MODULE 3: POISSONS AND LAPLACES EQUATION, STEADY MAGNETIC FIELD

STRUCTURE

1.1 Derivation of Poisson's equation and Laplace's equation
1.2 Uniqueness theorem,
1.3 Examples of the solutions Laplace Equations and Poisson's Equations

Objectives

1. To derive the Poissons and Laplaces equation
2. To derive the Uniqueness theorem
3. Application of Laplaces equation to parallel plate capacitor…

Laplace's & Poisson's equation:

Laplace's & Poisson's equation enable us to find potential fields within regions bounded by known potentials or charge densities.

Derivation of Laplace's & Poisson's equation:

From Gauss law in point form, we have

$$\nabla \cdot \mathbf{D} = \rho_v$$ --------------------------------(1).

By definition, D = €E. & from gradient relationship,

.

By substituting the above in equation 1, we get

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_v$$

Or --------------------2

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon}$$

For a homogeneous region in which € is constant. Equation 2 is poisson's equation.
In rectangular co-ordinates,

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$
$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Therefore,

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}}$$

If $\rho_V = 0$, indicating zero volume charge density, but allowing point charges, line charges & surface charge density to exist at singular locations as sources of the field, then

$$\nabla^2 V = 0$$

which is Laplace's equation. The ∇^2 operorator is called the Laplacian of V.

In rectangular coordinates Laplace equation is,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(cartesian)}$$

, In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}$$

& in spherical coordinates,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}$$

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Every conductor produces a field for which $\nabla^2 V$=0. In examples if it satisfies the boundary conditions and Laplace equation, then it is the only possible answer.

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Uniqueness theorem:

Let us assume we have two solutions of Laplace equation, V_1 and V_2, both general functions of the coordinates used. Therefore

$\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$. From which $\nabla^2(V_1 - V_2) = 0$

On the boundary, $V_{b1}=V_{b2}$. Let the difference between V_1 & V_2 be V_d. Therefore $V_d= V_1-V_2$. From Laplace equation,

$\nabla^2 V_d= \nabla^2V_2-\nabla^2V_1$. On the boundary V_d=0.

From Divergence theorem,

$$\boxed{\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_{\text{vol}} \nabla\cdot\mathbf{D}\, dv}$$

Using vector identity,

$$\nabla\cdot(V\mathbf{D}) \equiv V(\nabla\cdot\mathbf{D}) + \mathbf{D}\cdot(\nabla V)$$

We get,

$$\int_{\text{vol}} \nabla\cdot[(V_1 - V_2)\nabla(V_1 - V_2)]\, dv$$
$$\equiv \int_{\text{vol}} (V_1 - V_2)[\nabla\cdot\nabla(V_1 - V_2)]dv + \int_{\text{vol}} [\nabla(V_1 - V_2)]^2\, dv$$

As $V_1= V_2$,

$$\int_{\text{vol}} \nabla\cdot[(V_1 - V_2)\nabla(V_1 - V_2)]\, dv = \oint_S [(V_{1b} - V_{2b})\nabla(V_{1b} - V_{2b})]\cdot d\mathbf{S} = 0$$

Surface consists of boundaries and hence

$$\int_{vol} [\nabla(V_1 - V_2)]^2 \, dv = 0$$

Therefore

$$[\nabla(V_1 - V_2)]^2 = 0$$

And

$$\nabla(V_1 - V_2) = 0$$

As

$$V_1 - V_2 = V_{1b} - V_{2b} = 0$$

We obtain,

$$V_1 = V_2$$

2.8 Example of solution of Laplace's equation:

Example 1: For a Parallel plate capacitor:

Let us assume V is a function of x. Laplace's equation reduces to,

$$\frac{\partial^2 V}{\partial x^2} = 0$$

Since V is not a function of y & z.
Integrating the above equation twice we obtain,

$$V = Ax + B$$

Where A & B are integration constants.

If V=0 at x=0 and V= V_0 at x = d, then,

A= V_0/d and B = 0.

Therefore,

$$V = \frac{V_0 x}{d}$$

Hence we have,

$$V = V_0 \frac{x}{d}$$

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D}_S = \mathbf{D}\Big|_{x=0} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{a}_N = \mathbf{a}_x$$

$$D_N = -\epsilon \frac{V_0}{d} = \rho_S$$

$$Q = \int_S \frac{-\epsilon V_0}{d} dS = -\epsilon \frac{V_0 S}{d}$$

And the capacitance is

$$C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}$$

Example 2: **Capacitance of a co-axial cylindrical conductor:**

Assuming variation with respect to ρ Laplace equation becomes,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

Integrating twice on both sides we obtain,

$$\rho \frac{dV}{d\rho} = A,$$

$$V = A \ln \rho + B.$$

Assuming V = V0 at ρ = A and V= 0 at ρ = B, We get

$$V = V_0 \frac{\ln(b/\rho)}{\ln(b/a)}$$

$$\mathbf{E} = \frac{V_0}{\rho}\frac{1}{\ln(b/a)}\mathbf{a}_\rho$$

$$D_{N(\rho=a)} = \frac{\epsilon V_0}{a\ln(b/a)}$$

$$Q = \frac{\epsilon V_0 2\pi a L}{a\ln(b/a)}$$

$$\boxed{C = \frac{2\pi\epsilon L}{\ln(b/a)}}$$

Example 3: Spherical capacitor:

Assuming variation with respect to r Laplace equation becomes,

$$\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

Integrating twice on both sides we obtain,

$$\sin\theta\frac{dV}{d\theta} = A,$$

$$V = \int\frac{A\,d\theta}{\sin\theta} + B.$$

Assuming V = V_0 at θ = Π/2 and V= 0 at θ = α, We get

$$V = V_0\frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\alpha}{2}\right)}$$

$$\mathbf{E} = -\nabla V = \frac{1}{r}\frac{\partial V}{\partial\theta}\mathbf{a}_\theta = -\frac{V_0}{r\sin\theta\ln\left(\tan\frac{\alpha}{2}\right)}\mathbf{a}_\theta,$$

$$\rho_S = \frac{-\epsilon V_0}{\mathrm{r}\sin\alpha\ln\left(\tan\frac{\alpha}{2}\right)},$$

$$Q = \frac{-\epsilon V_0}{\sin\alpha\ln\left(\tan\frac{\alpha}{2}\right)}\int_0^\infty\int_0^{2\pi}\frac{r\sin\alpha\,d\phi\,dr}{r}$$

$$= \frac{-2\pi\epsilon_0 V_0}{\ln\left(\tan\frac{\alpha}{2}\right)}\int_0^\infty dr,$$

and

$$C \doteq \frac{2\pi\epsilon r_1}{\ln\left(\cot\frac{\alpha}{2}\right)}$$

.

Outcomes

The students are able to state and derive the poisons and laplace's equation and apply it to derive the capacitance of parallel plate capacitor, cylindrical conductor and spherical ring & show that Laplaces equation has only one solution

Recommended Questions

1. Derive Poisson's & Laplace's equation.

2. Using Laplace's equation , Prove that the potential distribution at any point in the region between two concentric cylinders of radii A & B as
V=Voln ῤ/B /ln A/B

3. State and prove uniqueness theorem

4. Derive for Capacitance of Parallel plate capacitor

5. Derive for Capacitance of Concentric spherical capacitor.

6. Let $V = 2xy^2z^3$ and $\varepsilon = \varepsilon_0$. Given point P(1,2,-1), Find (a) V at P; (b) E at P; (c) ρ_v at P; (d) the equation of the equipotential surface passing through P; (e) the equation of the streamline passing through P; (f) Does V satisfy the Laplaces Equation

Further Reading

TEXT BOOK:
1. Energy Electromagnetics, William H Hayt Jr . and John A Buck, Tata McGraw-Hill, 7th edition,2006.

REFERENCE BOOKS:
2. Electromagnetics with Applications, John Krauss and Daniel A Fleisch McGraw-Hill, 5th edition, 1999
3. Electromagnetic Waves And Radiating Systems, Edward C. Jordan and Keith G Balmain, Prentice – Hall of India / Pearson Education, 2nd edition, 1968.Reprint 2002
4. Field and Wave Electromagnetics, David K Cheng, Pearson Education Asia, 2nd edition, -1989, Indian Reprint – 2001.