MODULE 1 : Coulomb's Law, Electric Field Intensity and Flux Density

1.1   Experimental law of Coulomb

Coulomb's law states that the electrostatic force F between two point charges q1 and q2 is directly proportional to the product of the magnitude of the charges, and inversely proportional to the square of the distance between them., and it acts along the line joining the two charges. Then, as per the Coulomb's Law,

$$F \propto kq1q2$$
$$\text{Or} \quad F = (kq1q2)/(r^2) \text{ N}$$

Where k is the constant of proportionality whose value varies with the system of units. R^ is the unit vector along the line joining the two charges.

In SI unit, k= $\dfrac{1}{4\pi\epsilon_0}$.

Where ϵ_0 is called the permittivity of the free space.

It has an assigned value given as ϵ_0 =8.834$\times 10^{-12}$ F/m.

Force on a point charge:

The forces of attraction/repulsion between two point charges Q_1 and Q_2 (charges whose size is much smaller than the distance between them) are given by Coulomb's law:

$$\mathbf{F}_1 = k \cdot \frac{Q_1 Q_2}{R^2} \mathbf{a}_{21}$$

$$\mathbf{F}_2 = k \cdot \frac{Q_1 Q_2}{R^2} \mathbf{a}_{12}$$

where $k \approx 9 \times 10^9$ m/F in SI units, and R is the distance between the two charges. Here, $\mathbf{F}_1$ is the force exerted on Q_1, and $\mathbf{F}_2$ is the force acting on Q_2. The unit vector $\mathbf{a}_{21}$ points from charge 2 toward charge 1. Accordingly, $\mathbf{a}_{12} = -\mathbf{a}_{21}$.

Force on Q1 is given by

$$F1 = \frac{1}{4\pi\epsilon^0} \frac{Q_1 Q_2}{R^2} \mathbf{a}_{21} \qquad \text{Newtons}$$

Force due to several charges

Let there be many point charges q1,q2,q3.........qn at distances r1,r2,r3.....rn from charge q. The force F1, F2, F3..........Fn at the charges q1,q2,q3...........qn respectively are:

$$q\left\{\frac{q_1}{4\pi\varepsilon r_1}\hat{r} + \frac{q_1}{4\pi\varepsilon r_2}\hat{r}_2 \dots \dots \dots \dots \dots \dots \dots \dots \right\}$$

F=Fq1+Fq2+Fq3...............

Hence, $$F = q\left\{\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i\right\} \text{ N}$$

1.1   Objectives

After going through this section, the students are able
   to 1. State Coulombs law
Application of Coulombs Law to point charge as well as several charges

1.2   Electric field intensity

Electric field intensity at any point in an electric field is the force experienced by positive unit charge placed at that point.

Consider a charge Q located at a point A. At the point B in the electric fields set up by Q, it is required to find the electric field intensity E.

Let the charge at B be Δq and let the charge Q be fixed at A. Let r be the distance between A and B. As per the Coulomb's Law, the force between Q and q is given by:

$$F= Q\Delta q/(4\pi \epsilon^0 r^2) \; \hat{r} \; N$$

If it is a unit positive charge, then by definition, F in the above equation gives the magnitude of the electric field intensity E.

i.e. E=F when $\Delta q = 1.$

Therefore, the magnitude of the electric field
strength is: $E=Q/(4r \pi \epsilon_o^2)$

Let r be the unit vector along the line joining A and B. Thus, the vector relation between E is written as:

$$E=Q/(4\pi \epsilon_o r^2) \; V/m$$

1.2.1   Electric Field intensity due to several charges

Let there be many point charges q1,q2,q3......... qn at distances r1,r2,r3   .....rn be the corresponding unit vectors. The field E1, E2, E3.......... En at the charges q1,q2,q3........... qn respectively are:

$$\hat{r} +$$

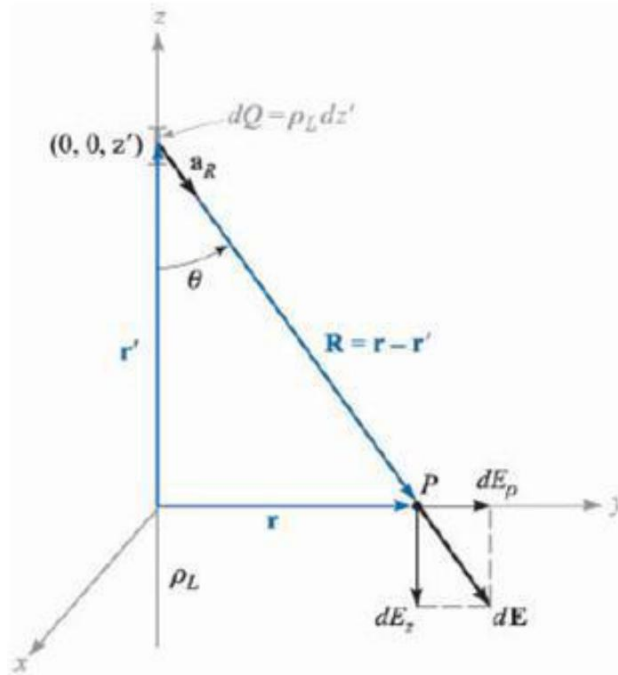E=Eq1+Eq2+Eq3   ....

Hence, $$E=\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{qi}{ri^2}\;\hat{r}i$$

1.2 .2   Electric field intensity at a point due to a infinite sheet of charge

Let us assume a straight line charge extending along Z axis in a cylindrical coordinate system from -∞ to +∞ as shown in the figure 1.1. Consider an incremental length dl at a point on the conductor. The incremental length has an incremental charge of dQ= ρl dl= ρldz' Coulombs.
Considering the charge dQ, the incremental field intensity at point p is given by,

$$dE = \frac{\rho_L dz'(r - r')}{4\pi\epsilon_0 |r - r'|^3}$$

Where

$$r = ya_y = \rho a_\rho$$
$$r' = z'a_z$$

,
and

$$r - r' = \rho a_\rho - z'a_z$$

Therefore,

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

Integrating the above and substituting z'=ρ cot θ, we get

and

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

1.2.3 Electric field intensity at a point due to a infinite sheet of charge:

Let us assume a infinite sheet of charge with surface charge density ρ_S as shown in the figure 1.2. Divide the sheet of charge into differential width strips. number of str Consider an incremental length dl at a point on the conductor. The line charge density $\rho_l = \rho_S dy'$.
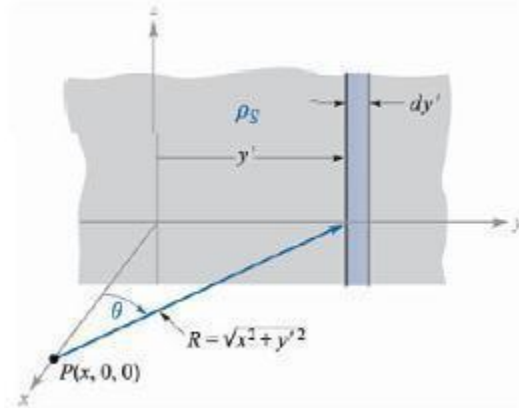
The differential Electric field intensity at point P,

$$dE_x = \frac{\rho_S \, dy'}{2\pi\epsilon_0 \sqrt{x^2 + y'^2}} \cos\theta = \frac{\rho_S}{2\pi\epsilon_0} \frac{x \, dy'}{x^2 + y'^2}$$

adding the effects of all the strips,

$$E_x = \frac{\rho_S}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x \, dy'}{x^2 + y'^2} = \frac{\rho_S}{2\pi\epsilon_0} \tan^{-1}\frac{y'}{x}\Big]_{-\infty}^{\infty} = \frac{\rho_S}{2\epsilon_0}$$

Therefore,

$$E = \frac{\rho_S}{2\epsilon_0} a_N$$

1.2.4 Electric field at a point on the axis of charged circular ring:

Let ρ be the charge density of the ring.

So, $\rho = dq/dl$

                           $dq = \rho dl$

Electric field due to an infinitely small element = $dE = dq/4\pi\varepsilon o\ r^2\ \hat{r}$

where $\hat{r}$ is the unit vector along AP.
dE can resolved into two rectangular components, dEx and dEy. Now, $dEx = dE\cos\theta$.
Taking the magnitude of dE from above, the equation becomes,

$$dEx = \frac{dq\cos\theta}{4\pi\varepsilon^0 r^2}$$

$$\cos\theta = \frac{x}{r}$$

substituting for dq from above, we have;

$$dEx = \frac{\rho h dl}{4\pi\varepsilon r^3}$$

The component dEy is directed downwards. If we consider an element of the ring at a point diametrically opposite to A, then its dEy component points upwards and hence, cancels with that due to element A. The dEx components add up.

$$\int dEy = 0.$$

The total field at P is the sum of the fields due to all the elements of the ring.
Therefore, $E = \int dE = \int dEx + \int dEy = \int dEx$

$$E = \int dEx = \frac{\rho x}{4\pi\varepsilon r^3}\int_0^{2\pi R} dl$$

$$= \frac{\rho x(2\pi R)}{4\pi\varepsilon r^3}$$

But, $r = (R^2 + x^2)^{\frac{1}{2}}$

Therefore, $E = \frac{2\pi R\rho x}{4\pi\varepsilon(R^2+x^2)^{3/2}}\ ax$

Where, a_X is the unit vector along the x axis.

1.2   Objectives

At the end of this section the students are able to
1. Define Electric field Intensity
2. Derive Electric field intensity at a due to several charges
3. Derive Electric field Intensity at a point due to sheet of charge
4. Derive Electric field intensity at a point on the axis of charged circular ring

1.3   Electric flux:

The concept of electric flux is useful in association with Gauss' law. The electric flux through a planar area is defined as the electric field times the component of the area perpendicular to the field. If the area is not planar, then the evaluation of the flux generally requires an area integral since the angle will be continually changing.

When the area A is used in a vector operation like this, it is understood that the magnitude of the vector is equal to the area and the direction of the vector is perpendicular to the area.
        Consider a concentric sphere having radius of 'a'm charged up to +Q C. This sphere is then placed in another sphere having a radius of 'b' m as shown in the figure 1.4.
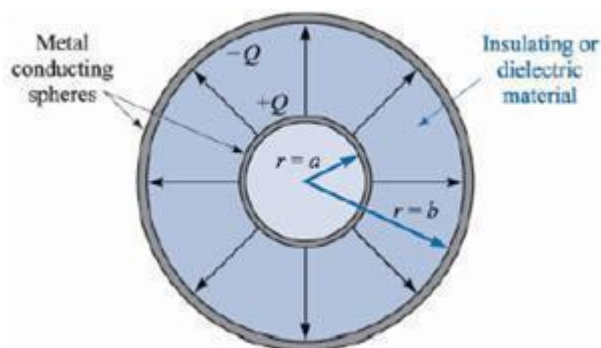
There is no electrical connection between them. The outer sphere is momentarily charged, then it found that the charge on the outer sphere is equal to the charge on the inner sphere. This is depicted by the radial lines. This is referred as displacement flux. Therefore,
$$\Psi = Q.$$

1.3.1   Electric flux density:

If +Q C of charge on the inner sphere produces the electric flux of ψ, tthen electric flux ψ uniformly distributed over the surface area $4\Pi a^2$ m^2 , where a is the radius of the inner sphere.

The electric flux density si given by

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

Similarly for the outer sphere,

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

If the inner sphere becomes smaller and smaller retaining a charge of Q C, it becomes a point charge. The flux density at appoint 'r' from the point charge is given by,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

The electric field intensity due to point charge in free space is given by,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

Therefore in free space,

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

1.3   Objective

After going through this section the students should be able to
1. Define Electric flux
2. Explain Electric flux density

1.4   Gauss law:

The Gauss's law states that. "The electric flux passing through any closed surface is equal to the total charge enclosed by the surface"

For the Gaussian-surface shown in the following figure, the Gauss' law can be expressed mathematically, .

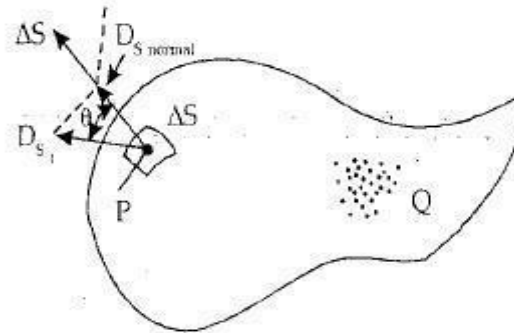$$\psi = \oint \vec{D}_s \cdot d\vec{s} = Q$$

Where

Ψ = flux passing through the closed surface

§s =1 surface integral

Ds =, flux density (vector quantity) normal to the
surface Q = Total charge enclosed in the surface
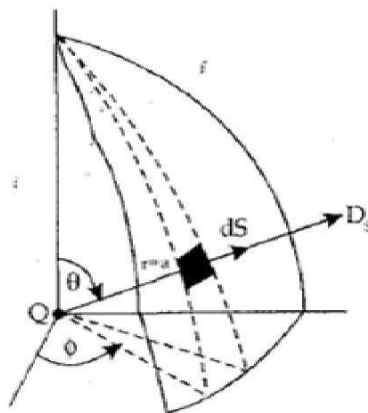

Gauss law for charge Q enclosed in a closed surface:

Let Q be the point charge placed at the origin of imaginary sphere in spherical co-
ordinate system with a radius of "a" as illustrated in the figure
The electrical field intensity cf the point charge is found to be equal to

$$\vec{E} = \frac{Q}{4\pi \epsilon_0 r^2} \vec{a}_r \qquad -----(1)$$

Where   r = Cl
 and we al so know that the relation between E and D as,

$$\vec{D} = \epsilon_0 E \qquad -----(2)$$

Therefore from (1) and (2) we get.

$$\vec{D} = \mathcal{E}_0 \frac{Q}{4\pi \mathcal{E}_0 \, r^2} \vec{a}_r$$

$$= \frac{Q}{4\pi a^2} \vec{a}_r$$

at the surface of the sphere,

$$\vec{D} = \frac{Q}{4\pi a^2} \vec{a}_r$$

The differential element of area on a spherical surface is, in spherical coordinate form is given by,

$$ds = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

Or $\quad d\vec{s} = a^2 \sin\theta \, d\theta \, d\phi \, \vec{a}_r$

Then the required integrand

$$= \vec{D}_s \cdot d\vec{s}$$

$$= \frac{Q}{4\pi a^2} \cdot a^2 \sin\theta \, d\theta \, d\phi \, \vec{a}_r \cdot \vec{a}_r$$

$$= \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi \quad (\because \vec{a}_r \cdot \vec{a}_r = 1 \text{ from vector basics})$$

Then the integration over the surface as required for Gauss' law.

$$\oint_S \vec{D}_s \cdot d\vec{s} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

The limits placed for integral indicate that the integration over the entire sphere in spherical co-ordinate system on integration we get

$$= \int_0^{2\pi} \frac{Q}{4\pi} (-\cos\theta)_0^\pi \, d\phi$$

$$= \int_0^{2\pi} \frac{Q}{2\pi} \, d\phi$$

$$= Q$$

Thus we get, comparing LHS of Gauss' law as

$$\boxed{\psi = Q}$$

This indicates that, Q coulombs of electric flux are crossing the surface as the enclosed charge is Q coulombs.

1.4 .1 Application of Gauss law:

In case of asymmetry, we need to choose a very closed surface such that D is almost constant over the surface. Consider any point P shown in the figure 1.6 located in the rectangular co-ordinate system.
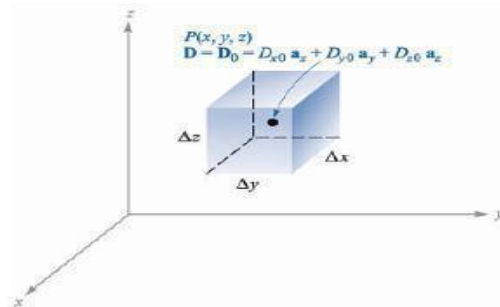
The value of D at point P, may be expressed in rectangular components as,
D=Dx0ax+Dy0ay+Dz0az. . From Gauss law, we have

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = Q}$$

In order to evaluate the integral over the closed surface, the integral must be broken into six integrals, one over each surface,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{front} + \int_{back} + \int_{right} + \int_{top} + \int_{bottom}$$

The surface element is very small & hence D is essentially constant ,

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

,

Similarly,

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

and,

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

.

Therefore collectively,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

or

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

Charge enclosed in volume Δv,

$$\boxed{\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v}$$

1.4   Objectives

At the end of this section the students are able to
1.   State and prove Gauss Law
2.   Apply Gauss law to find the charge enclosed in differential volume

1.5   Divergence:

From Gauss law, we know that,

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \doteq \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

And applying limits,

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\lim_{\Delta v\to 0}\frac{Q}{\Delta v}$$

The last term in the equation is the volume charge density, ρv.

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\rho_v$$

We shall write it as two separate equations,

And

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}$$

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\rho_v$$

Divergence is defined as,

$$\boxed{\text{div }\mathbf{D}=\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)}$$

$$\boxed{\text{Divergence of }\mathbf{A}=\text{div }\mathbf{A}=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{A}\cdot d\mathbf{S}}{\Delta v}}$$

Statement: The flux crossing the closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume, as the volume shrinks to zero.

Divergence in Cartesian system,

$$\boxed{\text{div }\mathbf{D}=\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\qquad\text{(cartesian)}}$$

Divergence in Cylindrical system,

$$\boxed{\text{div }\mathbf{D}=\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho)+\frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi}+\frac{\partial D_z}{\partial z}\qquad\text{(cylindrical)}}$$

Divergence in Spherical system,

$$\boxed{\text{div }\mathbf{D}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta)+\frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}\qquad\text{(spherical)}}$$

1.6.1 Maxwell's First equation:

From divergence theorem, we have

$$\text{div } \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v}$$

$$\text{div } \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\text{div } \mathbf{D} = \rho_v$$

From Gauss law,

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = Q$$

Per unit volume,

$$\frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \frac{Q}{\Delta v}$$

As the volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

Therefore,   div D = ρv.

1.6.2 Divergence theorem:

The del operator is defined as a vector operator.

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$
.

In Cartesian coordinate system,

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}(D_z)$$

Which is equal to,

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$
.

Therefore,

$$\text{div } \mathbf{D} = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

.

From Gauss law, we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

And by letting,

$$Q = \int_{vol} \rho_v \, dv \quad \& \quad \nabla \cdot \mathbf{D} = \rho_v$$
.

Hence we have,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{vol} \nabla \cdot \mathbf{D} \, dv$$

1.6   Objectives

At the end of this section the students are able to

1. Explain the concept of divergence

2. Derive Maxwell's First Equation

3. State and prove Divergence theorem

4.

5.

1.7   Recommended Questions

1. State Coulomb's law of force between any 2 point charges & indicate the units of the quantities involved.

2. Derive the general expression for electric field vector due to infinite line charge using Gauss law.

3. State and prove Gauss law.

4. Derive the general expression for E at a height h(h<a) , along the axis of the ring charge & normal to its plane.

5. From gauss law show that   .D=σv

6. State and prove divergence theorem for symmetric condition.

7. State and prove divergence theorem for asymmetric condition

1.8   Further Readings

1. Energy Electromagnetics, William H Hayt Jr. and John A Buck, Tata McGraw-Hill, 7th edition,2006.

2. Electromagnetics with Applications, John Krauss and Daniel A Fleisch McGraw-Hill, 5th edition, 1999

3. Electromagnetic Waves And Radiating Systems, Edward C. Jordan and Keith G Balmain, Prentice – Hall of India / Pearson Education, 2nd edition, 1968.Reprint 2002

4. Field and Wave Electromagnetics, David K Cheng, Pearson Education Asia, 2nd edition, 1989, Indian Reprint – 2001