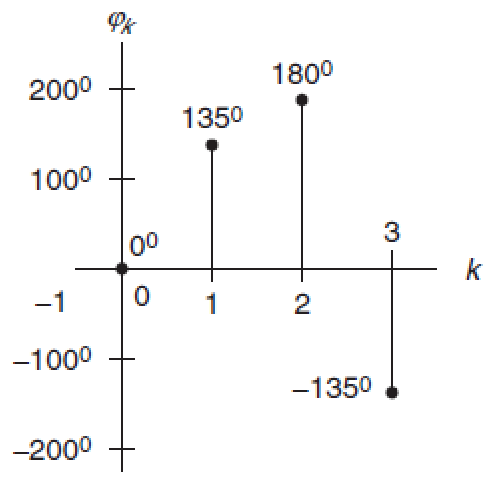
$$A_3 = \frac{1}{4}|X(3)| = 0.7071,\ \varphi_3 = \tan^{-1}\left(\frac{\text{Imag}[X(3)]}{\text{Real}[X(3)]}\right) = -135^0,$$

$$P_3 = \frac{1}{4^2}|X(3)|^2 = 0.5000.$$

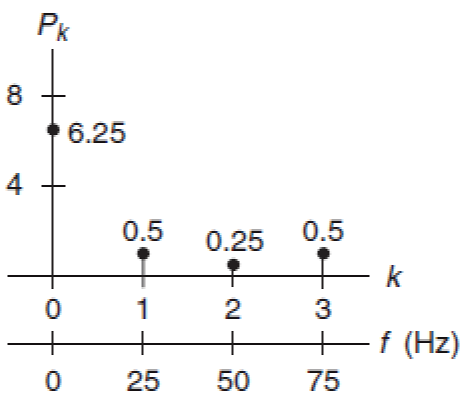
Example 5 - contd. (2)



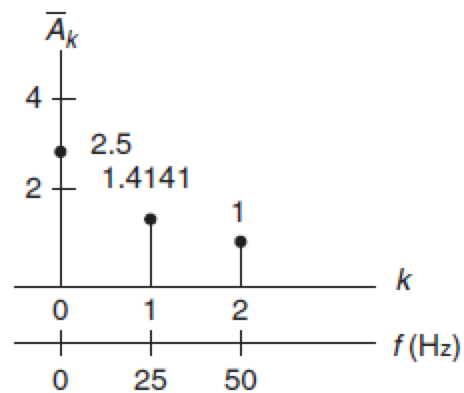
Amplitude Spectrum



Phase Spectrum



Power Spectrum



One sided Amplitude Spectrum

Example 6

Consider a digital sequence sampled at the rate of 10 kHz. If we use a size of 1,024 data points and apply the 1,024-point DFT to compute the spectrum,

a. Determine the frequency resolution.

b. Determine the highest frequency in the spectrum.

Solution:

a. $\Delta f = \frac{f_s}{N} = \frac{10000}{1024} = 9.776\,\text{Hz}.$

b. The highest frequency is the folding frequency, given by

$$f_{\max} = \frac{N}{2}\Delta f = \frac{f_s}{2}$$

$$= 512 \cdot 9.776 = 5000\,\text{Hz}$$

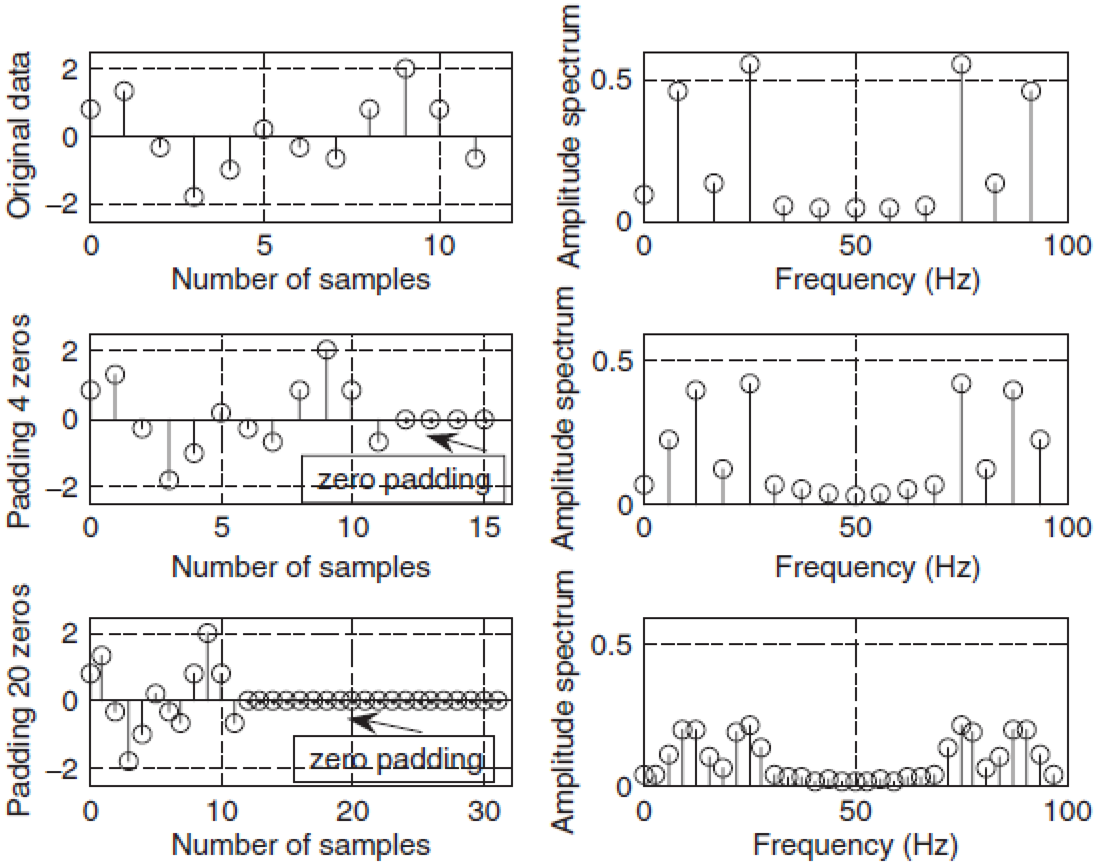
Zero Padding for FFT

FFT: Fast Fourier Transform.

→ A fast version of DFT; It requires signal length to be power of 2.

Therefore, we need to pad zero at the end of the signal.

However, it does not add any new information.



Example 7

Consider a digital signal has sampling rate = 10 kHz. For amplitude spectrum we need frequency resolution of less than 0.5 Hz. For FFT how many data points are needed?

Solution:

$$\Delta f = 0.5\,\text{Hz}$$

$$N = \frac{f_s}{\Delta f} = \frac{10000}{0.5} = 20000$$

For FFT, we need *N* to be power of 2.

$2^{14} = 16384 < 20000$ And $\boxed{2^{15} = 32768 > 20000}$

Recalculated frequency resolution,

$$\Delta f = \frac{f_s}{N} = \frac{10000}{32768} = 0.31\,\text{Hz}.$$

MATLAB Example - 1

$$x(n) = 2 \cdot \sin\left(2000\pi \frac{n}{8000}\right)$$

f_s

Use the MATLAB DFT to compute the signal spectrum with the frequency resolution to be equal to or less than 8 Hz.

$$N = \frac{f_s}{\Delta f} = \frac{8000}{8} = 1000$$

```
% Generate the sine wave sequence
fs = 8000;              %Sampling rate
N = 1000;               % Number of data points
x = 2* sin (2000* pi*[0:1:N - 1]/fs);
```

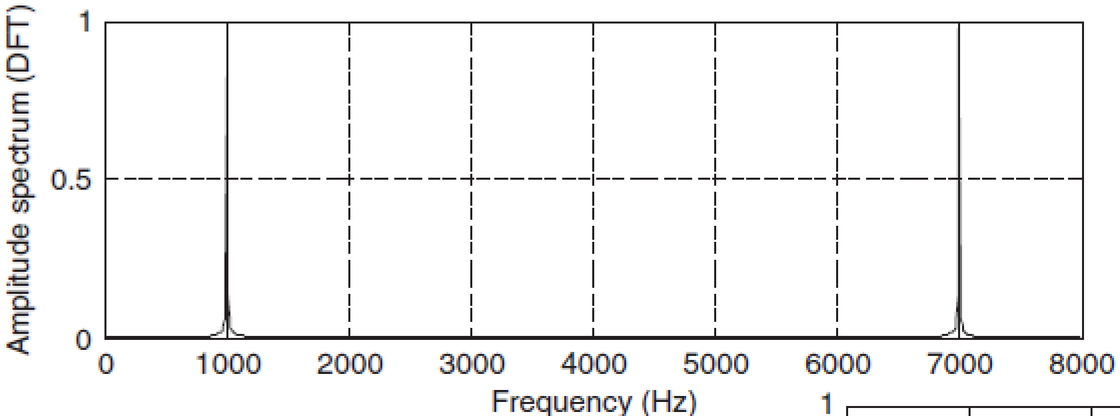
```
figure(1), plot(x);
```

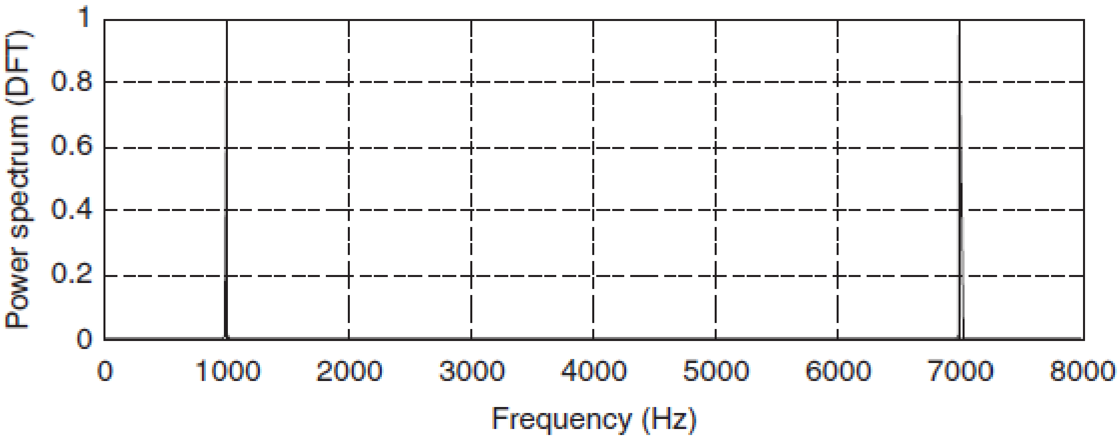
```
xf = abs(fft(x))/N;    %Compute the amplitude spectrum
```

```
P = xf.*xf;             %Compute the power spectrum
f = [0:1:N - 1]*fs/N;         %Map the frequency bin to the frequency (Hz)
```

MATLAB Example – contd. (1)

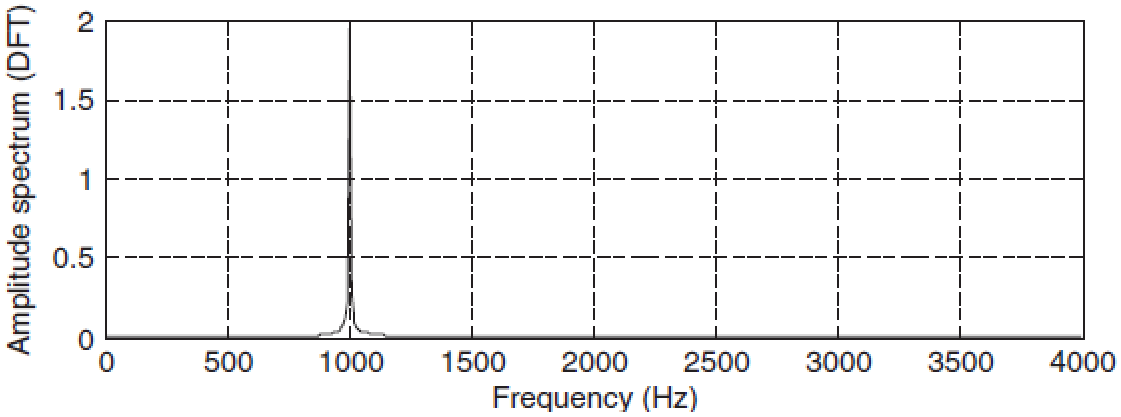
```
subplot(2,1,1); plot(f,xf);grid
xlabel('Frequency (Hz)'); ylabel('Amplitude spectrum (DFT)');
subplot(2,1,2);plot(f,P);grid
xlabel('Frequency (Hz)'); ylabel('Power spectrum (DFT)');
```



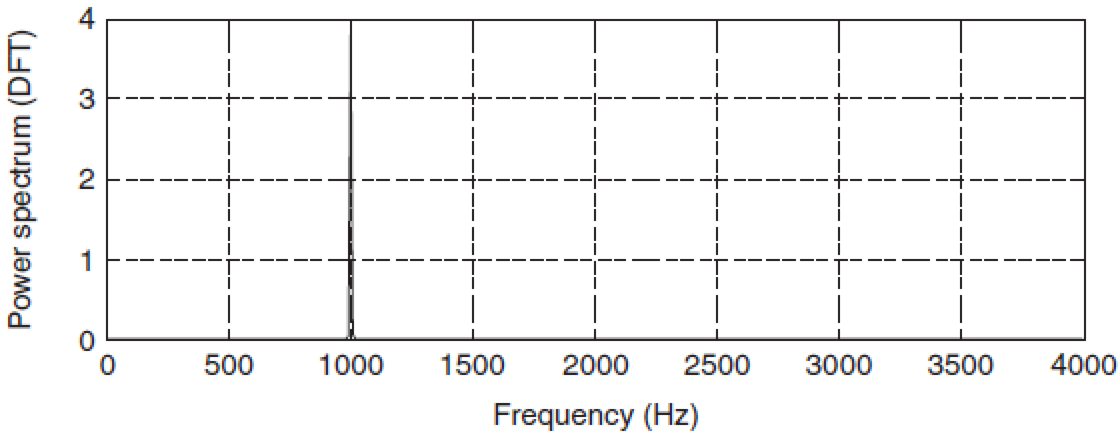


MATLAB Example – contd. (2)

```
% Convert it to one-sided spectrum
xf(2:N) = 2*xf(2:N);          % Get the single-sided spectrum
P = xf.*xf; % Calculate the power spectrum
f = [0:1:N/2]*fs/N % Frequencies up to the folding frequency
subplot(2,1,1); plot(f,xf(1:N/2+1));grid
xlabel('Frequency (Hz)'); ylabel('Amplitude spectrum (DFT)');
subplot(2,1,2);plot(f,P(1:N/2+1));grid
xlabel('Frequency (Hz)'); ylabel('Power spectrum (DFT)');
```



MATLAB Example – contd. (3)




```
% Zero padding to the length of 1024
x = [x, zeros(1,24)];
N = length(x);
xf = abs(fft(x))/N;          %Compute the amplitude spectrum with zero padding
P = xf.*xf;              %Compute the power spectrum
f = [0:1:N − 1]*fs/N;          %Map frequency bin to frequency (Hz)
subplot(2,1,1); plot(f,xf);grid
xlabel('Frequency (Hz)'); ylabel('Amplitude spectrum (FFT)');
subplot(2,1,2);plot(f,P);grid
xlabel('Frequency (Hz)'); ylabel('Power spectrum (FFT)');
```

…………

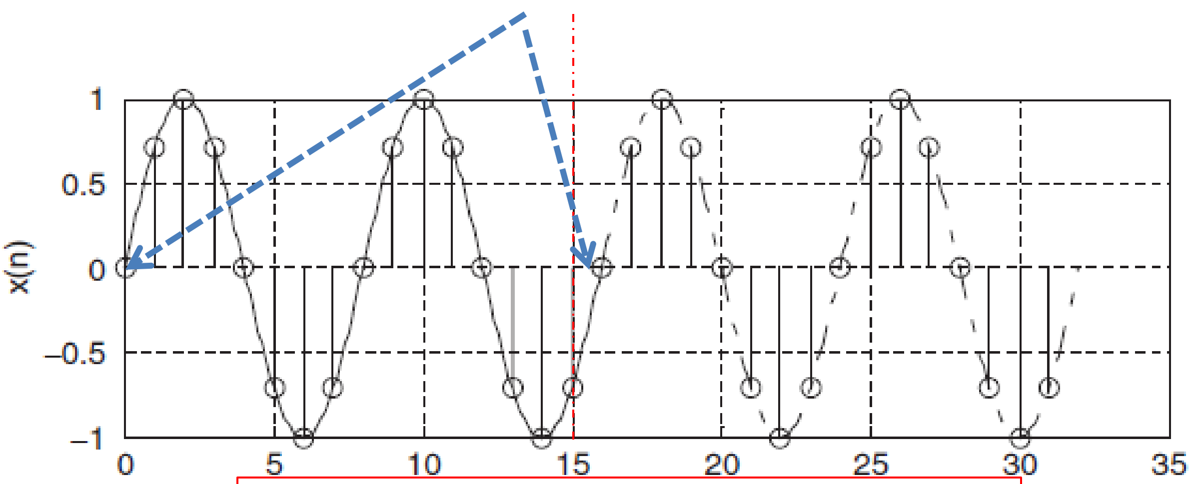
Effect of Window Size

When applying DFT, we assume the following:

1. Sampled data are periodic to themselves (repeat).

2. Sampled data are continuous to themselves and band limited to the folding frequency.

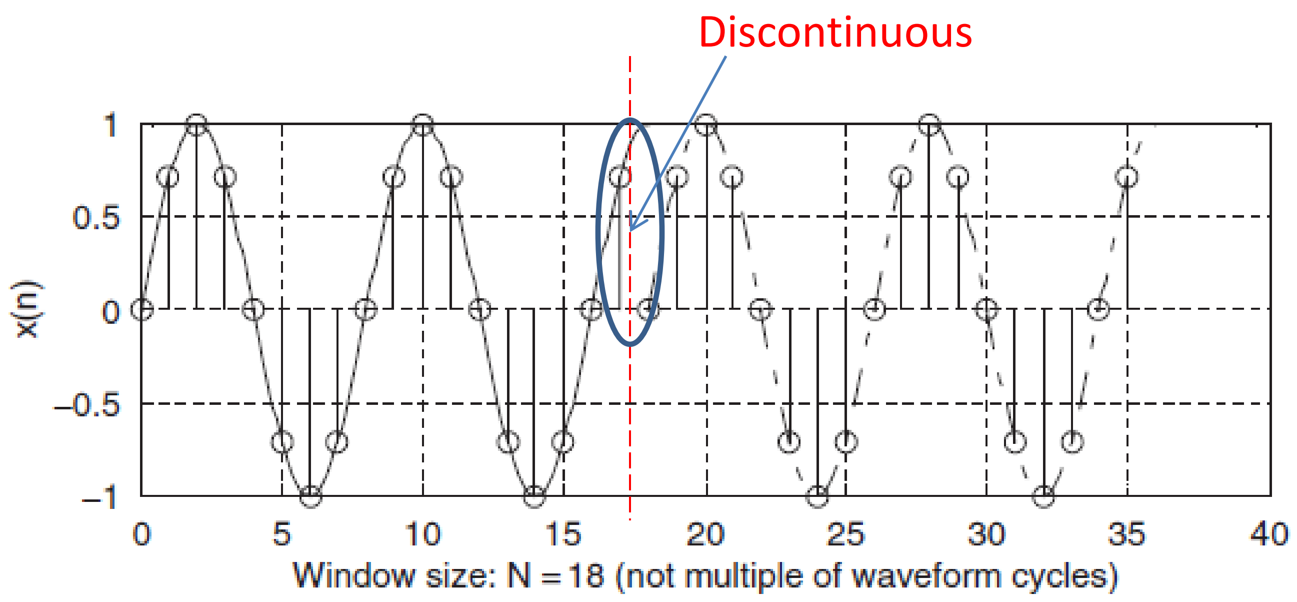
1 Hz sinusoid, with 32 samples



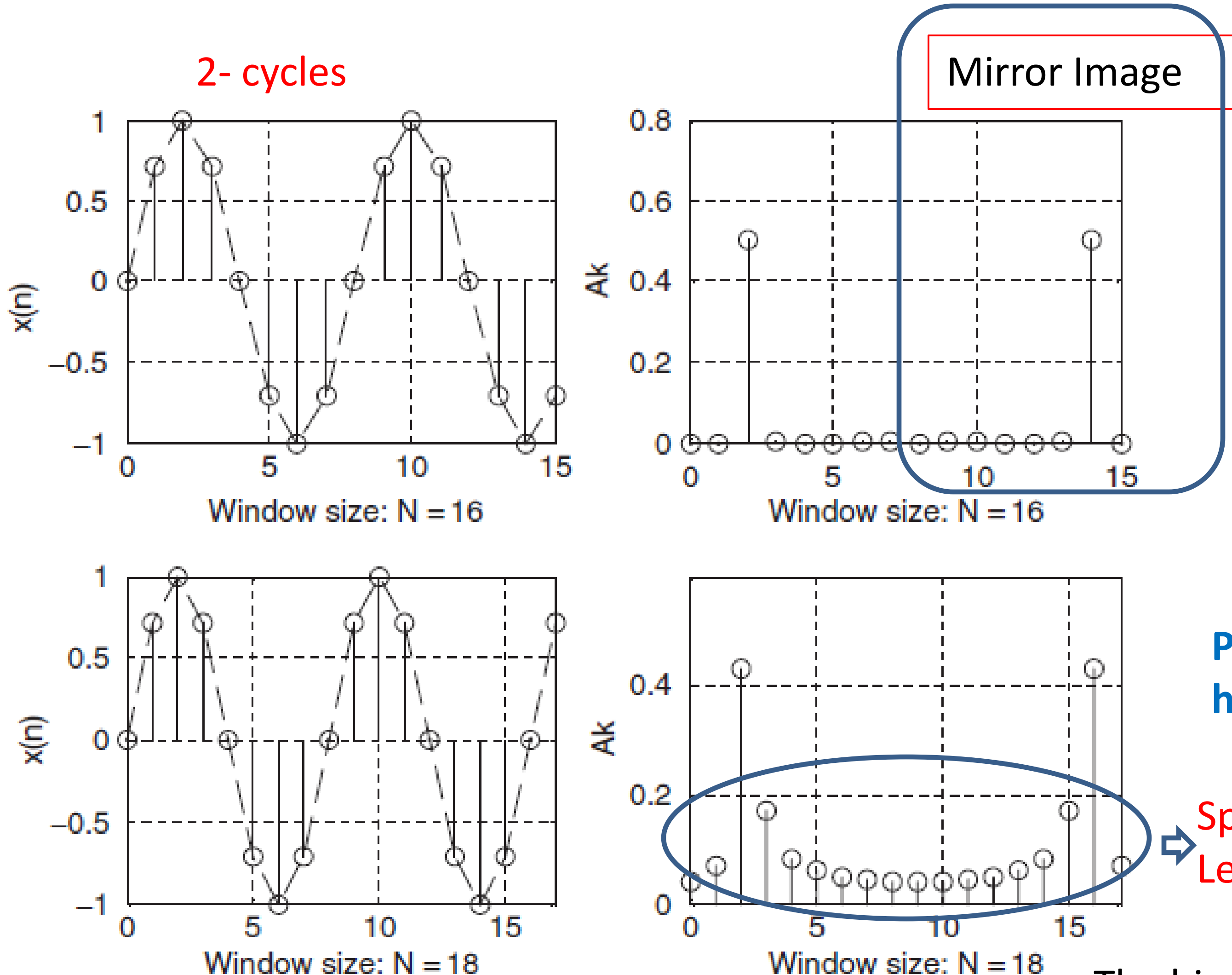
Window size: N = 16 (multiple of waveform cycles)

Effect of Window Size –contd. (1)

If the window size is not multiple of waveform cycles:



Effect of Window Size -contd. (2)



Produces single frequency

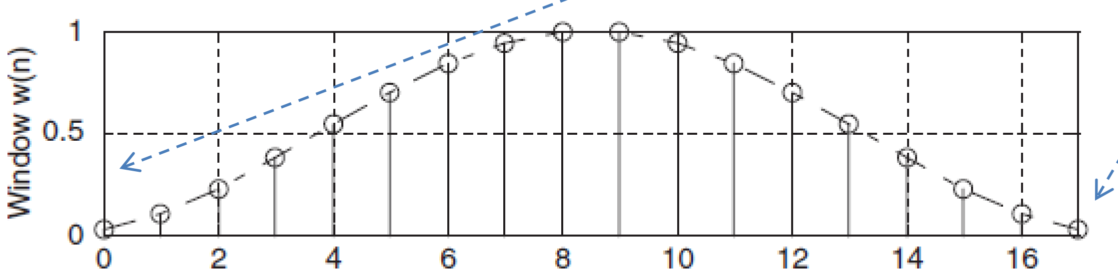
Produces many harmonics as well.

Spectral Leakage

The bigger the discontinuity, the more the leakage

Reducing Leakage Using Window

To reduce the effect of spectral leakage, a window function can be used whose amplitude tapers smoothly and gradually toward zero at both ends.



$$x_w(n) = x(n)w(n), \text{ for } n = 0, 1, \ldots, N-1.$$

Window function, $w(n)$
Data sequence, $x(n)$
Obtained windowed sequence, $x_w(n)$

Example 8

Given,

$x(2) = 1$ and $w(2) = 0.2265$;

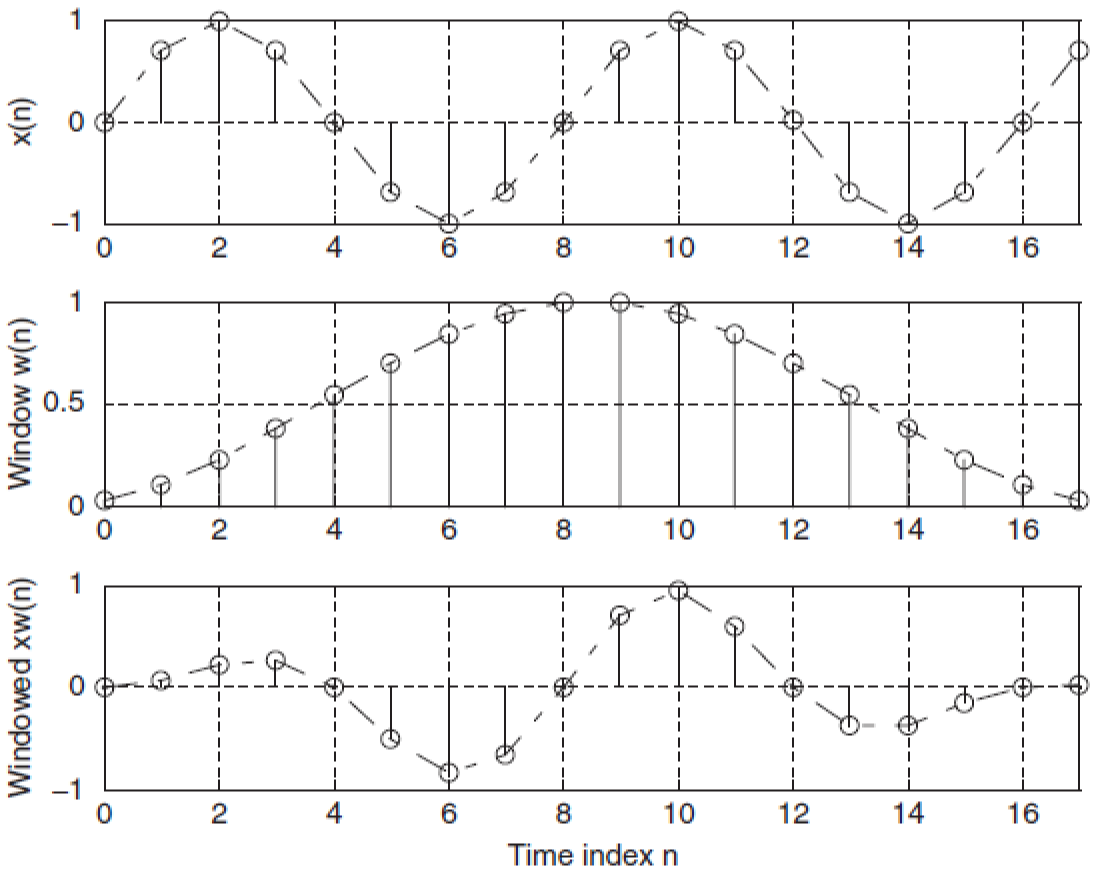
$x(5) = -0.7071$ and $w(5) = 0.7008$,

Calculate,

$x_w(2)$ and $x_w(5)$.

$$x_w(2) = x(2) \times w(2)$$
$$= 1 \times 0.2265 = 0.2265$$

$$x_w(5) = x(5) \times w(5)$$
$$= -0.7071 \times 0.7008 = -0.4956$$



Different Types of Windows

Rectangular Window (no window): $w_R(n) = 1 \qquad 0 \le n \le N-1$

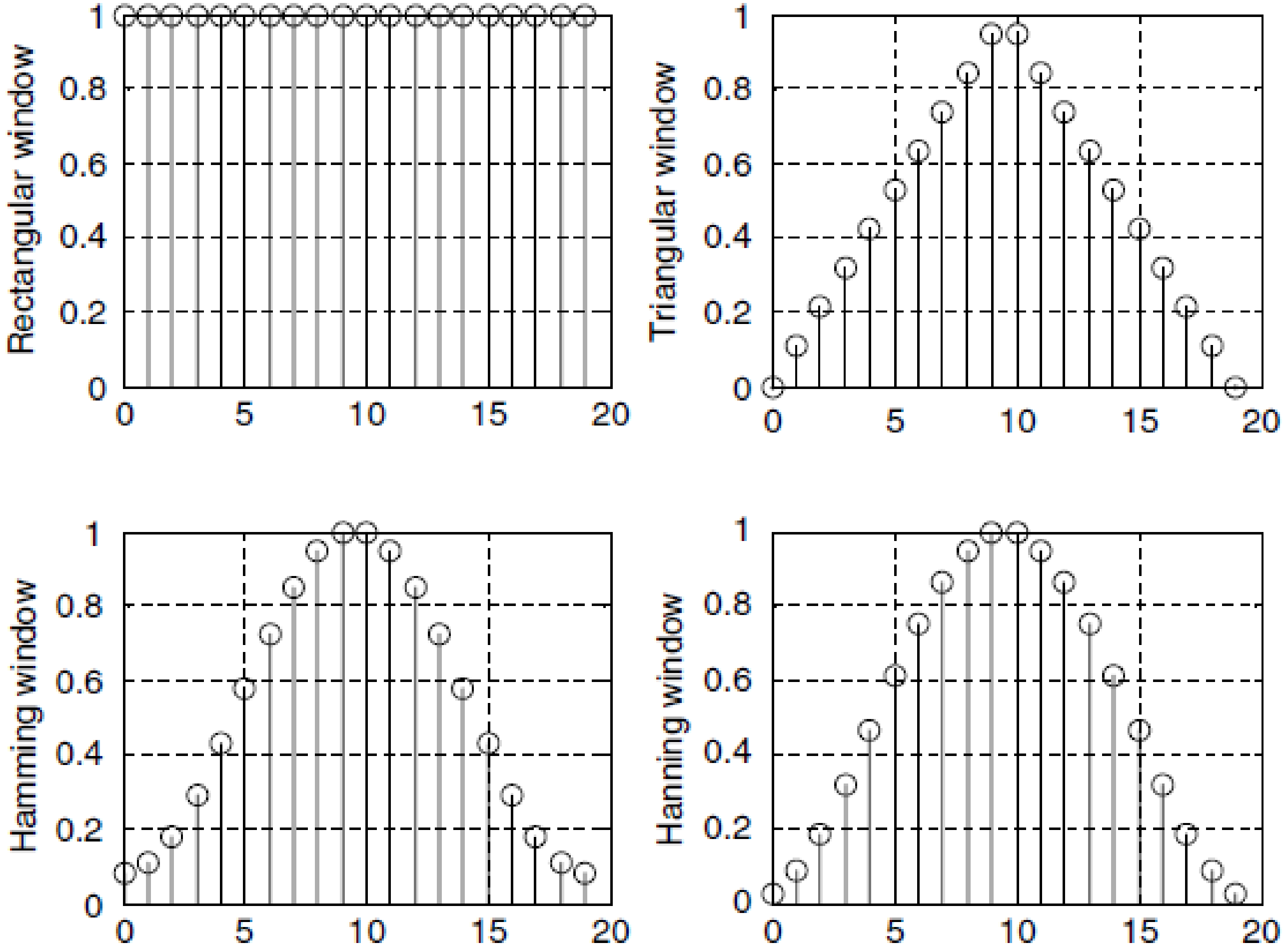
Triangular Window: $w_{tri}(n) = 1 - \dfrac{|2n - N + 1|}{N-1},\ 0 \le n \le N-1$

Hamming Window: $w_{hm}(n) = 0.54 - 0.46\cos\left(\dfrac{2\pi n}{N-1}\right),\ 0 \le n \le N-1$

Hanning Window: $w_{hn}(n) = 0.5 - 0.5\cos\left(\dfrac{2\pi n}{N-1}\right),\ 0 \le n \le N-1$

Different Types of Windows –contd.

Window size of 20 samples



Example 9

Problem:

Considering the sequence $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$, and given $f_s = 100$ Hz, $T = 0.01$ seconds, compute the amplitude spectrum, phase spectrum, and power spectrum

Using the Hamming window function.

Solution:

Since N = 4, Hamming window function can be found as:

$$w_{hm}(0) = 0.54 - 0.46\cos\left(\frac{2\pi \times 0}{4-1}\right) = 0.08$$

$$w_{hm}(1) = 0.54 - 0.46\cos\left(\frac{2\pi \times 1}{4-1}\right) = 0.77.$$

Similarly, $w_{hm}(2) = 0.77$, $w_{hm}(3) = 0.08$.

Example 9 - contd. (1)

Windowed sequence:

$$x_w(0) = x(0) \times w_{hm}(0) = 1 \times 0.08 = 0.08$$
$$x_w(1) = x(1) \times w_{hm}(1) = 2 \times 0.77 = 1.54$$
$$x_w(2) = x(2) \times w_{hm}(2) = 3 \times 0.77 = 2.31$$
$$x_w(0) = x(3) \times w_{hm}(3) = 4 \times 0.08 = 0.32.$$

DFT Sequence:

$$X(k) = x(0)\, W_N^{k0} + x(1)\, W_N^{k1} + x(2) W_N^{k2} + \ldots + x(N-1)\, W_N^{k(N-1)}$$

$$\Rightarrow \quad X(k) = x_w(0) W_4^{k\times 0} + x(1) W_4^{k\times 1} + x(2) W_4^{k\times 2} + x(3) W_4^{k\times 3}.$$

$$\Rightarrow \quad \begin{cases} X(0) = 4.25 \\ X(1) = -2.23 - j1.22 \\ X(2) = 0.53 \\ X(3) = -2.23 + j1.22 \end{cases}$$

$$\Delta f = \frac{1}{NT} = \frac{1}{4 \cdot 0.01} = 25\,\text{Hz}$$

Example 9 - contd. (2)

$$A_0 = \frac{1}{4}|X(0)| = 1.0625,\ \varphi_0 = \tan^{-1}\left(\frac{0}{4.25}\right) = 0^0,$$

$$P_0 = \frac{1}{4^2}|X(0)|^2 = 1.1289$$

$$A_1 = \frac{1}{4}|X(1)| = 0.6355,\ \varphi_1 = \tan^{-1}\left(\frac{-1.22}{-2.23}\right) = -151.32^0,$$

$$P_1 = \frac{1}{4^2}|X(1)|^2 = 0.4308$$

$$A_2 = \frac{1}{4}|X(2)| = 0.1325,\ \varphi_2 = \tan^{-1}\left(\frac{0}{0.53}\right) = 0^0,$$

$$P_2 = \frac{1}{4^2}|X(2)|^2 = 0.0176.$$

$$A_3 = \frac{1}{4}|X(3)| = 0.6355,\ \varphi_3 = \tan^{-1}\left(\frac{1.22}{-2.23}\right) = 151.32^0,$$

$$P_3 = \frac{1}{4^2}|X(3)|^2 = 0.4308.$$

MATLAB Example - 2

$$x(n) = 2 \cdot \sin\left(2000\pi \frac{n}{8000}\right)$$

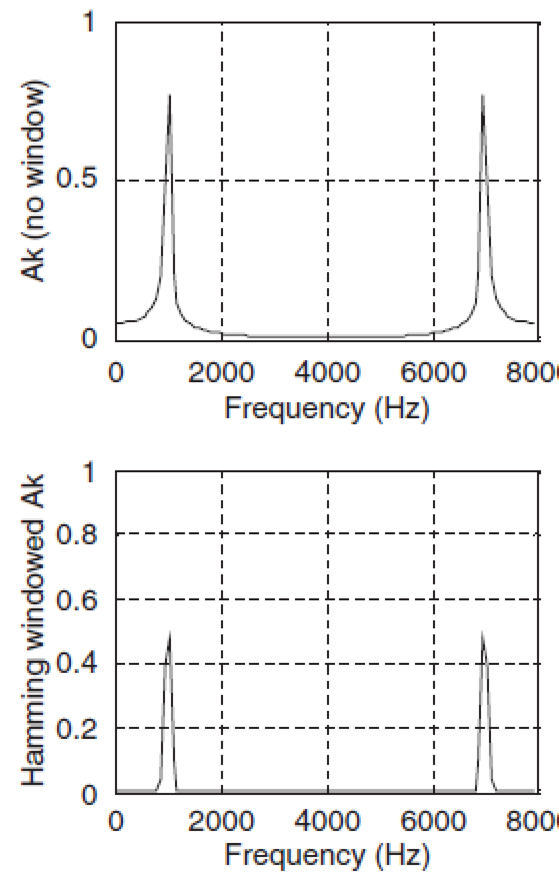
Compute the spectrum of a Hamming window function with a window size = 100.

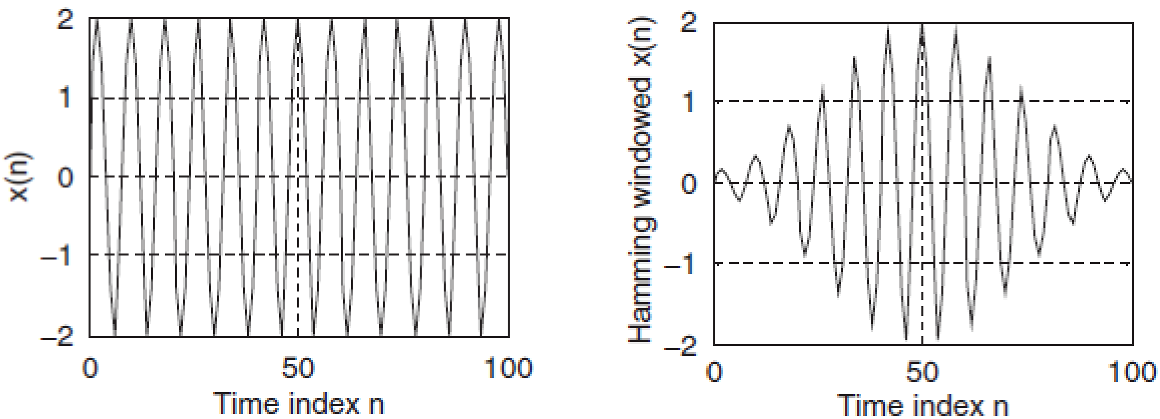
```
% Generate the sine wave sequence
fs = 8000; T = 1/fs;           % Sampling rate and sampling period


% Generate the sine wave sequence
x = 2* sin (2000*pi*[0:1:100]*T);
% Apply the FFT algorithm
N=length(x);
index_t = [0:1:N - 1];
f = [0:1:N - 1]*fs/N;
xf = abs(fft(x))/N;
```

```
%Using the Hamming window
x_hm = x.*hamming(N)';          %Apply the Hamming window function
xf_hm=abs(fft(x_hm))/N;            %Calculate the amplitude spectrum
```

MATLAB Example – 2 contd.

```
subplot(2,2,1);plot(index_t,x);grid
xlabel('Time index n'); ylabel('x(n)');
subplot(2,2,3); plot(index_t,x_hm);grid
xlabel('Time index n'); ylabel('Hamming windowed x(n)');
subplot(2,2,2);plot(f,xf);grid;axis([0 fs 0 1]);
xlabel('Frequency (Hz)'); ylabel('Ak (no window)');
subplot(2,2,4); plot(f,xf_hm);grid;axis([0 fs 0 1]);
xlabel('Frequency (Hz)'); ylabel('Hamming windowed Ak');
```





DFT Matrix

Frequency Spectrum | Multiplication Matrix | Time-Domain samples

$$
\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-2) \\ X(N-1) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & e^{-j\frac{2\pi}{N}} & e^{-j\frac{4\pi}{N}} & \cdots & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{2(N-1)\pi}{N}} \\
1 & e^{-j\frac{4\pi}{N}} & e^{-j\frac{8\pi}{N}} & \cdots & e^{-j\frac{4(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & e^{-j\frac{2(N-2)\pi}{N}} & e^{-j\frac{4(N-2)\pi}{N}} & \cdots & e^{-j\frac{2(N-2)^2\pi}{N}} & e^{-j\frac{2(N-2)(N-1)\pi}{N}} \\
1 & e^{-j\frac{2(N-1)\pi}{N}} & e^{-j\frac{4(N-1)\pi}{N}} & \cdots & e^{-j\frac{2(N-1)(N-2)\pi}{N}} & e^{-j\frac{(N-1)^2\pi}{N}}
\end{bmatrix}
\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-2) \\ x(N-1) \end{bmatrix}
$$

DFT Matrix

Let, $w_N = e^{-j2\pi/N}$

Then

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{(N-1)} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{(N-1)} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

DFT equation: $X(k) = \sum_{m=0}^{N-1} x(m) w_N^{mk} \qquad k = 0, \ldots, N-1$

DFT requires N^2 complex multiplications.

FFT

FFT: Fast Fourier Transform

A very efficient algorithm to compute DFT; it requires less multiplication.

The length of input signal, x(n) must be 2^m samples, where m is an integer.

Samples N = 2, 4, 8, 16 or so.

If the input length is not 2^m, append (pad) zeros to make it 2^m.

4	5	1	7	1

N = 5

4	5	1	7	1	0	0	0

N = 8, power of 2

DFT to FFT: Decimation in Frequency

DFT: $$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \text{ for } k = 0, 1, \ldots, N-1,$$

$$X(k) = x(0) + x(1) W_N^{k} + \ldots + x(N-1) W_N^{k(N-1)}$$

$$X(k) = \boxed{x(0) + x(1) W_N^{k} + \ldots + x\left(\frac{N}{2} - 1\right) W_N^{k(N/2-1)}} + \boxed{x\left(\frac{N}{2}\right) W^{kN/2} + \ldots + x(N-1) W_N^{k(N-1)}}$$

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \boxed{\sum_{n=N/2}^{N-1} x(n) W_N^{kn}}$$

$$X(k) = \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + W_N^{(N/2)k} \sum_{n=0}^{(N/2)-1} x\left(n + \frac{N}{2}\right) W_N^{kn}.$$

$$W_N^{N/2} = e^{-j\frac{2\pi(N/2)}{N}} = e^{-j\pi} = -1;$$

$$X(k) = \sum_{n=0}^{(N/2)-1} \left(x(n) + (-1)^k x\left(n + \frac{N}{2}\right)\right) W_N^{kn}$$

DFT to FFT: Decimation in Frequency

Now decompose into even (k = 2m) and odd (k = 2m+1) sequences.

$$X(2m) = \sum_{n=0}^{(N/2)-1} \left(x(n) + x\left(n + \frac{N}{2}\right)\right) W_N^{2mn}$$

$$X(2m+1) = \sum_{n=0}^{(N/2)-1} \left(x(n) - x\left(n + \frac{N}{2}\right)\right) W_N^{n} W_N^{2mn}$$

$$\boxed{W_N^2 = e^{-j\frac{2\pi \times 2}{N}} = e^{-j\frac{2\pi}{(N/2)}} = W_{N/2}}$$

$$X(2m) = \sum_{n=0}^{(N/2)-1} a(n) W_{N/2}^{mn} = DFT\{a(n) \ with \ (N/2) \text{ points}\}$$

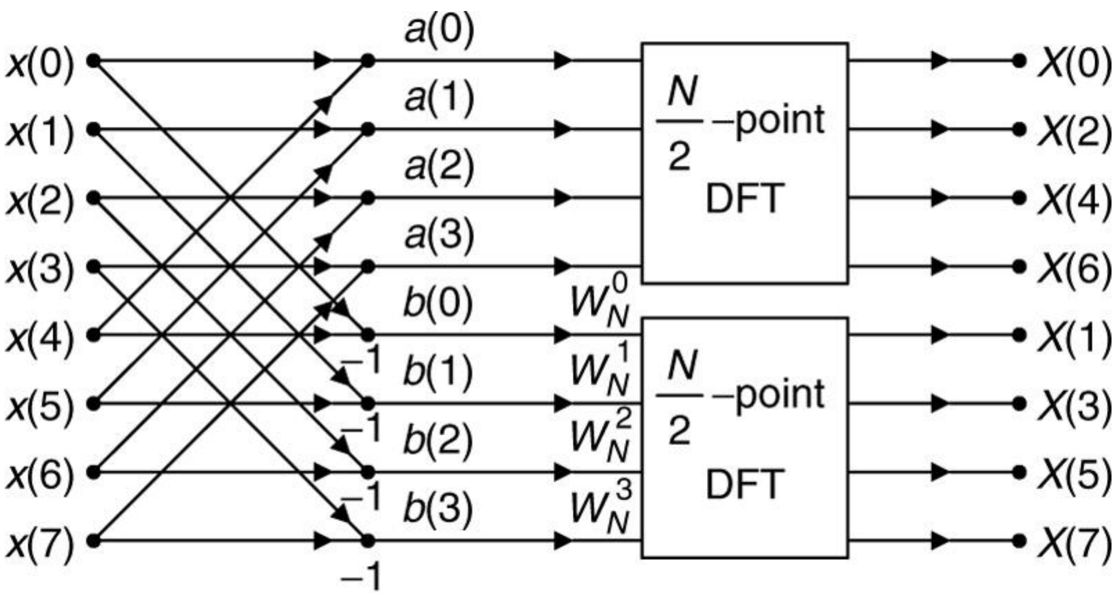
$$X(2m+1) = \sum_{n=0}^{(N/2)-1} b(n) W_N^{n} W_{N/2}^{mn} = DFT\{b(n) W_N^n \ with \ (N/2) \text{ points}\}$$

$$a(n) = x(n) + x\left(n + \frac{N}{2}\right), \text{ for } n = 0,1 \ldots, \frac{N}{2} - 1$$

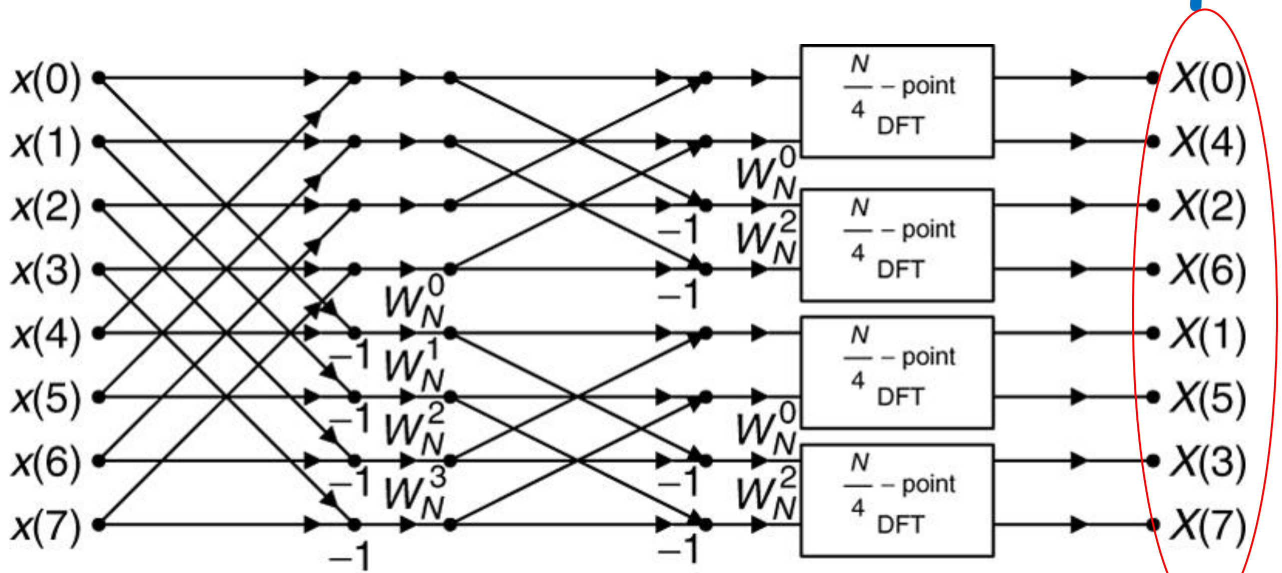
$$b(n) = x(n) - x\left(n + \frac{N}{2}\right), \text{ for } n = 0,1, \ldots, \frac{N}{2} - 1.$$

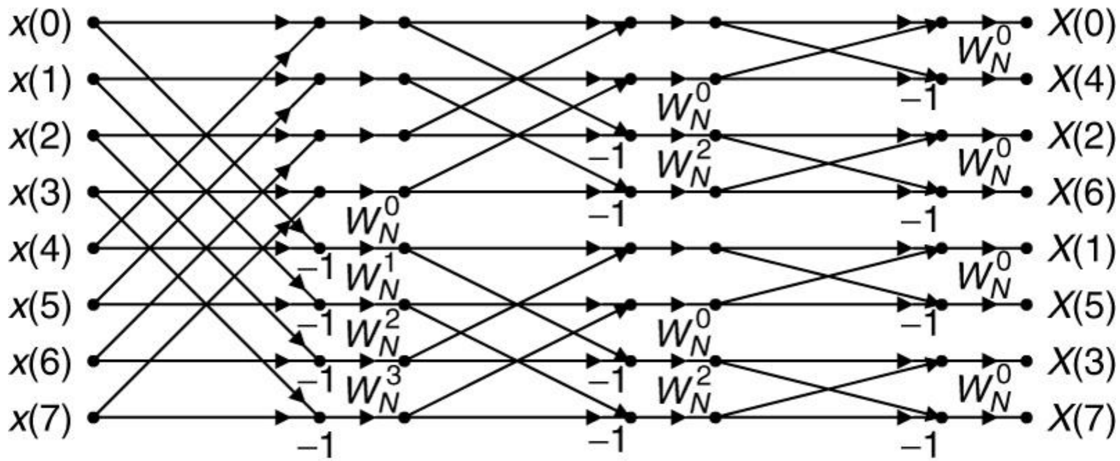
DFT to FFT: Decimation in Frequency

$$DFT\{x(n)\ with\ N\ \text{points}\} = \begin{cases} DFT\{a(n)\ with\ (N/2)\ \text{points}\} \\ DFT\{b(n)W_N^n\ with\ (N/2)\ \text{points}\} \end{cases}$$



DFT to FFT: Decimation in Frequency





12 complex multiplication

DFT to FFT: Decimation in Frequency

Binary	index	1st split	2nd split	3rd split	Bit reversal
000	0	0	0	0	000
001	1	2	4	4	100
010	2	4	2	2	010
011	3	6	6	6	011
100	4	1	1	1	001
101	5	3	5	5	101
110	6	5	3	3	011
111	7	7	7	7	111

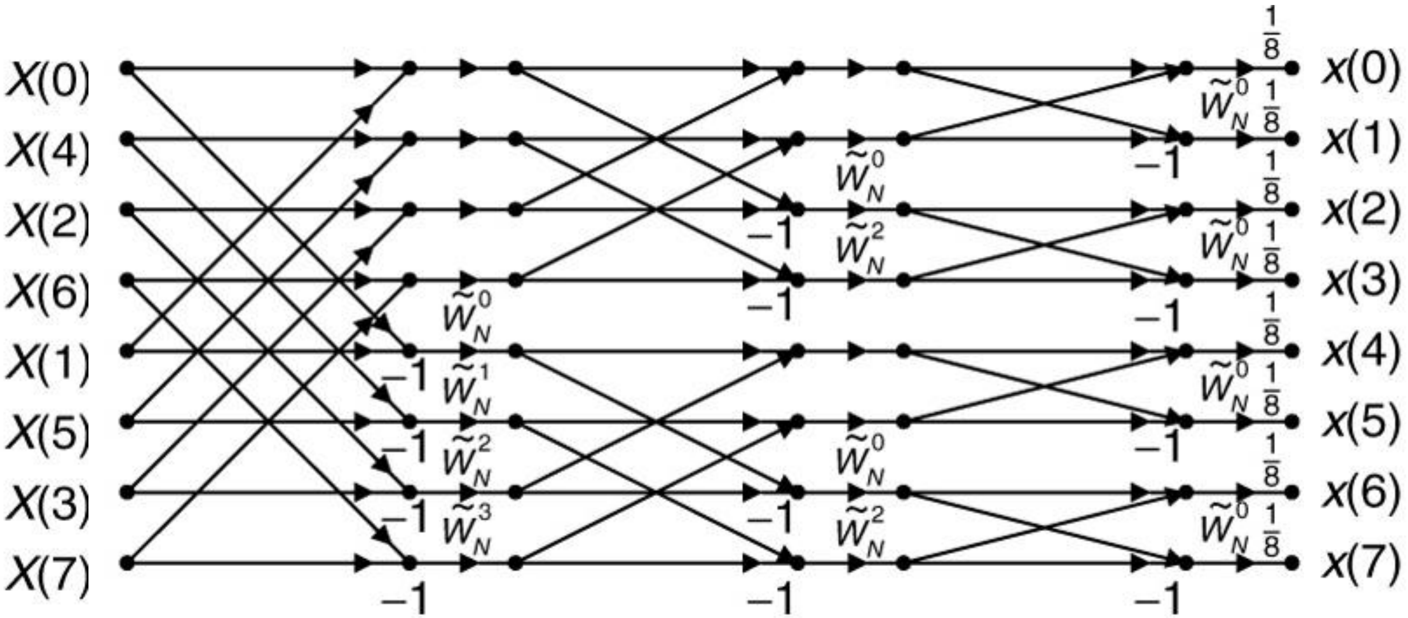
Complex multiplications of DFT $= N^2$, and

Complex multiplications of FFT $= \frac{N}{2}\log_2(N)$

For 1024 samples data sequence, DFT requires 1024×1024 = 1048576 complex multiplications. FFT requires (1024/2)log(1024) = 5120 complex multiplications.

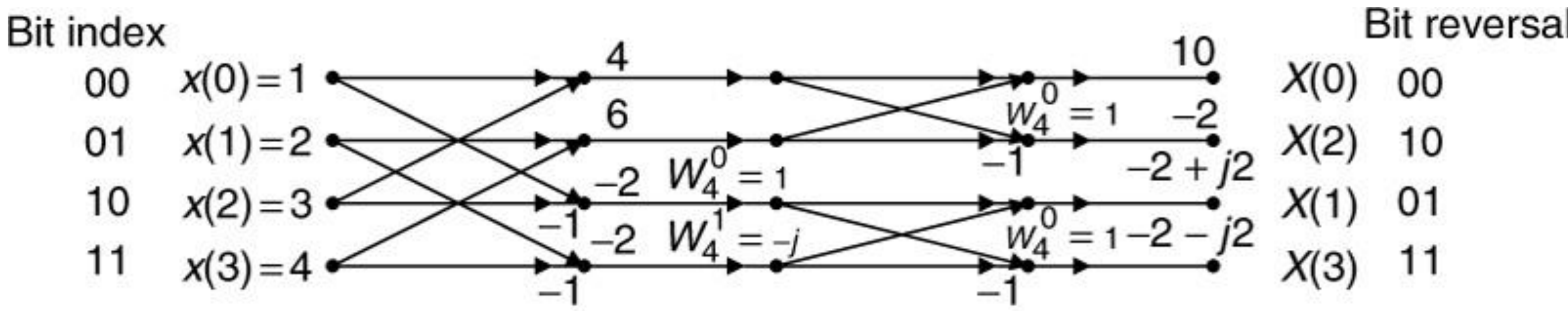
IFFT: Inverse FFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\tilde{W}_N^{kn}, \text{ for } k = 0, 1, \ldots, N-1.$$



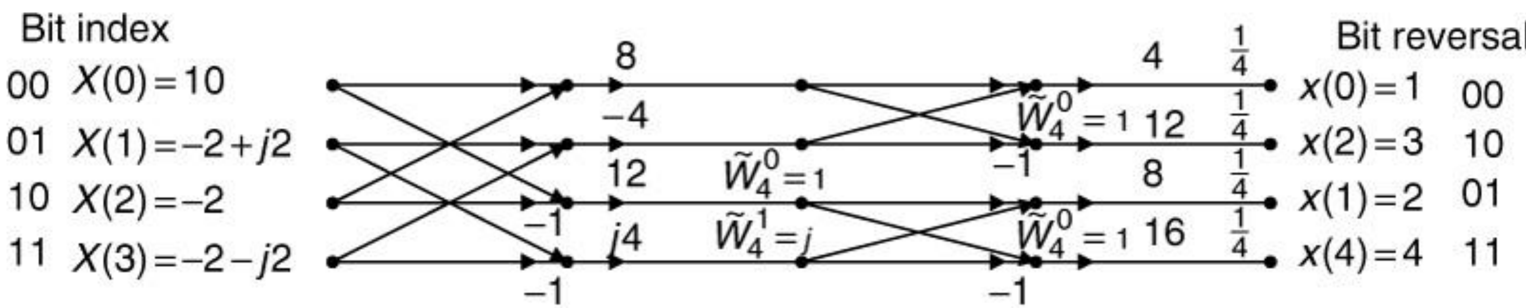
FFT and IFFT Examples

FFT



Number of complex multiplication = $\frac{N}{2}\log_2(N) = \frac{4}{2}\log_2(4) = 4.$

IFFT



DFT to FFT: Decimation in Time

Split the input sequence x(n) into the even indexed x(2m) and x(2m + 1), each with N/2 data points.

$$X(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_N^{2mk} + \sum_{m=0}^{(N/2)-1} x(2m+1) W_N^k W_N^{2mk},$$

for $k = 0, 1, \ldots, N-1.$

Using

$$w_N^2 = \left(e^{-j2\pi/N}\right)^2 = e^{-j2\pi/(N/2)} = w_{N/2}$$

$$X(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_{N/2}^{mk} + W_N^k \sum_{m=0}^{(N/2)-1} x(2m+1) W_{N/2}^{mk},$$

for $k = 0, 1, \ldots, N-1.$

DFT to FFT: Decimation in Time

Define new functions as

$$G(k) = \sum_{m=0}^{(N/2)-1} x(2m) W_{N/2}^{mk} = DFT\{x(2m)\ with\ (N/2) \text{ points}\}$$

$$H(k) = \sum_{m=0}^{(N/2)-1} x(2m+1) W_{N/2}^{mk} = DFT\{x(2m+1)\ with\ (N/2) \text{ points}\}.$$

As,

$$G(k) = G\left(k + \frac{N}{2}\right), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1$$

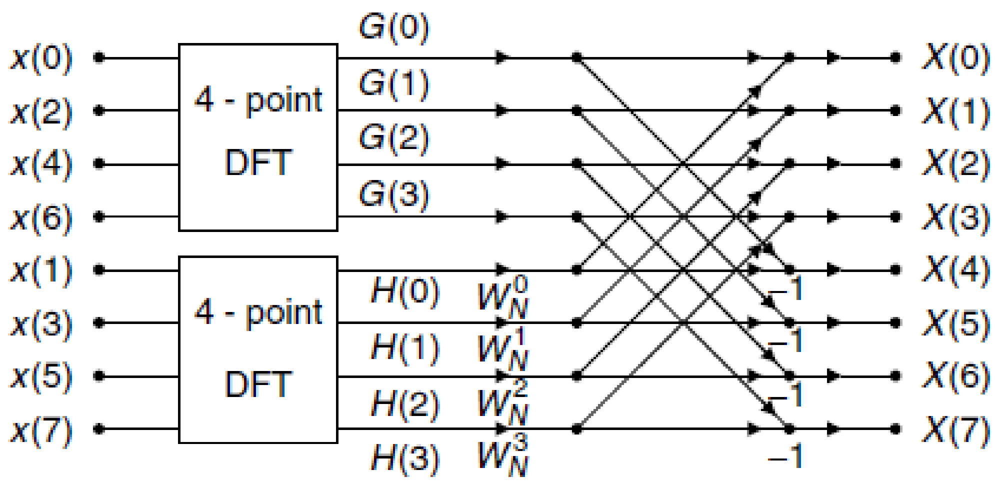
$$H(k) = H\left(k + \frac{N}{2}\right), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1.$$

$$\boxed{\begin{aligned} X(k) &= G(k) + W_N^k H(k), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1. \\ X\left(\frac{N}{2} + k\right) &= G(k) - W_N^k H(k), \text{ for } k = 0, 1, \ldots, \frac{N}{2} - 1. \end{aligned}}$$

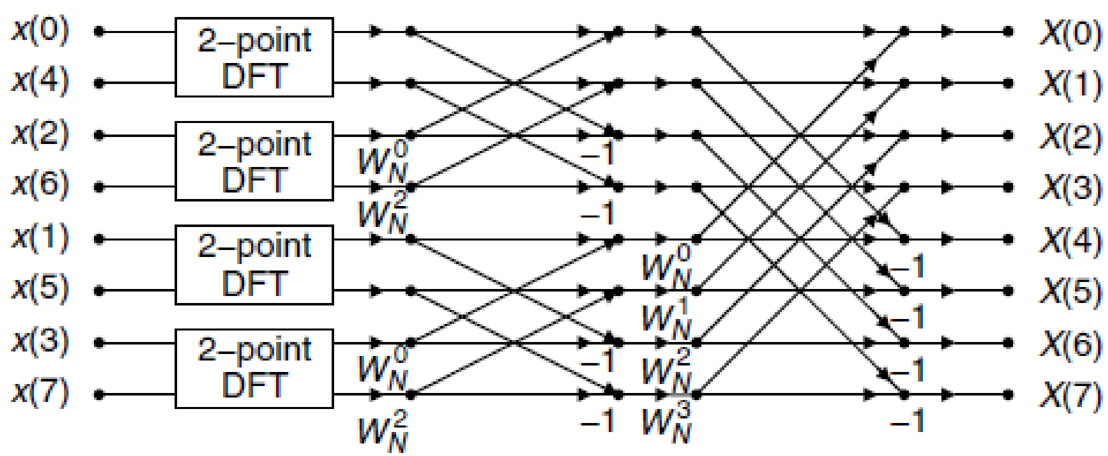
$$W_N^{(N/2+k)} = -W_N^k.$$

DFT to FFT: Decimation in Time

First iteration:

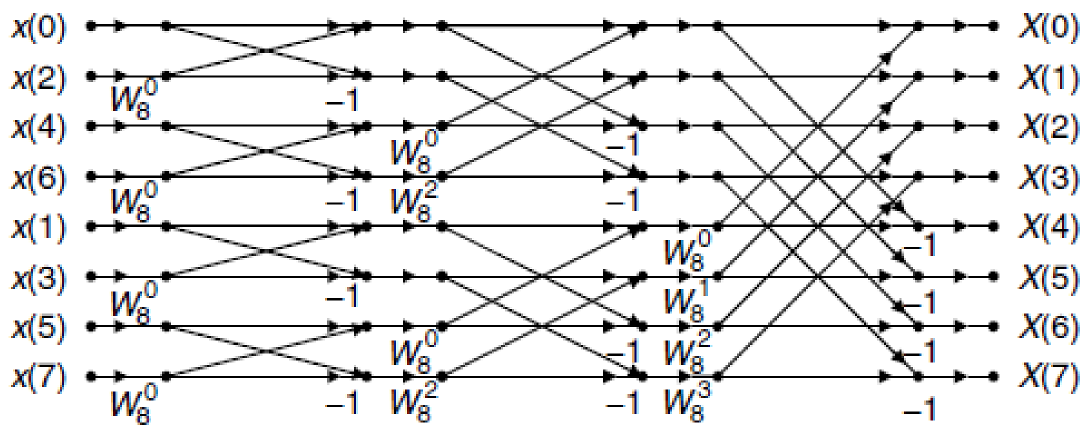


Second iteration:



DFT to FFT: Decimation in Time

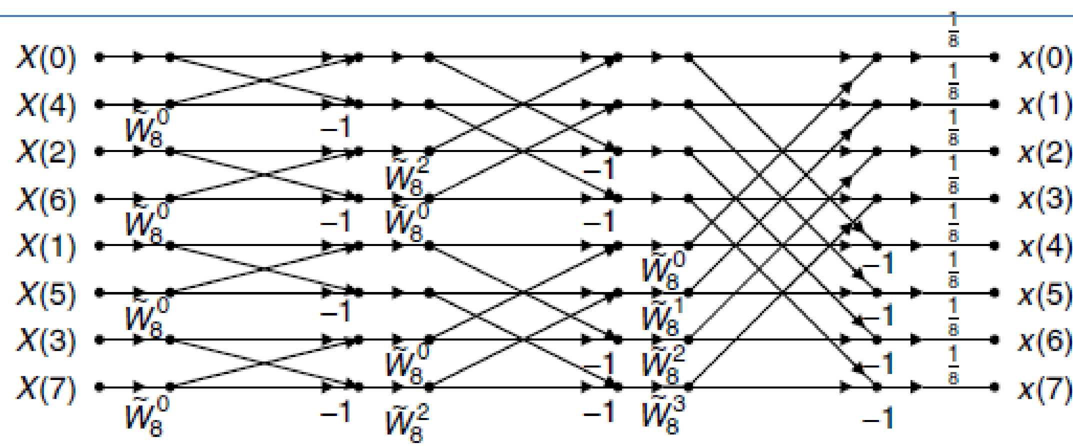
Third iteration:



$$W_N = e^{-\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

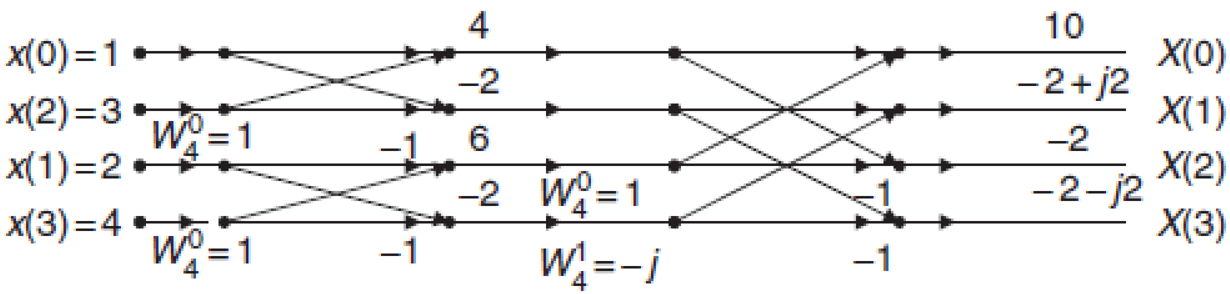
$$W_8^2 = e^{-\frac{2\pi\times 2}{8}} = e^{-\frac{\pi}{2}} = \cos(\pi/2) - j\sin(\pi/2) = -j$$

IFFT

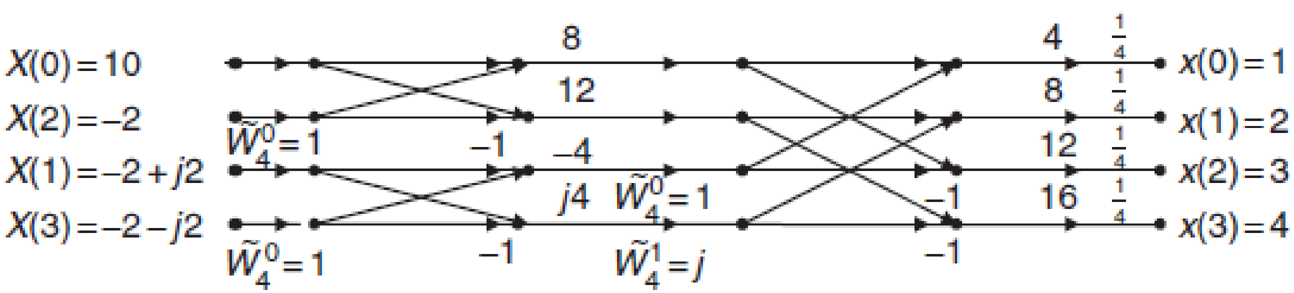


FFT and IFFT Examples

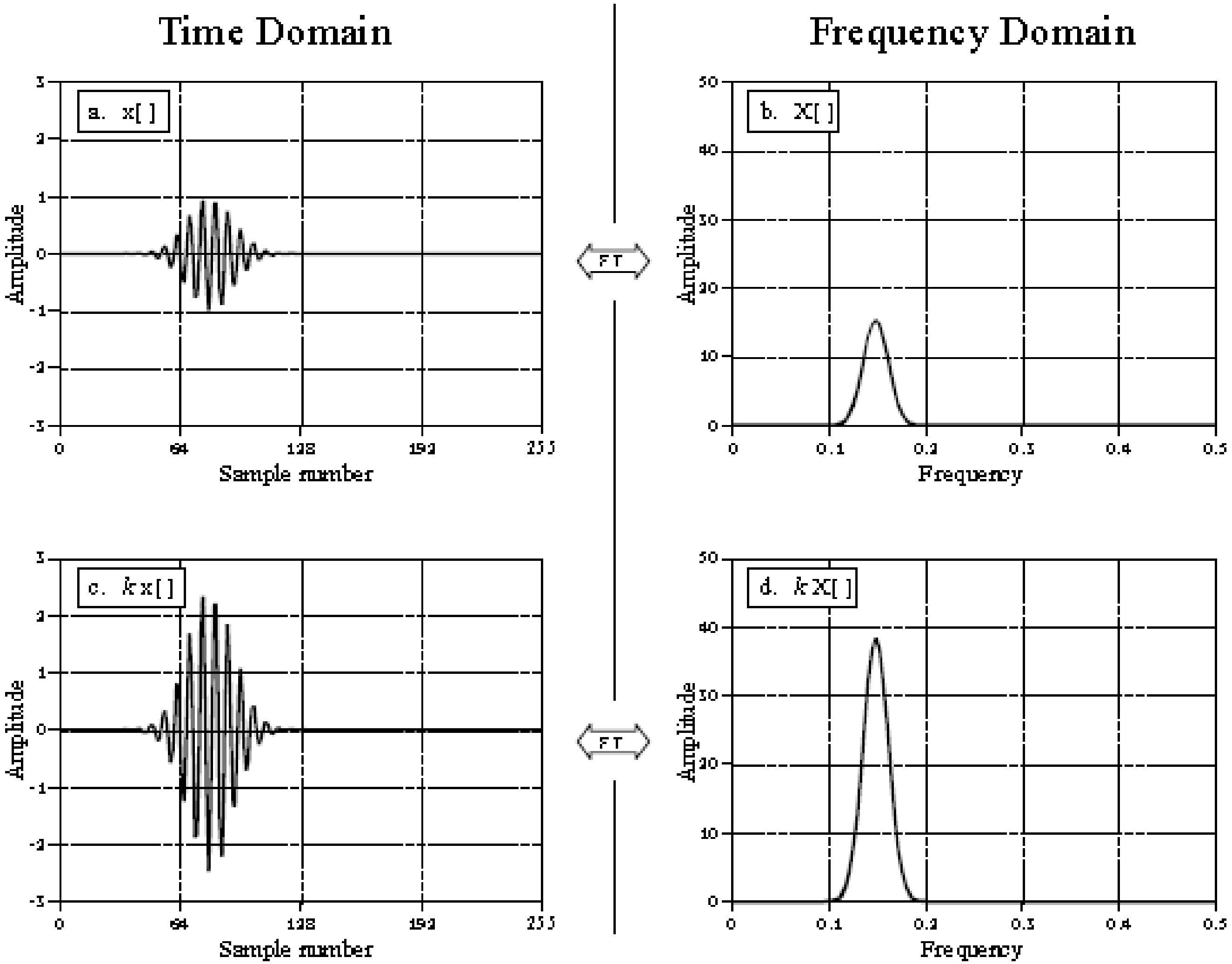
FFT



IFFT



Fourier Transform Properties (1)



FT is linear:

- **Homogeneity**
- **Additivity**

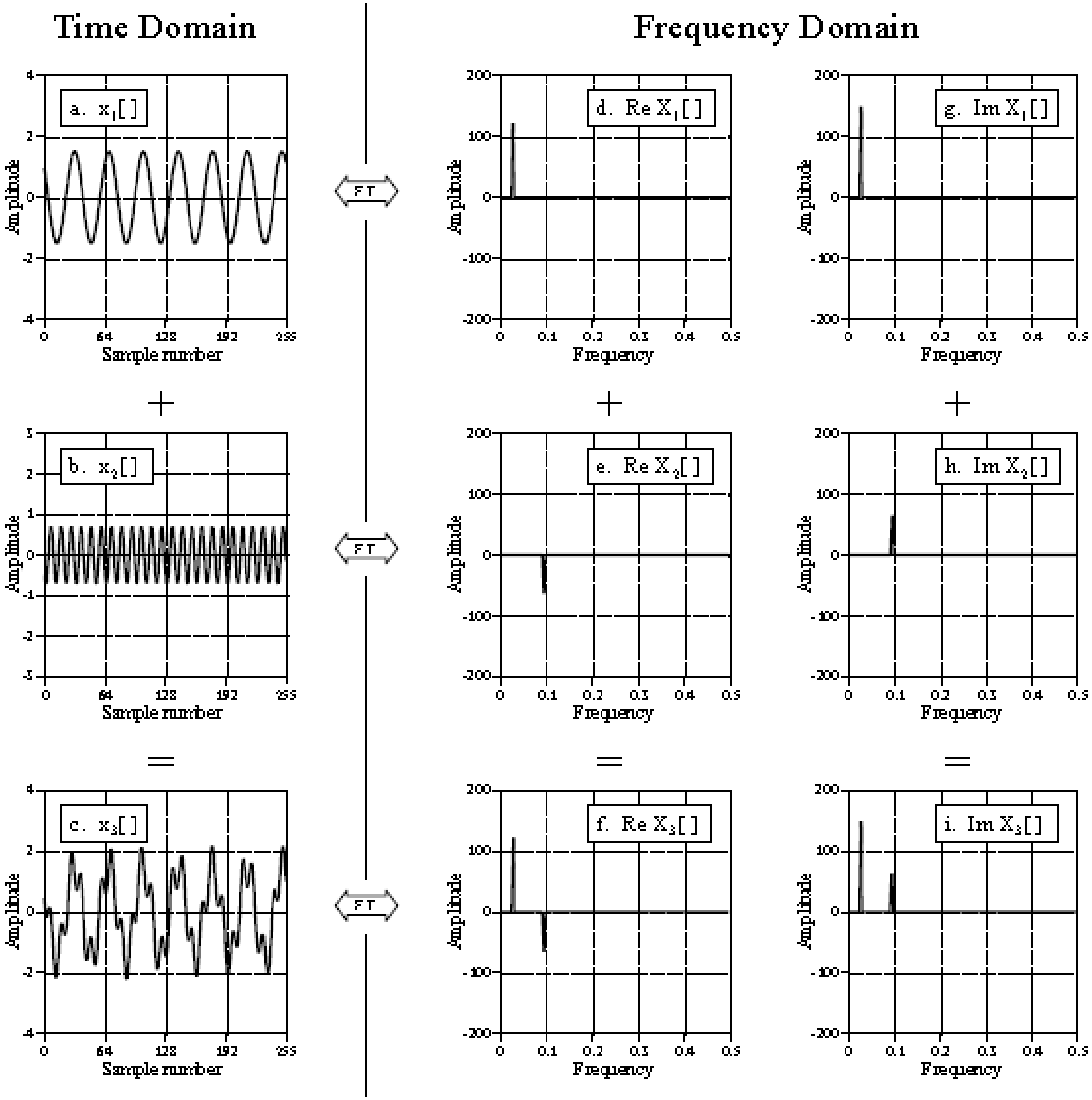
Homogeneity:

$x[\,] \xrightarrow{DFT} X[\,]$

$kx[\,] \xrightarrow{DFT} kX[\,]$

Frequency is not changed.

Fourier Transform Properties (2)



Additivity

$$
\begin{aligned}
&If: x_1[n] + x_2[n] = x_3[n] \\
&Then: \operatorname{Re} X_1[f] + \operatorname{Re} X_2[f] = \operatorname{Re} X_3[f] \\
&and \quad \operatorname{Im} X_1[f] + \operatorname{Im} X_2[f] = \operatorname{Im} X_3[f]
\end{aligned}
$$

Fourier Transform Pairs

Delta Function Pairs in Polar Form

Delta Function

Shifted Delta Function

Same Magnitude, Different Phase

Shifted Delta Function