Fourier representation for signals – 2

Fourier representation for signals – 2: Discrete and continuous Fourier transforms(derivations of transforms are excluded) and their properties.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

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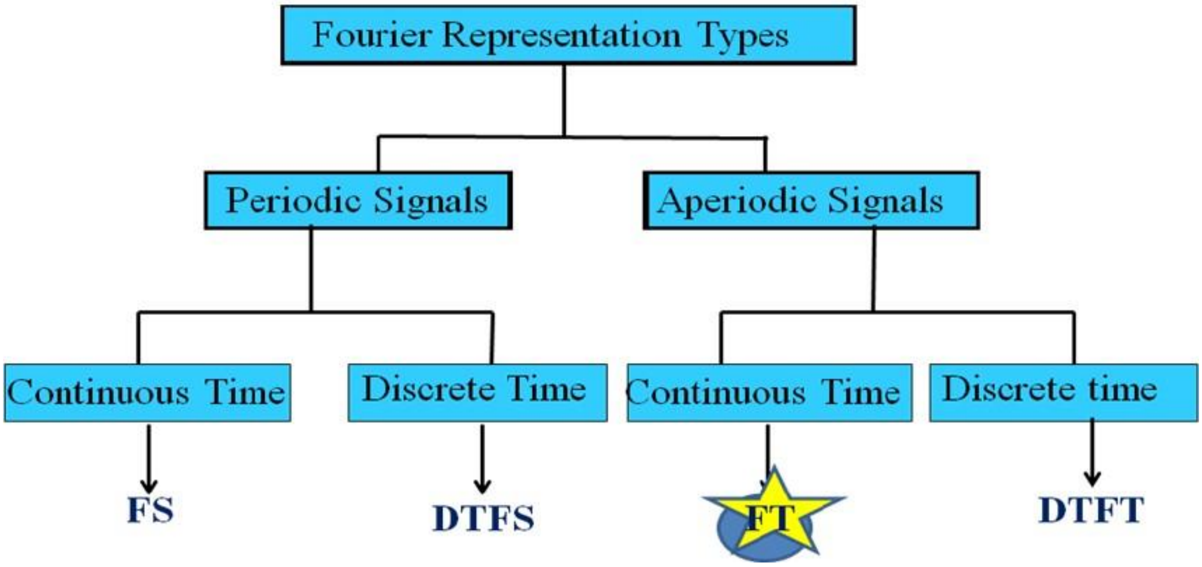
Fourier representation for signals

<u>Introduction:</u>

Fourier Representation for four Signal Classes



5.1 The Fourier transform

5.1.1 From Discrete Fourier Series to Fourier Transform:
Let ***x [n]*** be a nonperiodic sequence of finite duration. That is, for some positive integer ***N ,***

$$x[n] = 0 \qquad |n| > N_1$$

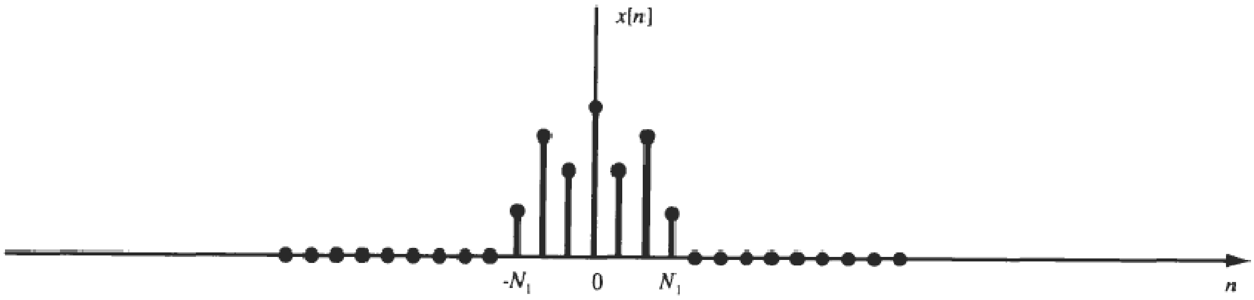
Such a sequence is shown in Fig. ***6-l(a).*** Let ***x,Jn]*** be a periodic sequence formed by repeating ***x [n]*** with fundamental period ***No*** as shown in Fig. 6-l(b). If we let ***No*** -, m, we have

$$\lim_{N_0 \to \infty} x_{N_0}[n] = x[n]$$

The discrete Fourier series of ***xNo[n]*** is given ***by***

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

$$c_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x_{N_0}[n]\, e^{-jk\Omega_0 n}$$



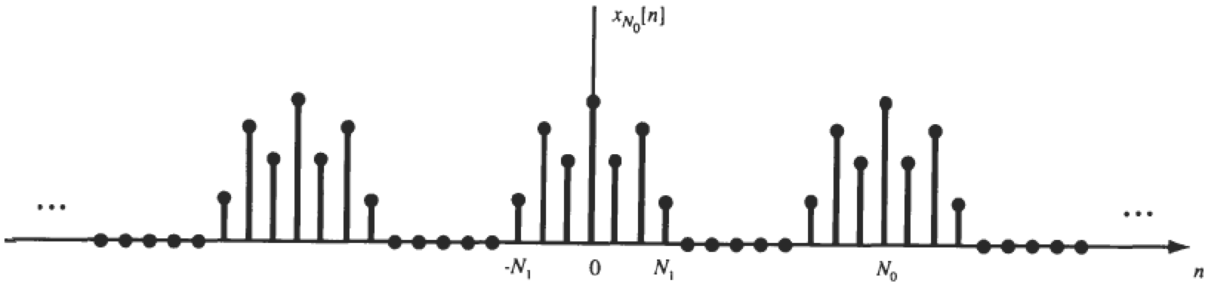


Fig. 6-1 (*a*) Nonperiodic finite sequence $x[n]$; (*b*) periodic sequence formed by periodic extension of $x[n]$.

$$c_k = \frac{1}{N_0} \sum_{n=-N_1}^{N_1} x[n]\, e^{-jk\Omega_0 n} = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\Omega_0 n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

the Fourier coefficients c_k can be expressed as

$$c_k = \frac{1}{N_0} X(k\Omega_0)$$

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} \frac{1}{N_0} X(k\Omega_0)\, e^{jk\Omega_0 n}$$

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=\langle N_0 \rangle} X(k\Omega_0)\, e^{jk\Omega_0 n}\Omega_0$$

Properties of the Fourier transform

Periodicity

As a consequence of Eq. (6.41), in the discrete-time case we have to consider values of R(radians) only over the range0 < Ω < 2π or π < Ω < π**,** while in the continuous-time case we have to consider values of ***0*** (radians/second) over the entire range –∞ < ω < ∞.

$$X(\Omega + 2\pi) = X(\Omega)$$

Linearity:

$$a_1 x_1[n] + a_2 x_2[n] \longleftrightarrow a_1 X_1(\Omega) + a_2 X_2(\Omega)$$

Time Shifting:

$$x[n-n_0] \leftrightarrow e^{-j\Omega n_0}X(\Omega)$$

Frequency Shifting:

$$e^{j\Omega_0 n}x[n] \leftrightarrow X(\Omega-\Omega_0)$$

Conjugation:

$$x^*[n] \leftrightarrow X^*(-\Omega)$$

Time Reversal:

$$x[-n] \leftrightarrow X(-\Omega)$$

Time Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Duality:

The duality property of a continuous-time Fourier transform is expressed as

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

There is no discrete-time counterpart of this property. However, there is a duality between the discrete-time Fourier transform and the continuous-time Fourier series. Let

$$x[n] \leftrightarrow X(\Omega)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$X(\Omega+2\pi) = X(\Omega)$$

Since Ω is a continuous variable, letting $\Omega = t$ and $n = -k$

$$X(t) = \sum_{k=-\infty}^{\infty} x[-k]\, e^{jkt}$$

Since X(t) is periodic with period To = 2 π and the fundamental frequency $\omega_0 = 2\pi/T_0 = 1$, Equation indicates that the Fourier series coefficients of X(t) will be x [- k] . This duality relationship is denoted by

$$X(t) \xleftrightarrow{\text{FS}} c_k = x[-k]$$

where FS denotes the Fourier series and ***c,*** are its Fourier coefficients.

Differentiation in Frequency:

$$nx[n] \leftrightarrow j\frac{dX(\Omega)}{d\Omega}$$

Differencing:

$$x[n]-x[n-1] \leftrightarrow (1-e^{-j\Omega})X(\Omega)$$

The sequence x[n] -x[n – 1] is called the first difference sequence. Equation is easily obtained from the linearity property and the time-shifting property .

Accumulation:

$$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \pi X(0)\,\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}X(\Omega) \qquad |\Omega| \le \pi$$

Note that accumulation is the discrete-time counterpart of integration. The impulse term on the right-hand side of Eq. ***(6.57)*** reflects the dc or average value that can result from the accumulation.

Convolution:

$$x_1[n] * x_2[n] \leftrightarrow X_1(\Omega)X_2(\Omega)$$

As in the case of the z-transform, this convolution property plays an important role in the study of discrete-time LTI systems.

Multiplication:

$$x_1[n]x_2[n] \leftrightarrow \frac{1}{2\pi}X_1(\Omega)\otimes X_2(\Omega)$$

where @ denotes the periodic convolution defined by

$$X_1(\Omega)\otimes X_2(\Omega)=\int_{2\pi} X_1(\theta)X_2(\Omega-\theta)\,d\theta$$

The multiplication property (6.59) is the dual property of Eq. (6.58).

Parseval's Relations:

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n]=\frac{1}{2\pi}\int_{2\pi} X_1(\Omega)X_2(-\Omega)\,d\Omega$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2=\frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2\,d\Omega$$

Summary

Property	$x(t), y(t)$	$X(j\omega), Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{j\omega}{a})$
Convolution	$x(t) ** y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$X(j\omega) ** Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$

Recommended Questions

1. Obtain the Fourier transform of the signal e $^{-at}$. u(t) and plot spectrum.

2. Determine the DTFT of unit step sequence x(n) = u(n) its magnitude and phase.

3. The system produces the output of yet) = e^{-t} u(t), for an input of x(t) = e-2t.u(t). Determine impulse response and frequency response of the system.
4. The input and the output of a causal LTI system are related by differential equation
   $\frac{d2y(t)}{dt2}+ \frac{6dy(t)}{dt}+ 8y(t)= 2x(t)$
   i) Find the impulse response of this system
   ii) What is the response of this system if x(t) = te^{-at} u(t)?

5. Discuss the effects of a time shift and a frequency shift on the Fourier representation.

6. Use the equation describing the DTFT representation to determine the time-domain signals corresponding to the following DTFTs :
   i) $X(e^{j\Omega})$= Cos(Ω)+j Sin(Ω)
   ii) $X(e^{j\Omega})$={1, for π/2<Ω< π; 0 otherwise and $X(e^{j\Omega})$=-4 Ω

7. Use the defining equation for the FT to evaluate the frequency-domain representations for the following signals:
   i) X(t)= e^{-3t}u(t-1)
   ii) X(t)=e^{-t} Sketch the magnitude and phase spectra.

8. Show that the real and odd continuous time non periodic signal has purely imaginary Fourier transform. (4 Marks)

Fourier Series and LTI System

- Fourier series representation can be used to construct any periodic signals in discrete as well as continuous-time signals of practical importance.
- We have also seen the response of an LTI system to a linear combination of complex exponentials taking a simple form.
- Now, let us see how Fourier representation is used to analyze the response of LTI System.

Consider the CTFS synthesis equation for x(t) given by
Suppose we apply this signal as an input to an LTI System with impulse respose h(t). Then, since each of the complex exponentials in the expression is an eigen function of the system. Then, with $sk = jk\omega_o$, it follows that the output is

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(e^{jk\omega o})e^{jk\omega ot}$$

Thus y(t) is periodic with frequency as x(t). Further, if ak is the set of Fourier series coefficients for the input x(t), then $\{a_k H(e^{jk\omega o})\}$ is the set of coefficient for the y(t). Hence in LTI, modify each of the Fourier coefficient of the input by multiplying by the frequency response at the corresponding frequency.

Example:

Consider a periodic signal x(t), with fundamental frequency 2π, that is expressed in the form

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t} \quad \text{.......(1)}$$

$$where, \quad a_0=1, \quad a_1=a_{-1}=1/4\ , \ a_2=a_{-2}=1/2, \ a_3=a_{-3}=1/3,$$

Suppose that the this periodic signal is input to an LTI system with impulse response To calculate the FS Coeff. Of o/p y(t), lets compute the frequency response.The impulse response is therefore,

$$H(j\omega) = \int_0^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau \qquad = -\frac{1}{1+j\omega} e^{-\tau}e^{-j\omega\tau}\Big|_0^{\infty}$$

and

$$H(j\omega) = \frac{1}{1+j\omega}$$

Y(t) at ωo = 2π . We obtain,

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t} \quad with\ b_k = a_k H(jk2\pi), so\ that$$

$$b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right) b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right) b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right)$$

$$b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right) b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right) \quad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right)$$

$$b_0 = 1$$

The above o/p coefficients. Could be substituted in

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}$$

Finding the Frequency Response

We can begin to take advantage of this way of finding the output for any input once we have H(ω).
To find the frequency response H(ω) for a system, we can:

1. Put the input $x(t) = e^{i\omega t}$ into the system definition
2. Put in the corresponding output $y(t) = H(\omega)\, e^{i\omega t}$
3. Solve for the frequency response H(ω). (The terms depending on t will cancel.)

Example:

Consider a system with impulse response

$$h(t) = \begin{cases} \frac{1}{5} & \text{for } t \in [0,5] \\ 0 & \text{otherwise} \end{cases}$$

Find the output corresponding to the input x(t) = cos(10 t).

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{\tau=0}^{5} \frac{1}{5} \cos(10(t-\tau))\, d\tau$$

$$y(t) = \frac{1}{5}\left(-\frac{1}{10}\sin(10(t-\tau))\right)\Bigg|_{\tau=0}^{5} = \frac{1}{50}\left(\sin(10t) - \sin(10(t-5))\right)$$

Differential and Difference Equation Descriptions

Frequency Response is the system's steady state response to a sinusoid. In contrast to differential and difference-equation descriptions for a system, the frequency response description cannot represent initial conditions, it can only describe a system in a steady state condition. The differential-equation representation for a continuous-time system is

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{N} b_k \frac{d^k}{dt^k} x(t)$$

$$since, \frac{d}{dt} g(t) \xleftrightarrow{FT} j\omega G(j\omega)$$

Rearranging the equation we get

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

The frequency of the response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Hence, the equation implies the frequency response of a system described by a linear constant-coefficient differential equation is a ratio of polynomials in jω.

The difference equation representation for a discrete-time system is of the form.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Take the DTFT of both sides of this equation, using the time-shift property.

$$g[n-k] \xleftrightarrow{DTFT} e^{-jk\omega} G(e^{j\omega})$$

To obtain

$$\sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k Y\left(e^{j\omega}\right) = \sum_{k=0}^{N} a_k \left(e^{-j\omega}\right)^k X\left(e^{j\omega}\right)$$

- Rewrite this equation as the ratio

$$\frac{Y\left(e^{j\omega}\right)}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k \left(e^{j\omega}\right)^k}{\sum_{k=0}^{N} a_k \left(e^{j\omega}\right)^k}$$

- The frequency response is the polynomial in $e^{j\omega}$

$$H(e^{j\omega}) = \frac{Y\left(e^{j\omega}\right)}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k \left(e^{j\omega}\right)^k}{\sum_{k=0}^{N} a_k \left(e^{j\omega}\right)^k}$$

Differential Equation Descriptions

Ex: Solve the following differential Eqn using FT.

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 5y(t) = 3\frac{d}{dt}x(t) + x(t)$$

For all t where, $x(t) = (1 + e^{-t})u(t)$

Soln:we have

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 5y(t) = 3\frac{d}{dt}x(t) + x(t)$$

FT gives,

$$[(j\omega)^2 + 4(j\omega) + 5]Y(j\omega) = (3j\omega + 1)X(j\omega)$$

$$and\ x(t) = (1+e^{-t})u(t) \qquad x(t) = u(t) + (e^{-t})u(t)$$

$$X(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega+1)}\ since\ u(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}$$

$$and (e^{-t})u(t) \xleftrightarrow{FT} \frac{1}{j\omega+1}$$

Hence we have
$$X(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega+1)}$$

And $[(j\omega)^2 + 4(j\omega) + 5]Y(j\omega) = (3j\omega+1)X(j\omega)$

i.e

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega)^2 + 4(j\omega) + 5]}X(j\omega)$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]}\left[\frac{1}{j\omega} + \pi\delta(\omega) + \frac{1}{(j\omega+1)}\right]$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega)^2 + 4(j\omega) + 5]}\left[\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{(j\omega+1)}\right]$$

$$\boxed{Y(j\omega) = Y(1) + Y(2) + Y(3)}$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]j\omega} + \frac{\pi}{5}\delta(\omega) + \frac{(3j\omega+1)}{[(j\omega+2)^2+1](j\omega+1)}$$

$$Y(j\omega) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]j\omega} + \frac{(3j(\omega=0)+1)\pi[\delta(0)=1]}{[(j(\omega=0)+2)^2+1]j(\omega=0)} + \frac{(3j\omega+1)}{[(j\omega+2)^2+1](j\omega+1)}$$

$$Y(1) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1]j\omega} Y(1) = \frac{A}{j\omega} + \frac{Bj\omega + C}{[(j\omega+2)^2+1]}$$

$Performing\ partial\ fraction\ we\ get$ $A = \frac{1}{5}, B = -\frac{1}{5}, C = \frac{11}{5}$

$$\boxed{Y(1) = \frac{1/5}{j\omega} + \frac{-1/5 j\omega + 11/5}{[(j\omega+2)^2+1]}}$$

$Similarly$

$$Y(3) = \frac{(3j\omega+1)}{[(j\omega+2)^2+1](j\omega+1)}$$

$$Y(3) = \frac{R}{(j\omega+1)} + \frac{Pj\omega + Q}{[(j\omega+2)^2+1]}$$

$Performing\ partial\ fraction\ we\ get$ $R = -1, P = 1, Q = 6$

$$\boxed{Y(3) = \frac{-1}{(j\omega+1)} + \frac{j\omega+6}{[(j\omega+2)^2+1]}}$$

$$Y(3) = \frac{-1}{(j\omega+1)} + \frac{j\omega+6}{[(j\omega+2)^2+1]} \quad Y(j\omega) = Y(1) + Y(2) + Y(3)$$

Hence, we have

$$Y(1) = \frac{1/5}{j\omega} + \frac{-1/5j\omega + 11/5}{[(j\omega+2)^2+1]}$$

$$Y(2) = \frac{\pi}{5}\delta(\omega)$$

Readjusting

$$Y(j\omega) = \frac{1/5}{j\omega} + \frac{-1/5j\omega+11/5}{[(j\omega+2)^2+1]} + \frac{\pi}{5}\delta(\omega) + \frac{-1}{(j\omega+1)} + \frac{j\omega+6}{[(j\omega+2)^2+1]}$$

$$\boxed{Y(j\omega) = \frac{1}{5}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{(j\omega+1)} + \frac{1}{5}\left[\frac{4j\omega+41}{[(j\omega+2)^2+1]}\right]}$$

$$Y(j\omega) = \frac{1/5}{j\omega} + \frac{\pi}{5}\delta(\omega) + \frac{11/5 - 1/5j\omega}{[(j\omega+2)^2+1]} + \frac{j\omega+6}{[(j\omega+2)^2+1]} - \frac{1}{(j\omega+1)}$$

we know that,

$$e^{-\beta t}\cos\omega_o t u(t) \xleftrightarrow{FT} \frac{\beta + j\omega}{[(\beta+j\omega)^2 + \omega_o{}^2]}$$

$$e^{-\beta t}\sin\omega_o t u(t) \xleftrightarrow{FT} \frac{\omega_o}{[(\beta+j\omega)^2 + \omega_o{}^2]}$$

Readjusting the last term, we get

$$Y(j\omega) = \frac{1}{5}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{(j\omega+1)} + \frac{4}{5}\left[\frac{j\omega+2}{[(j\omega+2)^2+1]}\right] + \frac{33}{5}\left[\frac{1}{[(j\omega+2)^2+1]}\right]$$

Now, taking the inverse Fourier Transform, we get

$$\boxed{y(t) = \frac{1}{5}u(t) - e^{-t}u(t) + \frac{4}{5}e^{-2t}\cos t\, u(t) + \frac{33}{5}e^{-2t}\sin t\, u(t)}$$

Differential Equation Descriptions

- Ex: Find the frequency response and impulse response of the system described by the differential equation.

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 2\frac{d}{dt}x(t) + x(t)$$

Here we have N=2, M=1. Substituting the coefficients of this differential equation in

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Differential Equation Descriptions

We obtain

$$H(j\omega) = \frac{2j\omega + 1}{(j\omega)^2 + 3j\omega + 2}$$

The impulse response is given by the inverse FT of H(jω). Rewrite H(jω) using the partial fraction expansion.

$$H(j\omega) = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 2}$$

Solving for A and B we get, A=-1 and B=3. Hence

$$H(j\omega) = \frac{-1}{j\omega + 1} + \frac{3}{j\omega + 2}$$

The inverse FT gives the impulse response

$$\boxed{h(t) = 3e^{-2t}u(t) - e^{-t}u(t)}$$

Difference Equation

Ex: Consider an LTI system characterized by the following second order linear constant coefficient difference equation.

$$y[n] = 1.3433y[n-1] - 0.9025y[n-2] + x[n] - 1.4142x[n-1] + x[n-2]$$

Find the frequency response of the system.

Soln:

$$y[n] = 1.3433y[n-1] - 0.9025y[n-2] + x[n] - 1.4142x[n-1] + x[n-2]$$

$$Y(e^{j\omega}) = 1.3433(e^{-j\omega})Y(e^{j\omega}) - 0.9025(e^{-j2\omega})Y(e^{j\omega}) + X(e^{j\omega}) - 1.4142(e^{-j\omega})X(e^{j\omega}) + (e^{-j2\omega})X(e^{j\omega})$$

$$we\ know, y[n-k] \xleftrightarrow{DTFT} e^{-jk\omega}Y(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$= \frac{1 - 1.4142e^{-j\omega} + e^{-j2\omega}}{1 - 1.3433e^{-j\omega} + 0.9025e^{-j2\omega}}$$

Ex: If the unit impulse response of an LTI System is h(n)=α^nu[n], find the response of the system to an input defined by $x[n] = \beta^n u[n]$, where $\beta, \alpha < 1$ *and* $\alpha \neq \beta$

Soln:

$$y[n] = h[n] * x[n]$$

Taking DTFT on both sides of the equation, we get

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \qquad Y(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \text{x} \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \text{x} \frac{1}{1 - \beta e^{-j\omega}} = \frac{A}{1 - \alpha e^{-j\omega}} \text{x} \frac{B}{1 - \beta e^{-j\omega}}$$

where A and B are constants to be found by using partial fractions

$$Let, \quad e^{-j\omega} = v \qquad Then, Y(e^{j\omega}) = \frac{A}{1 - \alpha v} \text{x} \frac{B}{1 - \beta v}$$

By performing partial fractions, we get $A = \frac{\alpha}{\alpha - \beta}$, B$= \frac{-\beta}{\alpha - \beta}$

$$Therefore, Y(e^{j\omega}) = \frac{\frac{\alpha}{\alpha - \beta}}{1 - \alpha\, e^{-j\omega}} \text{x} \frac{\frac{-\beta}{\alpha - \beta}}{1 - \beta\, e^{-j\omega}}$$

Taking inverse DTFT, we get

$$\boxed{y[n] = \left[\frac{\alpha}{\alpha - \beta}\alpha^n - \frac{\beta}{\alpha - \beta}\alpha^n\right] u[n]}$$

Sampling

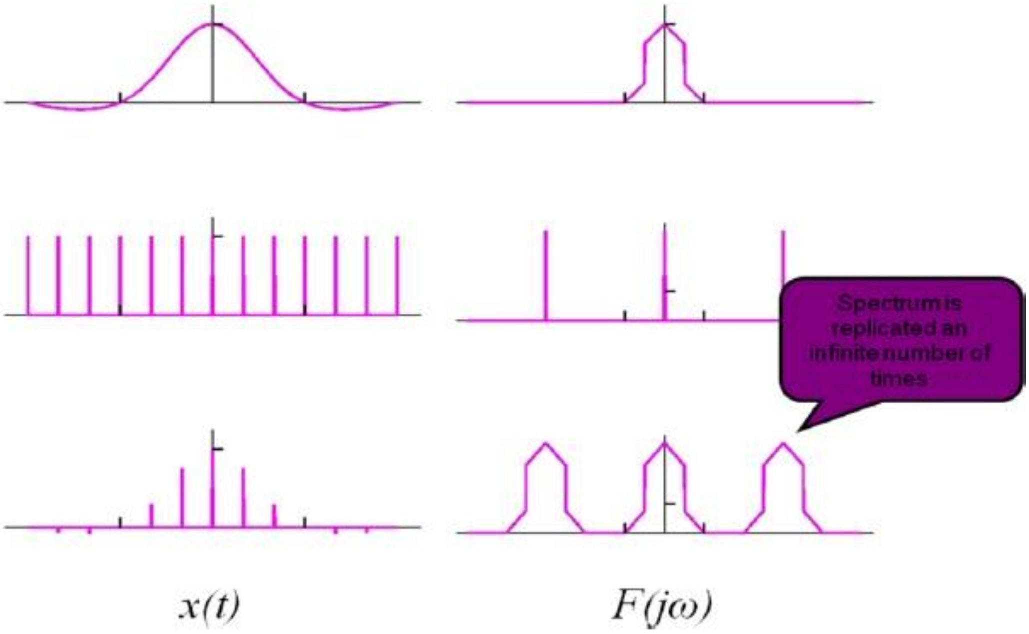
In this chapter let us understand the meaning of sampling and which are the different methods of sampling. There are the two types. Sampling Continuous-time signals and **Sub-sampling**. In this again we have *Sampling Discrete-time signals*.

Sampling Continuous-time signals

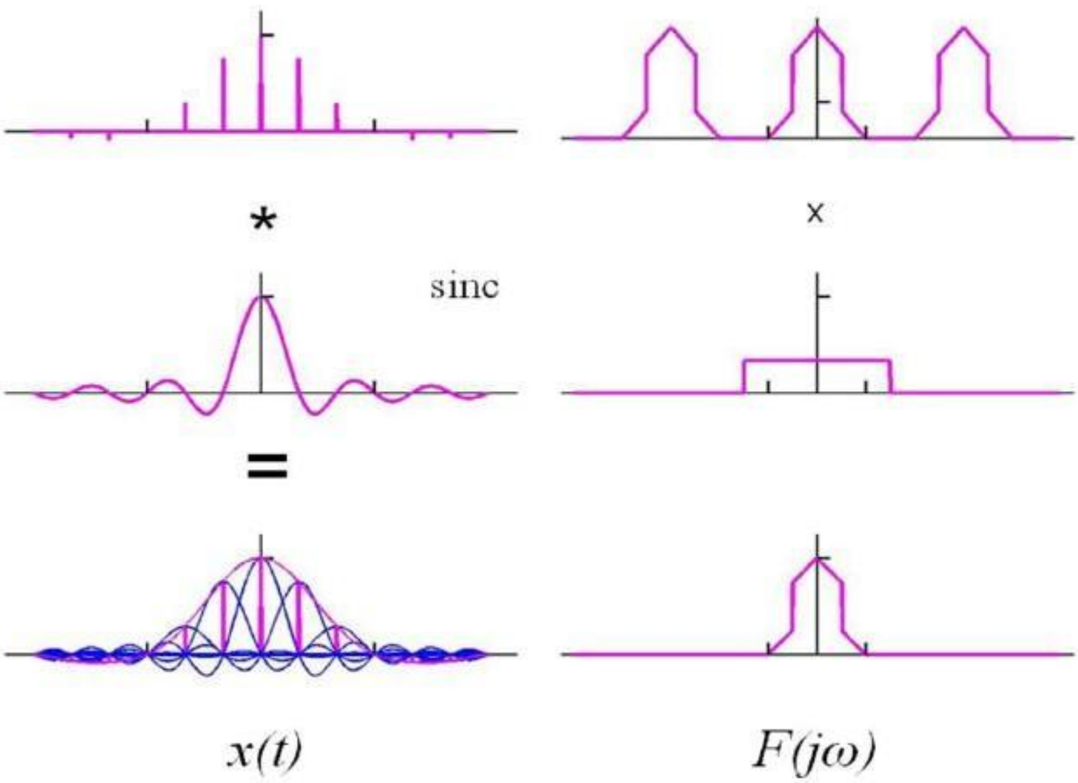
Sampling of continuous-time signals is performed to process the signal using digital processors. The sampling operation generates a discrete-time signal from a continuous-time signal.DTFT is used to analyze the effects of uniformly sampling a signal.Let us see, how a DTFT of a sampled signal is related to FT of the continuous-time signal.

- Sampling: Spatial Domain: A continuous signal x(t) is measured at fixed instances spaced apart by an interval 'T'. The data points so obtained form a discrete signal x[n]=x[nT]. Here, ΔT is the sampling period and 1/ ΔT is the sampling frequency.Hence, sampling is the multiplication of the signal with an impulse signal.

- **Sampling theory**



- **Reconstruction theory**



Sampling: Spatial Domain

From the Figure we can see

Where x[n] is equal to the samples of x(t) at integer multiples of a sampling interval T

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n)\delta(t - n\tau)$$

Now substitute $x(nT)$ *for* $x[n]$ *to obtain*

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n\tau)\,\delta(t - n\tau)$$

$$since\ x(t)\delta(t - n\tau) = x(n\tau)\delta(t - n\tau)$$

we may rewrite $x_\delta(t)$ *as a product of time functions*

$$x_\delta(t) = x(t)p(t) \qquad where, \qquad p(t) = \delta(t - n\tau)$$

Hence, Sampling is the multiplication of the signal with an impulse train.

The effect of sampling is determined by relating the FT of $x_\delta(t)$ to the FT of $x(t)$. Since Multiplication in the time domain corresponds to convolution in the frequency domain, we have

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

Substituting the value of $P(j\omega)$ as the FT of the pulse train i.e

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

We get,

$$P(j\omega) = \frac{2\pi}{\tau} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

$$where, \omega_s = \frac{2\pi}{\tau}, is\ the\ sampling\ frequency. Now$$

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{\tau} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

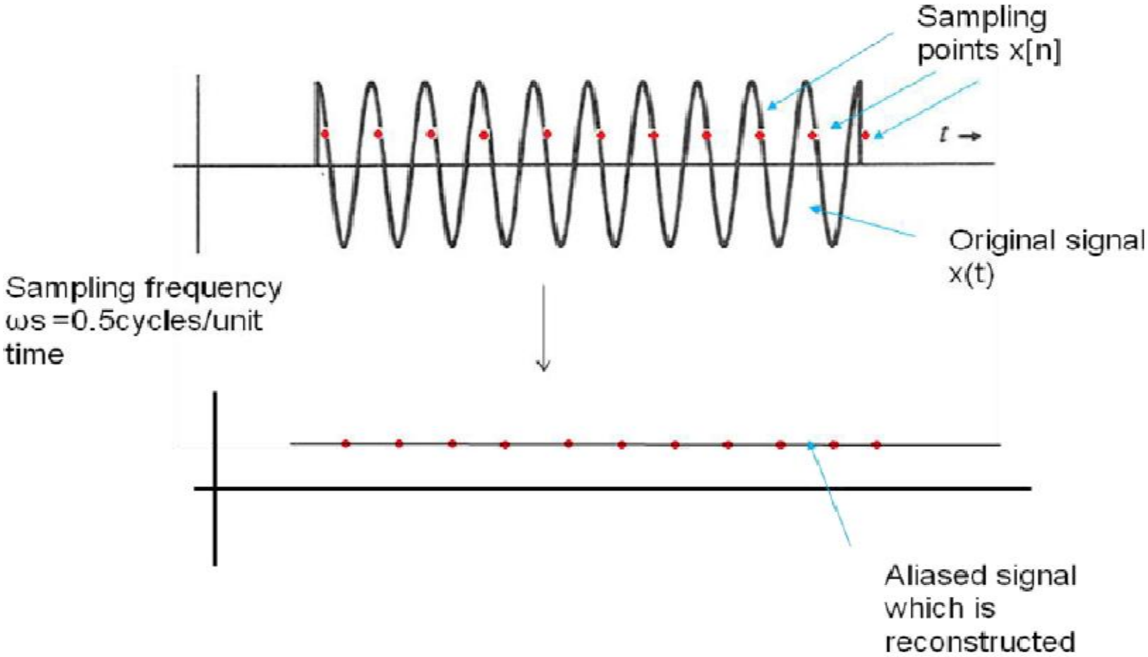
$$\boxed{X_\delta(j\omega) = \frac{1}{\tau} \sum_{n=-\infty}^{+\infty} X(j(\omega - k\omega_s))}$$

The FT of the sampled signal is given by an infinite sum of shifted version of the original signals **FT** and the offsets are integer multiples of ω_s.

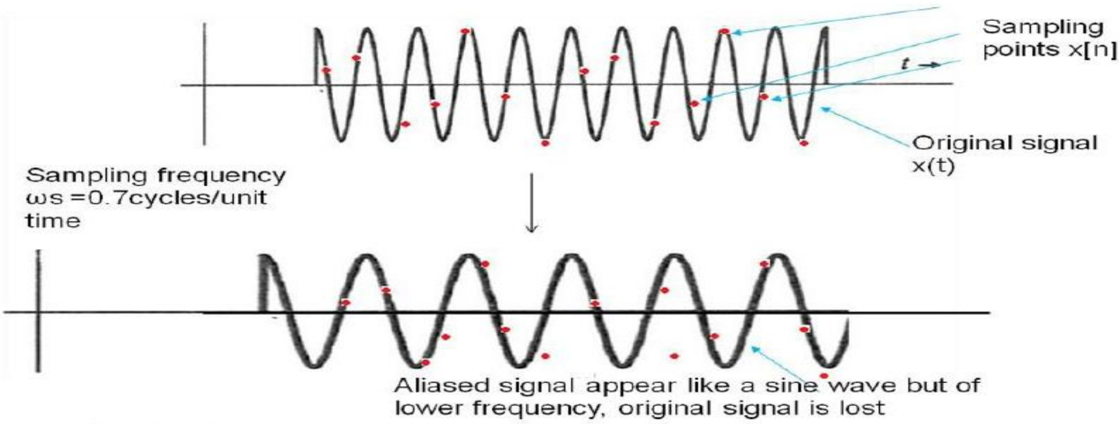
Aliasing : an example

Frequency of original signal is 0.5 oscillations per time unit). Sampling frequency is also 0.5 oscillations per time unit). Original signal cannot be recovered.

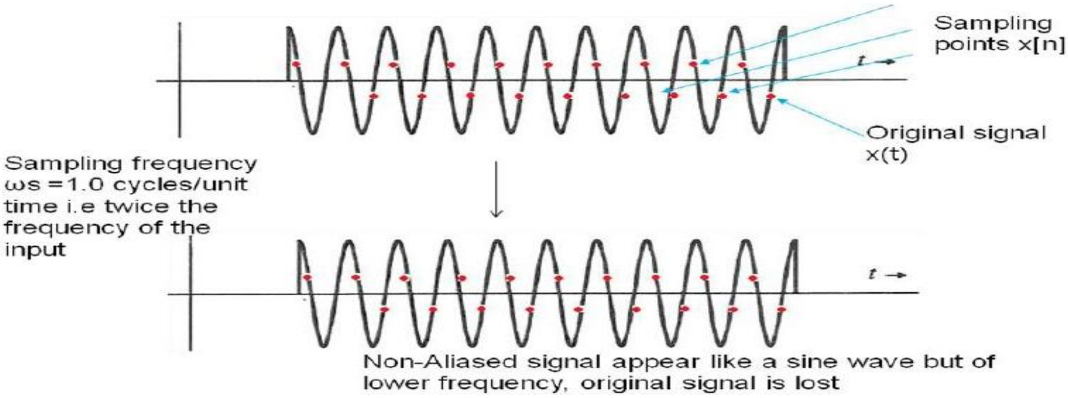
Aliasing Ex:1



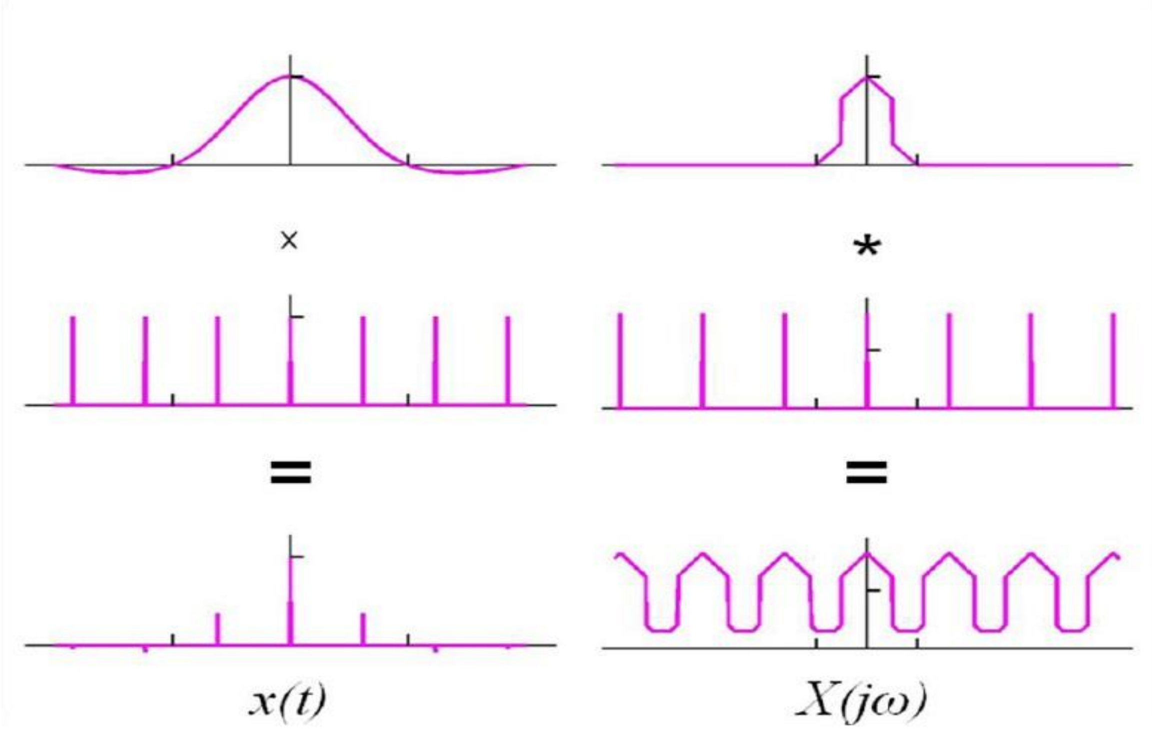
Aliasing Ex:2



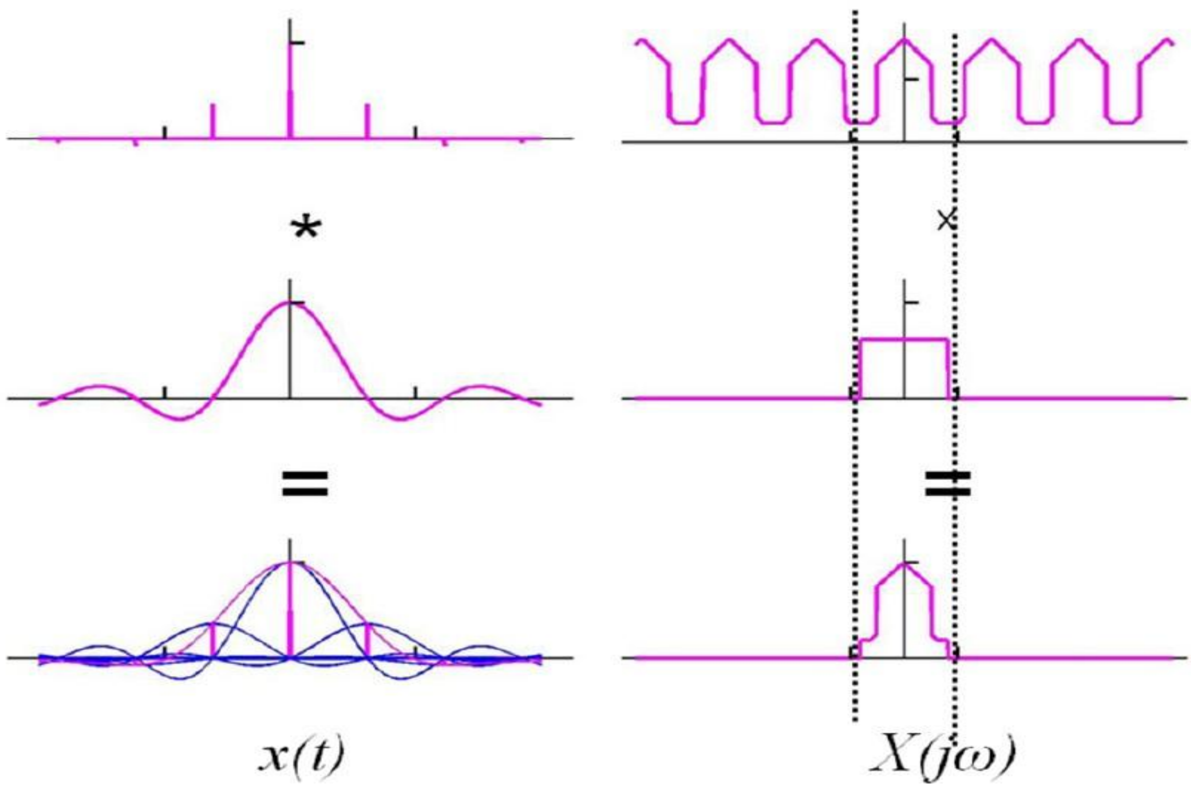
Non-Aliasing: Ex 3



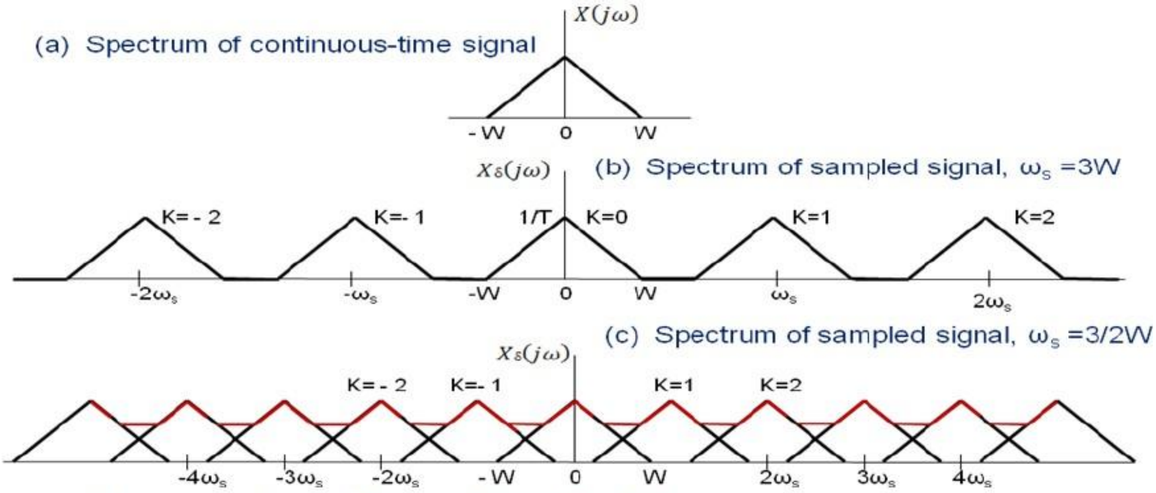
Sampling below the Nyquist rate



Reconstruction below the Nyquist rate



FT of sampled signal for different sampling frequency



- Reconstruction problem is addressed as follows.
- Aliasing is prevented by choosing the sampling interval T so that $\omega_s > 2W$, where W is the highest frequency component in the signal.
- This implies we must satisfy $T < \pi/W$.
- Also, DTFT of the sampled signal is obtained from $X_\delta(j\omega)$ using the relationship $\Omega = \omega T$, that is

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = X_\delta(j\omega)\,|_{\omega\,=\,\Omega/\tau}$$

- This scaling of the independent variable implies that $\omega = \omega_s$ corresponds to $\Omega = 2\pi$

Subsampling: Sampling discrete-time signal

- FT is also used in discrete sampling signal.
- Let $y[n] = x[qn]$ be a subsampled version x[n], where q is a positive integer.
- Relating DTFT of y[n] to the DTFT of x[n], by using FT to represent x[n] as a sampled versioned of a continuous time signal x(t).
- Expressing now y[n] as a sampled version of the sampled version of the same underlying CT x(t) obtained using a sampling interval q that associated with x[n]
- We know to represent the sampling version of x[n] as the impulse sampled CT signal with sampling interval T.

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n)\delta(t - n\tau)$$

- Suppose, x[n] are the samples of a CT signal x(t), obtained at integer multiples of T. That is, x[n]=x[nT]. Let $x(t) \xleftrightarrow{FT} X(j\omega)$ and applying it to obtain

$$\boxed{X_\delta(j\omega) = \frac{1}{\tau}\sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))}$$

- Since y[n] is formed using every qth sample of x[n], we may also express y[n] as a sampled version of x(t).we have $y[n] = x[qn] = x(nq\tau)$

- Hence, active sampling rate for y]n] is T'=qT. Hence

$$y_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\tau') \xleftrightarrow{FT} Y_\delta(j\omega) = \frac{1}{\tau'} \sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s'))$$

- Hence substituting T'=q T, and $\omega_s' = \omega_s/q$

$$Y_\delta(j\omega) = \frac{1}{q\tau} \sum_{k=-\infty}^{+\infty} X(j(\omega - \frac{k}{q}\omega_s))$$

- We have expressed both $Y_\delta(j\omega)$ *and* $X_\delta(j\omega)$ as a function of .
- Expressing $X(j\omega)$ as a function of $X_\delta(j\omega)$. Let us write k/q as a proper function, we get

$$\frac{k}{q} = l + \frac{m}{q},$$

where l is the integer portion of $\frac{k}{q}$, *and m is the remainder*

allowing k to range from $-\infty$ *to* $+\infty$ *corresponds*

to having l range from $-\infty$ *to* $+\infty$ *and m from* 0 *to* $q-1$

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-i} \left\{ \frac{1}{\tau} \sum_{l=-\infty}^{+\infty} X_\delta\left(j\left(\omega - l\omega_s - \frac{m}{q}\,\omega_s\right)\right)\right\}$$

$$Y_\delta(j\omega) = \frac{1}{q} \sum_{m=0}^{q-i} X_\delta\left(j\left(\omega - \frac{m}{q}\,\omega_s\right)\right)$$

which represents a sum of shifted versions of

$X_\delta(j\omega)$ *normalized by q.*

Converting from the FT representation back to DTFT

and substituting $\Omega = \omega\tau'$ *above*

and also $X(e^{j\Omega}) = X_\delta(j\Omega/\tau)$ *, we write this result as*

$$\boxed{Y_\delta(e^{j\Omega}) = \frac{1}{q} \sum_{m=0}^{q-i} X_q(e^{j(\Omega - m2\pi)})}$$

where, $X_q(e^{j\Omega}) = X(e^{j\Omega/q})$ — *a scaled DTFT version*

Recommended Questions

1. Find the frequency response of the RLC circuit shown in the figure. Also find the impulse response of the circuit

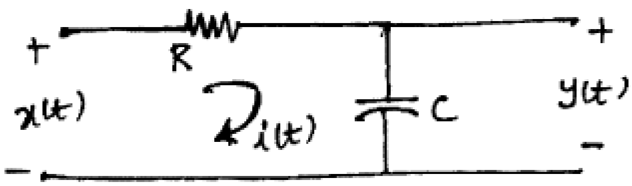


Fig.Q6(b)

2. The input and output of causal LTI system are described by the differential equation.

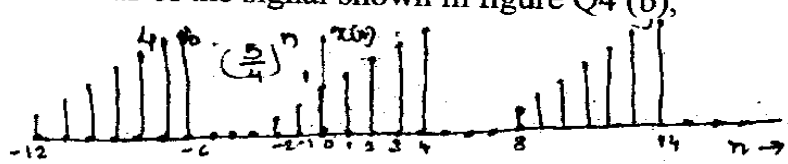
$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

i) Find the frequency response of the system
ii) Find impulse response of the system
iii) What is the response of the system if $x(t) = te^{-t}u(t)$. **(10 Marks)**

3. If $x(t) \leftrightarrow X(f)$. Show that $x(t)\cos w_0 t \leftrightarrow 1/2[X(f-f_0)+X(f-f_0)]$ where $w0=2\pi f_0$

4. The input $x(t) = e^{-3t}u(t)$ when applied to a system, results in an output $y(t) = e^{-t}u(t)$. Find the frequency response and impulse response of the system. **(07 Marks)**

5. Find the DTFS co-efficients of the signal shown in figure Q4 (b),



6. State sampling theorem. Explain sampling of continuous time signals with relevant expressions and figures.

7. Find the Nyquist rate for each of the following signals:
i) $x(t) = \text{sinc}(200t)$ ii) $x(t) = \text{sinc}^2(500t)$