Introduction to Digital Control System

1. Introduction:

A digital control system model can be viewed from different perspectives including control algorithm, computer program, conversion between analog and digital domains, system performance etc. One of the most important aspects is the sampling process level. In continuous time control systems, all the system variables are continuous signals. Whether the system is linear or nonlinear, all variables are continuously present and therefore known (available) at all times. A typical continuous time control system is shown in Figure 1.

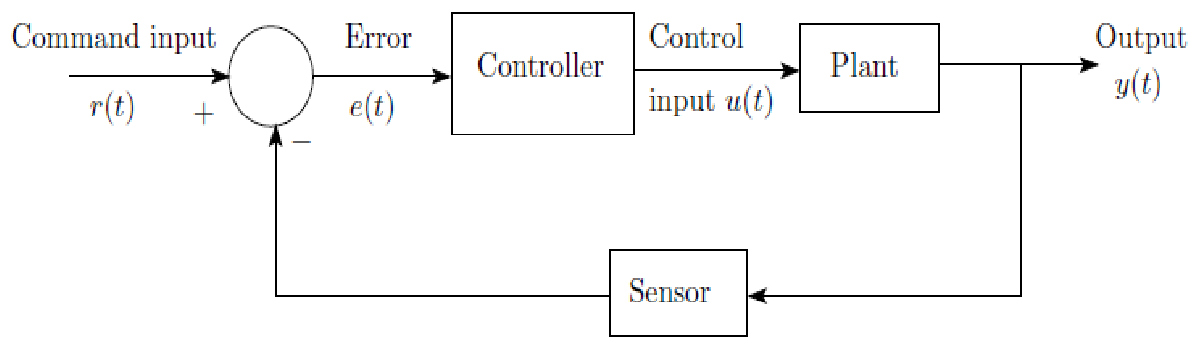


Figure 1: A typical closed loop continuous time control system

In a digital control system, the control algorithm is implemented in a digital computer. The error signal is discretized and fed to the computer by using an **A/D** (analog to digital) converter. The controller output is again a discrete signal which is applied to the plant after using a **D/A** (digital to analog) converter. General block diagram of a digital control system is shown in **Figure 2.**

e(t) is sampled at intervals of **T**. In the context of control and communication, sampling is a process by which a continuous time signal is converted into a sequence of numbers at discrete time intervals. It is a fundamental property of digital control systems because of the discrete nature of operation of digital computer.

Figure 3 shows the structure and operation of a finite pulse width sampler, where (a) represents the basic block diagram and (b) illustrates the function of the same. T is the sampling period and p is the sample duration

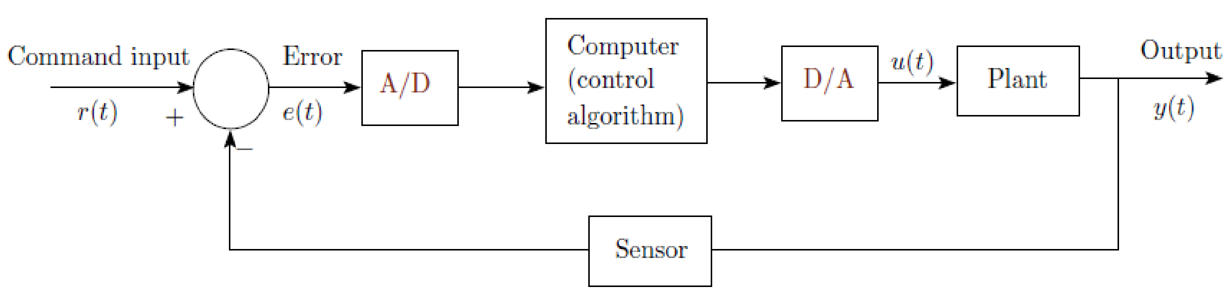


Figure 2: General block diagram of a digital control system

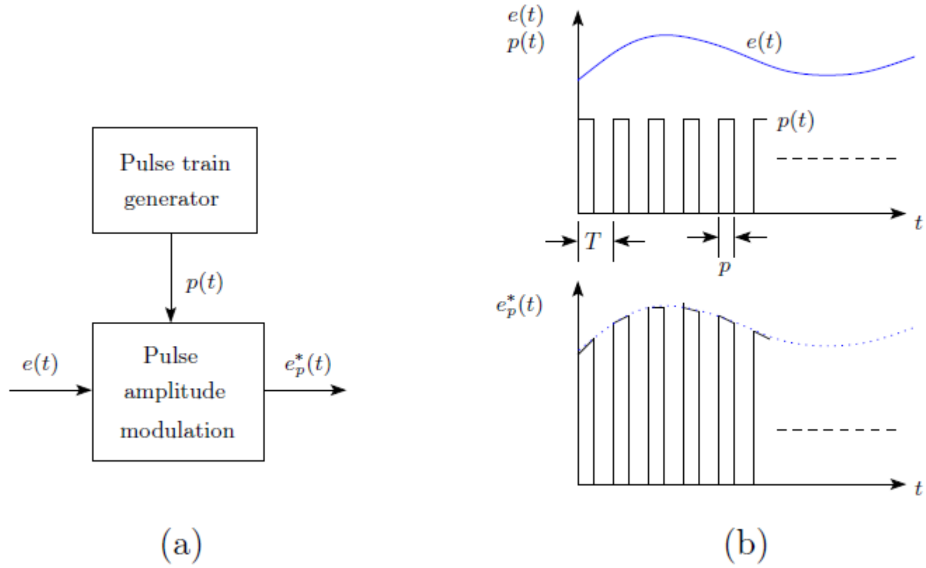


Figure 3: Basic structure and operation of a finite pulse width sampler

In the early development, an analog system, not containing a digital device like computer, in which some of the signals were sampled, was referred to as a sampled data system. With the advent of digital computer, the term discrete-time system denoted a system in which all its signals are in a digital coded form. Most practical systems today are of hybrid nature, i.e., contains both analog and digital components.

Before proceeding to any depth of the subject we should first understand the reason behind going for a digital control system. Using computers to implement controllers has a number of advantages. Many of the difficulties involved in analog implementation can be avoided. Few of them are enumerated below.

1. Probability of accuracy or drift can be removed.
2. Easy to implement sophisticated algorithms.
3. Easy to include logic and nonlinear functions.
4. Re-configurability of the controllers.

1.1 A Naive Approach to Digital Control

One may expect that a digital control system behaves like a continuous time system if the sampling period is sufficiently small. This is true under reasonable assumptions. A crude way to obtain digital control algorithms is by writing the continuous time control law as a differential equation and approximating the derivatives by differences and integrations by summations. This will work when the sampling period is very small. However various parameters, like over shoot, settling time will be slightly higher than those of the continuous time control.

Example: PD controller

A continuous time PD controller can be discretized as follows:

$$
\begin{aligned}
u(t) &= K_p e(t) + K_d \frac{de(t)}{dt} \\
\Rightarrow u(kT) &= K_p e(kT) + K_d \frac{[e(kT) - e((k-1)T)]}{T}
\end{aligned}
$$

where k represents the discrete time instants and T is the discrete time step or the sampling period. We will see later the control strategies with different behaviors, for example deadbeat control, can be obtained with computer control which are not possible with a continuous time control.

1.2 Aliasing

Stable linear systems have property that the steady state response to sinusoidal excitations is sinusoidal with same frequency as that of the input. But digital control systems behave in a much more complicated way because sampling will create signals with new frequencies. Aliasing is an effect of the sampling that causes different signals to become indistinguishable. Due to aliasing, the signal reconstructed from samples may become different than the original continuous signal. This can drastically deteriorate the performance if proper care is not taken.

2. Inherently Sampled Systems

Sampled data systems are natural descriptions for many phenomena. In some cases sampling occurs naturally due to the nature of measurement system whereas in some cases it occurs because information is transmitted in pulsed form. The theory of sampled data systems thus has many applications.

a. Radar: When a radar antenna rotates, information about range and direction is naturally obtained once per revolution of the antenna.

b. Economic Systems: Accounting procedures in economic systems are generally tied to the calendar. Information about important variables is accumulated only at certain times, e.g., daily, weekly, monthly, quarterly or yearly even if the transactions occur at any point of time.

c. Biological Systems: Since the signal transmission in the nervous system occurs in pulsed form, biological systems are inherently sampled.

All these discussions indicate the need for a separate theory for sampled data control systems or digital control systems.

3. How Was Theory Developed?

a. Sampling Theorem: Since all computer controlled systems operate at discrete times only, it is important to know the condition under which a signal can be retrieved from its values at discrete points. Nyquist explored the key issue and Shannon gave the complete solution which is known as Shannon’s sampling theorem. We will discuss Shannon’s sampling theorem in proceeding lectures.

b. Difference Equations and Numerical Analysis: The theory of sampled-data system is closely related to numerical analysis. Difference equations replaced the differential equations in continuous time theory. Derivatives and integrals are evaluated numerically by approximating them with differences and sums.

c. Transform Methods: Z-transform replaced the role of Laplace transform in continuous domain.

d. State Space Theory: In late 1950's, a very important theory in control system was developed which is known as state space theory. The discrete time representations of state models are obtained by considering the systems only at sampling points.

4. Discrete time system representations

As mentioned above, discrete time systems are represented by difference equations. We will focus on LTI systems unless mentioned otherwise.

4.1 Approximation for numerical differentiation

1. Using backward difference

(a) First order

$$\begin{aligned}\text{Continuous: } u(t) &= \dot{e}(t)\\ \text{Discrete: } u(kT) &= \frac{e(kT)-e((k-1)T)}{T}\end{aligned}$$

(b) Second order

$$\begin{aligned}\text{Continuous: } u(t) &= \ddot{e}(t)\\ \text{Discrete: } u(kT) &= \frac{\dot{e}(kT)-\dot{e}((k-1)T)}{T}\\ &= \frac{e(kT)-e((k-1)T)-e((k-1)T)+e((k-2)T)}{T^2}\\ &= \frac{e(kT)-2e((k-1)T)+e((k-2)T)}{T^2}\end{aligned}$$

2. Using forward difference

(a) First order

$$\begin{aligned}\text{Continuous: } u(t) &= \dot{e}(t)\\ \text{Discrete: } u(kT) &= \frac{e((k+1)T)-e(kT)}{T}\end{aligned}$$

(b) Second order

$$\begin{aligned}\text{Continuous: } u(t) &= \ddot{e}(t)\\ \text{Discrete: } u(kT) &= \frac{\dot{e}((k+1)T)-\dot{e}(kT)}{T}\\ &= \frac{e((k+2)T)-2e((k+1)T)+e(kT)}{T^2}\end{aligned}$$

4.2 Approximation for numerical integration

The numerical integration technique depends on the approximation of the instantaneous continuous time signal. We will describe the process of backward rectangular integration technique.

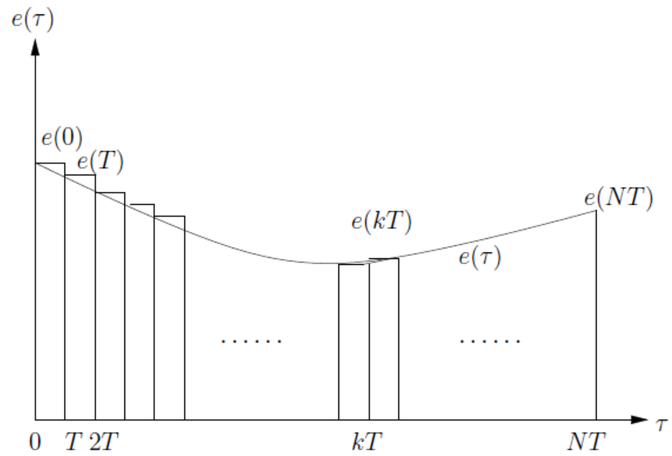


Figure 4: Concept behind Numerical Integration

As shown in Figure 4, the integral function can be approximated by a number of rectangular pulses and the area under the curve can be represented by summation of the areas of all the small rectangles. Thus,

$$\begin{aligned}
\text{if } u(t) &= \int_0^t e(\tau)d\tau \\
\Rightarrow u(NT) &= \int_0^{NT} e(\tau)d\tau \\
&\cong \sum_{k=0}^{N-1} e(kT)\Delta t \\
&= \sum_{k=0}^{N-1} e(kT)T
\end{aligned}$$

where $k = 0, 1, 2, \cdots, N-1$, $\Delta t = T$ and $N > 0$. From the above expression,

$$\begin{aligned}
u((N-1)T) &= \int_0^{(N-1)T} e(\tau)d\tau \\
&= \sum_0^{N-2} e(kT)T \\
\Rightarrow u(NT) - u((N-1)T &= Te((N-1)T) \\
\text{or, } u(NT) &= u((N-1)T) + Te((N-1)T)
\end{aligned}$$

The above expression is a recursive formulation of backward rectangular integration where the expression of a signal at a given time explicitly contains the past values of the signal. Use of this recursive equation to evaluate the present value of u(NT) requires to retain only the immediate past sampled value e((N −1)T) and the immediate past value of the integral u((N −1)T), thus saving the storage space requirement.

In forward rectangular integration, we start approximating the curve from top right corner. Thus the approximation is

$$u(NT) = \sum_{k=1}^{N} e(kT)T$$

The recursive relation of the forward rectangular integration is:

$$u(NT) = u((N-1)T) + Te(NT)$$

Polygonal or trapezoidal integration is another numerical integration technique where the total area is divided into a number of trapezoids and expressed as the sum of areas of individual trapezoids.

Example 1: Consider the following continuous time expression of a PID controller:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \frac{de(t)}{dt}$$

Where u (t) is the controller output and e (t) is the input to the controller. Considering t = NT, find out the recursive discrete time formulation of u (NT) by approximating the derivative by backward difference and integral by backward rectangular integration technique.

Solution: u (NT) can be approximated as

$$u(NT) = K_p e(NT) + K_i \sum_{k=0}^{N-1} e(kT)T + K_d \frac{e(NT) - e((N-1)T)}{T}$$

Similarly $u((N-1)T)$ can be written as

$$u((N-1)T) = K_p e((N-1)T) + K_i \sum_{k=0}^{N-2} e(kT)T + K_d \frac{e((N-1)T) - e((N-2)T)}{T}$$

Subtracting $u((N-1)T)$ from $u(NT)$,

$$
\begin{aligned}
u(NT) - u((N-1)T) &= K_p e(NT) + K_i \sum_{k=0}^{N-1} e(kT)T + K_d \frac{e(NT) - e((N-1)T)}{T} - \\
&\quad K_p e((N-1)T) - K_i \sum_{k=0}^{N-2} e(kT)T - K_d \frac{e((N-1)T) - e((N-2)T)}{T} \\
\Rightarrow u(NT) &= u((N-1)T) + K_p\left[e(NT) - e((N-1)T)\right] + K_i T e((N-1)T) \\
&\quad + K_d \frac{e(NT) - 2e((N-1)T) + e((N-2)T)}{T}
\end{aligned}
$$

which is the required recursive relation.

Similarly, if we use forward difference and forward rectangular integration, we would get the recursive relation as

$$
\begin{aligned}
u(NT) &= u((N-1)T) + K_p\left[e(NT) - e((N-1)T)\right] + K_i T e(NT) \\
&\quad + K_d \frac{e((N+1)T) - 2e(NT) + e((N-1)T)}{T}
\end{aligned}
$$

4.3 Difference Equation Representation

The general linear difference equation of an n^{th} order causal LTI SISO system is:

$$
\begin{aligned}
&y((k+n)T) + a_1 y((k+n-1)T) + a_2 y\left((k+n-2)T\right) + \cdots + a_n y\left(kT\right) \\
&\qquad = b_0 u((k+m)T) + b_1 u\left((k+m-1)T\right) + \ldots + b_m u\left(kT\right)
\end{aligned}
$$

where y is the output of the system and u is the input to the system and $m \leq n$. This inequality is required to avoid anticipatory or non-causal model.

Example 2: If you express the recursive relation for PID control in general difference equation form, is the system causal?
Solution: The output of the PID controller is u and the input is e. When approximated with forward difference and forward rectangular integration, u(NT) is found as:

$$
\begin{aligned}
u(NT) &= u((N-1)T) + K_p\left[e(NT) - e((N-1)T)\right] + K_i T e(NT) \\
&\quad + K_d \frac{e((N+1)T) - 2e(NT) + e((N-1)T)}{T}
\end{aligned}
$$

By putting N = k+1 and comparing with general difference equation, we can say n = 1 whereas m = 2. Thus the system is non-causal. However, when the approximation uses backward difference and backward rectangular integration, the approximated model becomes causal.

5. Mathematical Modeling of Sampling Process

Sampling operation in sampled data and digital control system is used to model either the sample and hold operation or the fact that the signal is digitally coded. If the sampler is used to represent S/H (Sample and Hold) and A/D (Analog to Digital) operations, it may involve delays, finite sampling duration and quantization errors. On the other hand if the sampler is used to represent digitally coded data the model will be much simpler. Following are two popular sampling operations:

1. Single rate or periodic sampling

2. Multi-rate sampling

We would limit our discussions to periodic sampling only.

5.1 Finite pulse width sampler

In general, a sampler is the one which converts a continuous time signal into a pulse modulated or discrete signal. The most common type of modulation in the sampling and hold operation is the pulse amplitude modulation.

The symbolic representation, block diagram and operation of a sampler are shown in Figure 5. The pulse duration is p second and sampling period is T second. Uniform rate sampler is a linear device which satisfies the principle of superposition. As in Figure 5, p (t) is a unit pulse train with period T.

$$p(t) = \sum_{k=-\infty}^{\infty} [u_s(t - kT) - u_s(t - kT - p)]$$

where us(t) represents unit step function. Assume that leading edge of the pulse at t = 0 coincides with t = 0. Thus f^*_p (t) can be written as

$$f_p^*(t) = f(t) \sum_{k=-\infty}^{\infty} [u_s(t - kT) - u_s(t - kT - p)]$$

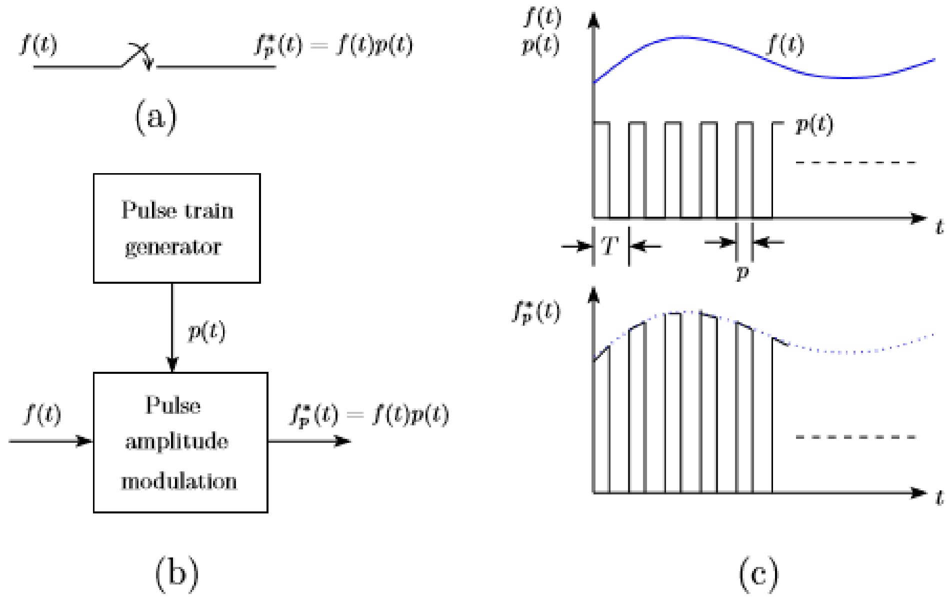


Figure 5: Finite pulse width sampler: (a) Symbolic representation (b) Block diagram (c) Operation.

Frequency domain characteristics:

Since p(t) is a periodic function, it can be represented by a Fourier series, as

$$p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_s t}$$

where

$w_s = \dfrac{2\pi}{T}$ is the sampling frequency and C_n's are the complex Fourier series coefficients.

$$C_n = \frac{1}{T}\int_0^T p(t)e^{-jnw_s t}dt$$

Since $p(t) = 1$ for $0 \leq t \leq p$ and 0 for rest of the period,

$$\begin{aligned} C_n &= \frac{1}{T}\int_0^p e^{-jnw_s t}dt \\ &= \left[\frac{1}{-jnw_s T}.e^{-jnw_s t}\right]_0^p \\ &= \frac{1 - e^{-jnw_s p}}{jnw_s T} \end{aligned}$$

C_n can be rearranged as,

$$\begin{aligned} C_n = &= \frac{e^{-jnw_sp/2}(e^{jnw_sp/2} - e^{-jnw_sp/2})}{jnw_sT} \\ &= \frac{2je^{-jnw_sp/2}\sin(nw_sp/2)}{jnw_sT} \\ &= \frac{p}{T}\frac{\sin(nw_sp/2)}{nw_sp/2}e^{-jnw_sp/2} \end{aligned}$$

Since $f_p^*(t)$ is also periodic, it can be written as

$$f_p^*(t) = \sum_{n=-\infty}^{\infty} C_n f(t) e^{jnw_st}$$

$$\begin{aligned} \Rightarrow F_p^*(jw) &= \mathcal{F}[f_p^*(t)], \quad \text{where } \mathcal{F} \text{ represents Fourier transform} \\ &= \int_{-\infty}^{\infty} f_p^*(t)e^{-jwt}dt \end{aligned}$$

Using complex shifting theorem of Fourier transform

$$\mathcal{F}\left[e^{jnw_st}f(t)\right] = F(jw - jnw_s)$$

$$\Rightarrow F_p^*(jw) = \sum_{n=-\infty}^{\infty} C_n F(jw - jnw_s)$$

Since n is from $-\infty$ to ∞, the above equation can also be written as

$$F_p^*(jw) = \sum_{n=-\infty}^{\infty} C_n F(jw + jnw_s)$$

where,

$$\begin{aligned} C_o &= \lim_{n \to 0} C_n \\ &= \frac{p}{T} \\ F_p^*(jw)|_{n=0} &= C_0 F(jw) = \frac{p}{T}F(jw) \end{aligned}$$

The above equation implies that the frequency contents of the original signal f(t) are still present in the sampler output except that the amplitude is multiplied by the factor $\frac{p}{T}$. For $n \neq 0$, C_n is a complex quantity, the magnitude of which is,

$$|C_n| = \frac{p}{T}\left|\frac{\sin(nw_sp/2)}{nw_sp/2}\right|$$

Magnitude of $F_p^*(jw)$

$$\begin{aligned}\left|F_p^*(jw)\right| &= \left|\sum_{n=-\infty}^{\infty} C_n F(jw+jnw_s)\right| \\ &\leq \sum_{n=-\infty}^{\infty} |C_n||F(jw+jnw_s)|\end{aligned}$$

Sampling operation retains the fundamental frequency but in addition, sampler output also contains the harmonic components.

$$F(jw+jnw_s) \quad \text{for} \quad n = \pm 1, \pm 2, ...$$

According to Shannon's sampling theorem, "if a signal contains no frequency higher than wc rad/sec, it is completely characterized by the values of the signal measured at instants of time separated by T = π/wc sec."

Sampling frequency rate should be greater than the Nyquist rate which is twice the highest frequency component of the original signal to avoid aliasing.

If the sampling rate is less than twice the input frequency, the output frequency will be different from the input which is known as aliasing. The output frequency in that case is called alias frequency and the period is referred to as alias period.

The overlapping of the high frequency components with the fundamental component in the frequency spectrum is sometimes referred to as folding and the frequency $\frac{ws}{2}$ is often known as folding frequency. The frequency w_c is called Nyquist frequency.

A low sampling rate normally has an adverse effect on the closed loop stability. Thus, often we might have to select a sampling rate much higher than the theoretical minimum.

5.2 Flat-top approximation of finite-pulse width sampling

The Laplace transform of f^*_p (t) can be written as

$$F_p^*(s) = \sum_{n=-\infty}^{\infty} \frac{1-e^{-jnw_s p}}{jnw_s T} F(s+jnw_s)$$

If the sampling duration p is much smaller than the sampling period T and the smallest time constant of the signal f(t), the sampler output can be approximated by a sequence of rectangular pulses since the variation of f(t) in the sampling duration will be less significant. Thus for k = 0, 1, 2........ $f^*_p(t)$ can be expressed as an infinite series

$$f_p^*(t) = \sum_{k=0}^{\infty} f(kT)\left[u_s(t-kT) - u_s(t-kT-p)\right]$$

Taking Laplace transform,

$$F_p^*(s) = \sum_{k=0}^{\infty} f(kT)\left[\frac{1-e^{-ps}}{s}\right] e^{-kTs}$$

Since p is very small, e^{-ps} can be approximated by taking only the first 2 terms, as

$$\begin{aligned} 1 - e^{-ps} &= 1 - [1 - ps + \frac{(ps)^2}{2!}.......] \\ &\cong ps \\ \text{Thus,}\ \ F_p^*(s) &\cong p\sum_{k=0}^{\infty} f(kT)e^{-kTs} \end{aligned}$$

In time domain,

$$f_p^*(t) \;=\; p\sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

where, δ(t) represents the unit impulse function. Thus the finite pulse width sampler can be viewed as an impulse modulator or an ideal sampler connected in series with an attenuator with attenuation p.

5.3 The ideal sampler

In case of an ideal sampler, the carrier signal is replaced by a train of unit impulse as shown in Figure 6. The sampling duration p approaches 0, i.e., its operation is instantaneous.

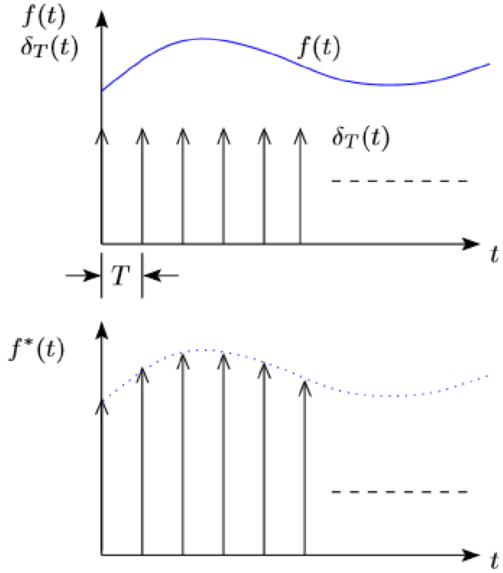


Figure 6: Ideal sampler operation

The output of an ideal sampler can be expressed as

$$
\begin{aligned}
f^*(t) &= \sum_{k=0}^{\infty} f(kT)\delta(t-kT) \\
\Rightarrow F^*(s) &= \sum_{k=0}^{\infty} f(kT)e^{-kTs}
\end{aligned}
$$

One should remember that practically the output of a sampler is always followed by a hold device which is the reason behind the name sample and hold device. Now, the output of a hold device will be the same regardless the nature of the sampler and the attenuation factor p can be dropped in that case. Thus the sampling process can be be always approximated by an ideal sampler or impulse modulator.

6. Data Reconstruction

Most of the control systems have analog controlled processes which are inherently driven by analog inputs. Thus the outputs of a digital controller should first be converted into analog signals before being applied to the systems. Another way to look at the problem is that the high frequency components of f(t) should be removed before applying to analog devices. A low pass filter or a data reconstruction device is necessary to perform this operation.

In control system, hold operation becomes the most popular way of reconstruction due to its simplicity and low cost. Problem of data reconstruction can be formulated as:

“ Given a sequence of numbers, f(0),f(T),f(2T),⋯ ,f(kt),⋯, a continuous time signal f(t), t ≥ 0, is to be reconstructed from the information contained in the sequence.”

Data reconstruction process may be regarded as an extrapolation process since the continuous data signal has to be formed based on the information available at past sampling instants. Suppose the original signal f(t) between two consecutive sampling instants kT and (k + 1)T is to be estimated based on the values of f(t) at previous instants of kT, i.e., (k − 1)T, (k − 2)T, ⋯0.

Power series expansion is a well-known method of generating the desired approximation which yields

$$
\begin{aligned}
f_k(t) &= f(kT) + f^{(1)}(kT)(t-kT) + \frac{f^{(2)}(kT)}{2!}(t-kT)^2 + \ldots.. \\
\text{where, } f_k(t) &= f(t) \ \text{ for } \ kT \le t \le (k+1)T \ \text{ and} \\
f^{(n)}(kT) &= \frac{d^n f(t)}{dt^n}\Big|_{t=kT} \quad \text{for } \ n = 1, 2, \ldots
\end{aligned}
$$

Since the only available information about f(t) is its magnitude at the sampling instants, the derivatives of f(t) must be estimated from the values of f(kT), as

$$
\begin{aligned}
f^{(1)}(kT) &\cong \frac{1}{T}[f(kT) - f((k-1)T)] \\
\text{Similarly,} \quad f^{(2)}(kT) &\cong \frac{1}{T}[f^{(1)}(kT) - f^{(1)}((k-1)T)] \\
\text{where,} \quad f^{(1)}((k-1)T) &\cong \frac{1}{T}[f((k-1)T) - f((k-2)T)]
\end{aligned}
$$

6.1 Zero Order Hold

Higher the order of the derivatives to be estimated is, larger will be the number of delayed pulses required. Since time delay degrades the stability of a closed loop control system, using higher order derivatives of f(t) for more accurate reconstruction often causes serious stability problem. Moreover a high order extrapolation requires complex circuitry and results in high cost.

For the above reasons, use of only the first term in the power series to approximate f(t) during the time interval kT ≤ t < (k+1)T is very popular and the device for this type of extrapolation is known as zero-order extrapolator or zero order hold. It holds the value of f(kT) for kT ≤ t < (k + 1)T until the next sample f((k + 1)T) arrives. Figure 7 illustrates the operation of a ZOH where the green

line represents the original continuous signal and brown line represents the reconstructed signal from ZOH.

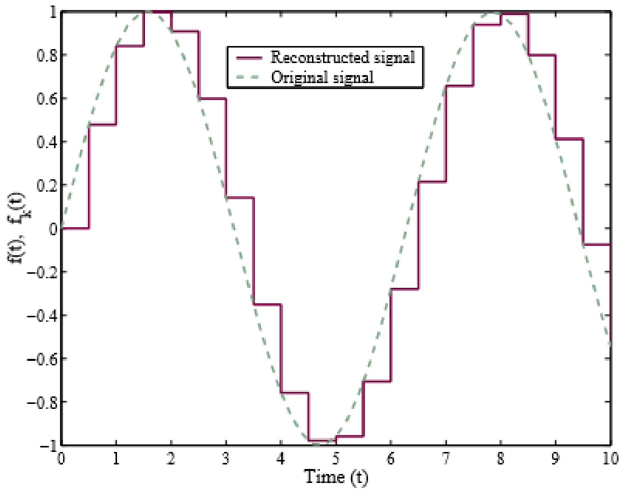


Figure 7: Zero order hold operation

The accuracy of zero order hold (ZOH) depends on the sampling frequency. When T → 0, the output of ZOH approaches the continuous time signal. Zero order hold is again a linear device which satisfies the principle of superposition.

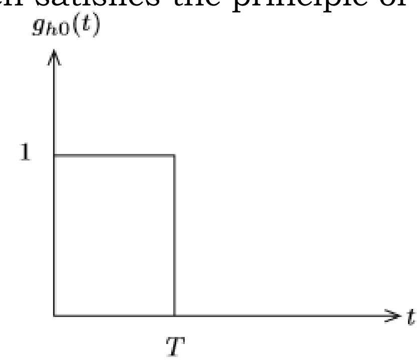


Figure 8: Impulse response of ZOH

The impulse response of a ZOH, as shown in Figure 8, can be written as

$$
\begin{aligned}
g_{ho}(t) &= u_s(t) - u_s(t-T) \\
\Rightarrow G_{ho}(s) &= \frac{1-e^{-TS}}{s} \\
G_{ho}(jw) &= \frac{1-e^{-jwT}}{jw} = T\,\frac{\sin(wT/2)}{wT/2}e^{-jwT/2}
\end{aligned}
$$

Since $T = \dfrac{2\pi}{w_s}$, we can write

$$G_{ho}(jw) \quad = \quad \frac{2\pi}{w_s}\frac{\sin(\pi w/w_s)}{\pi w/w_s}\, e^{-j\pi w/w_s}$$

Magnitude of $G_{ho}(jw)$:

$$|G_{ho}(jw)| = \frac{2\pi}{w_s}\left|\frac{\sin(\pi w/w_s)}{\pi w/w_s}\right|$$

Phase of $G_{ho}(jw)$:

$$\angle G_{h0}(jw) = \angle \sin(\pi w/w_s) - \frac{\pi w}{w_s} \quad \text{rad}$$

The sign of $\angle \sin(\pi w/w_s)$ changes at every integral value of $\dfrac{\pi w}{w_s}$. The change of sign from $+$ to − can be regarded as a phase change of −1800. Thus the phase Characteristics of ZOH is linear with jump discontinuities of −1800 at integral multiple of ws. The magnitude and phase characteristics of ZOH are shown in Figure 9.

At the cut off frequency $w_c = \dfrac{w_s}{2}$, magnitude is 0.636. When compared with an ideal low pass filter, we see that instead of cutting of sharply at $w = \frac{w_s}{2}$, the amplitude characteristics of $G_{ho}(jw)$ is zero at $\dfrac{w_s}{2}$ and integral multiples of w_s.

<u>6.2 First Order Hold</u>

When the 1st two terms of the power series are used to extrapolate f(t), over the time interval kT < t < (k + 1)T, the device is called a first order hold (FOH). Thus

$$\begin{aligned} f_k(t) &= f(kT) + f^1(kT)(t - kT) \\ \text{where, } f^1(kT) &= \frac{f(kT) - f((k-1)T)}{T} \\ \Rightarrow f_k(t) &= f(kT) + \frac{f(kT) - f((k-1)T)}{T}(t - kT) \end{aligned}$$

Impulse response of FOH is obtained by applying a unit impulse at t = 0, the corresponding output is obtained by setting k = 0,1,2,.....

$$\text{for } k = 0, \text{ when } 0 \le t < T, \quad f_0(t) = f(0) + \frac{f(0) - f(-T)}{T}t$$

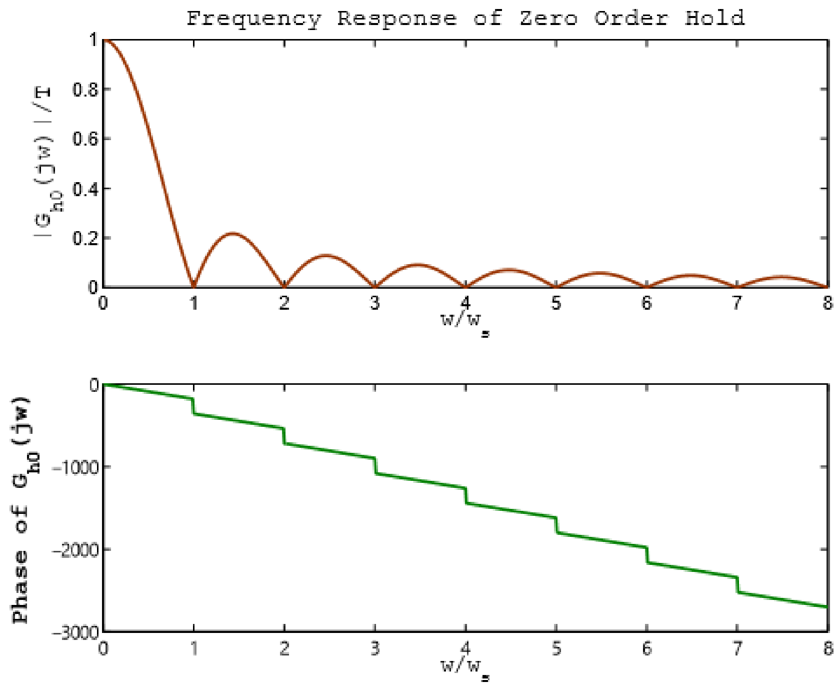


Figure 9: Frequency response of ZOH

$f(0) = 1$ [impulse unit] $f(-T) = 0$ $f_{h1}(t) = 1 + \frac{t}{T}$ in this region. When $T \le t < 2T$

$$f_1(t) = f(T) + \frac{f(T) - f(0)}{T}(t - T)$$

Since, $f(T) = 0$ and $f(0) = 1$, $f_{h1}(t) = 1 - \frac{t}{T}$ in this region. $f_{h1}(t)$ is 0 for $t \ge 2T$, since

f(t) = 0 for t ≥ 2T. Figure 10 shows the impulse response of first order hold.

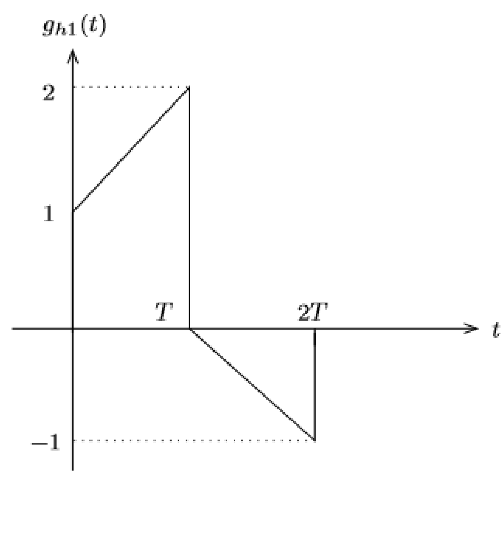


Figure 10: Impulse response of First Order Hold

If we combine all three regions, we can write the impulse response of a first order hold as,

$$
\begin{aligned}
g_{h1}(t) &= (1+\frac{t}{T})u_s(t) + (1-\frac{t}{T})u_s(t-T) - (1+\frac{t}{T})u_s(t-T) - (1-\frac{t}{T})u_s(t-2T) \\
&= (1+\frac{t}{T})u_s(t) - 2\frac{t}{T}u_s(t-T) - (1-\frac{t}{T})u_s(t-2T)
\end{aligned}
$$

One can verify that according to the above expression, when $0 \le t < T$, only the first term produces a nonzero value which is nothing but $(1+t/T)$. Similarly, when $T \le t < 2T$, first two terms produce nonzero values and the resultant is $(1 - t/T)$. In case of $t \ge 2T$, all three terms produce nonzero values and the resultant is 0.

The transfer function of a first order hold is:

$$
G_{h1}(s) = \frac{1+Ts}{T}\left[\frac{1-e^{-Ts}}{s}\right]^2
$$

Frequency Response $\quad G_{h1}(jw) = \dfrac{1+jwT}{T}\left[\dfrac{1-e^{-jwT}}{s}\right]^2$

$$
\begin{aligned}
\text{Magnitude: } |G_{h1}(jw)| &= \left|\frac{1+jwT}{T}\right| |G_{h0}(jw)|^2 \\
&= \frac{2\pi}{w_s}\sqrt{1+\frac{4\pi^2 w^2}{w_s^2}}\left|\frac{\sin(\pi w/w_s)}{\pi w/w_s}\right|^2 \\
\text{Phase: } \angle G_{h1}(jw) &= \tan^{-1}(2\pi w/w_s) - 2\pi w/w_s \quad \text{rad}
\end{aligned}
$$

The frequency response is shown in Figure 11.

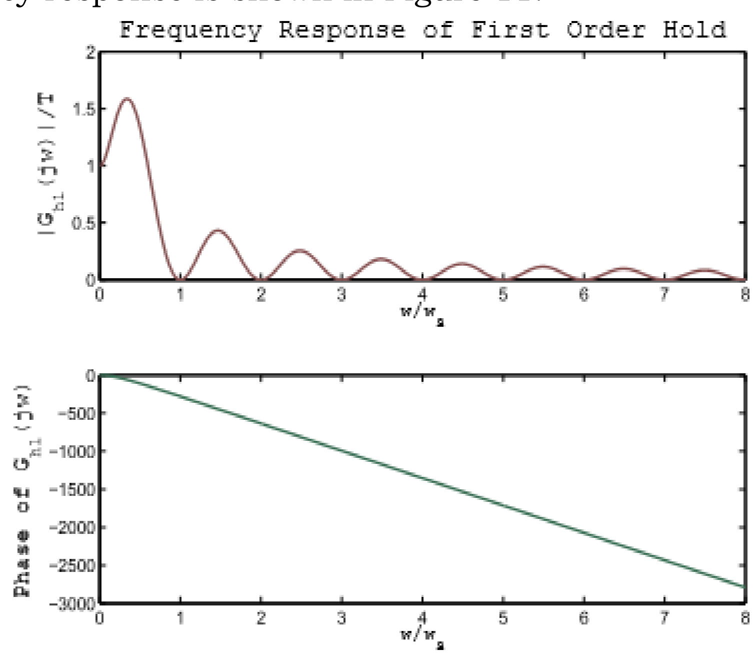


Figure 11: Frequency response of FOH

Figure 12 shows a comparison of the reconstructed outputs of ZOH and FOH.

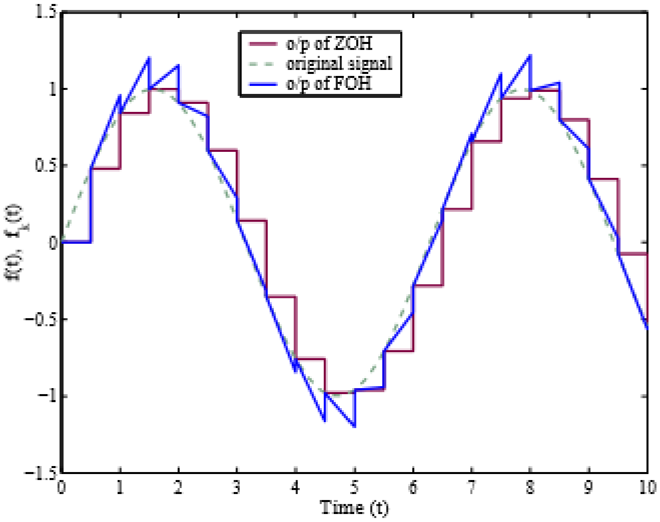


Figure 12: Operation of ZOH and FOH

State Space Analysis:-

State:- The state of the dynamic system is the smallest set of variables called the state variables such that the knowledge of these variable at $t = t_0$, together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system for $t \geq t_0$. The state of the system refers to the past, present and future conditions of the system.

State Variables:- the state variables of a dynamic system are the minimal set of variables selected such that the knowledge of these variables at any time $t = t_0$, together with the knowledge of the inputs for $t \geq t_0$ completly determines the dynamic behaviour of the system for $t \geq t_0$; i.e, the state variables of a dynamic system are the smallest set of variables that determine the state of the dynamic system.

State Space:- The n dimensional space whose coordinate axis consists of the x_1 axis, x_2 axis ---- x_n axis, where $x_1, x_2,$ ---- x_n are state variables is called a state-space. Any state can be represented by a point in the state space.

State Vector :- A state vector is an $n \times 1$ vector that determines uniquely the system state $x(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $u(t)$ for $t \geq t_0$ is specified.

Input Vector :- An input vector is a $m \times 1$ column vector in which the elements are the m inputs u_1, u_2 ----- u_m,

Output vector :- An output vector is a $p \times 1$ column vector in which the elements are the p outputs y_1, y_2 ---- y_p.

System matrix 'A' :- The system matrix 'A' is $n \times n$ constant matrix relating the state variables of the system to the first derivaties of the state variables.

Input matrix B :- The input matrix 'B' is an $n \times m$ constant matrix relating the input variables of the system to the first derivaties of the state variables.

Output matrix C :- The output matrix 'C' is a $p \times n$ constant matrix relating the state variables to the outputs.

Transition matrix 'D' :- The transition matrix 'D' is a $p \times m$ constant matrix. relating the inputs of the system.

State Equations:- The state Equations are a set of first-order differential Equations, where in, the first derivatives of the state Variables are Expressed in terms of the state variables and the Inputs of the system.

→ $\dot{x}(t) = Ax(t) + Bu(t)$ is Known as the state Equation of the system

Output Equations:- The Output Equations are the Equations in which the Outputs of the system are Expressed in terms of the state Variables and the Inputs to the system.

$y(t) = Cx(t) + Du(t)$:- is Known as the Output Equation of the system.

Dynamic Equations:- The state Equations and the Output Equations together are Called the dynamic Equations of the system.

State model:- The state Equations and the Output Equations together are Called the state model of the system. It is a state Variable representation of the system. It is given by

$$\dot{x}(t) = A\,x(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Eigen values:- The roots of the characteristic Equation of a state model, $|\lambda I - A| = 0$ are often called the Eigen values of the matrix A.

Eigen vector:- Any non-zero Vector p_i that satisfies the matrix Equation $|\lambda I - A| p_i = 0$, wher λ_i $i = 1, 2, \ldots m$ denotes the Eigen values of 'A' is called the Eigenvector of 'A' associated with the Eigen value λ_i.

Modal matrix:- The matrix formed by placin the Eigen vectors together (column-wise) is called the modal matrix.

Diagonalizing matrix:- The modal matrix used to diagonalize the System matrix 'A' is called the diagonalizing matrix.

Vander Monde Matrix:- The Vander Monde matrix is a special form of modal matrix which can be written directly, if the system matrix 'A' is in Companion form and all its Eigen values are distint.

Modified Vander Monde matrix:- The modified Vander monde matrix is a special form of modal matrix which can be Written directly, if the System matrix 'A' is in companion form and some of it's Eigen values are repeated.

Advantage of State Variable Analysis

* It can be applied to non-linear system.
* It can be applied to tile invarient systems.
* It can be applied to multiple input multiple output systems.
* It gives idea about the internal state of the system

State transition Matrix: (STM) :- The state transition matrix is defined as a matrix that satisfies the linear homogenous state Equation $\dot{x}(t) = A\,x(t)$. So it represents the free response of the system.

The STM is the inverse laplace transform of the resolvant matrix.

The matrix exponential e^{At} carries out transition of state from the initial state $x(0)$ at $t=0$, to a state $X(t)$ at time t, Hence it is known as the state transition matrix.

Properities of State transition Matrix:-

if $\phi(t) = e^{At}$

a) $\phi(0) = e^{A(0)} = 1 = I$

b) $\phi^n(t) = \phi(nt)$

c) $\phi^{-1}(t) = \phi(-t)$

d) $\phi(t_1)\,\phi(t_2) = \phi(t_1+t_2)$

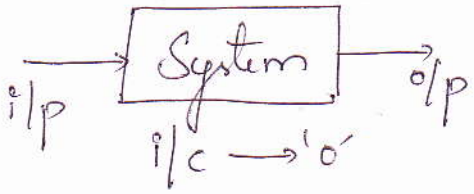
5) $\phi(t_2-t_1)\,\phi(t_1-t_0) = \phi(t_2-t_0)$

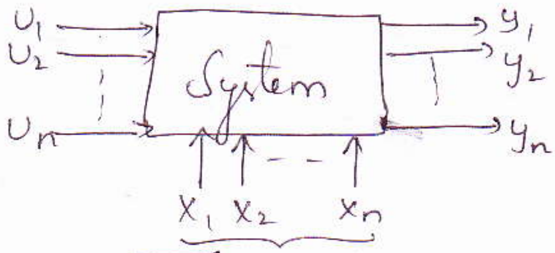
Module - 5

State Space Analysis

Control System

- Classical control System
- Modern Control system (or) Digital control system



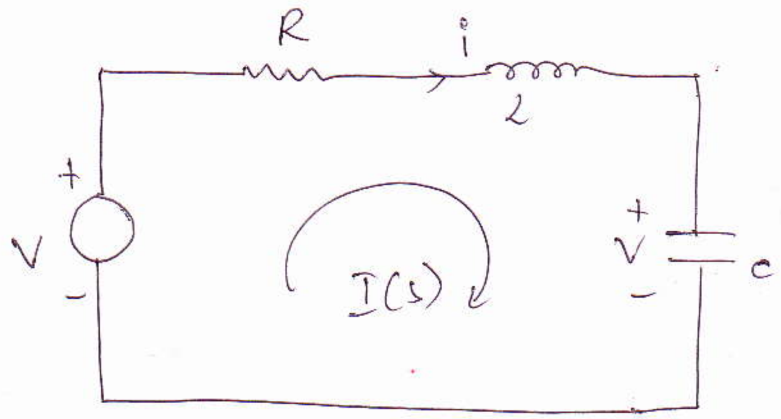


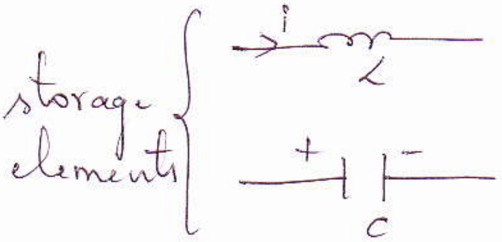
$$T.F = \frac{\mathcal{L}[o/p]}{\mathcal{L}[i/p]}$$

Initial condition are these → state variables

Matrix approach

Initial condition exist's due to storage elements in electrical network





Any system in state space can be represented by

two equations (i) state equation
(ii) Output equation

→ State equation is given by $\underbrace{\dot{X}(t)}_{\text{state model}} = AX(t) + Bu(t)$

→ Output eqn is given by $y(t) = CX(t) + Du(t)$

State Equation

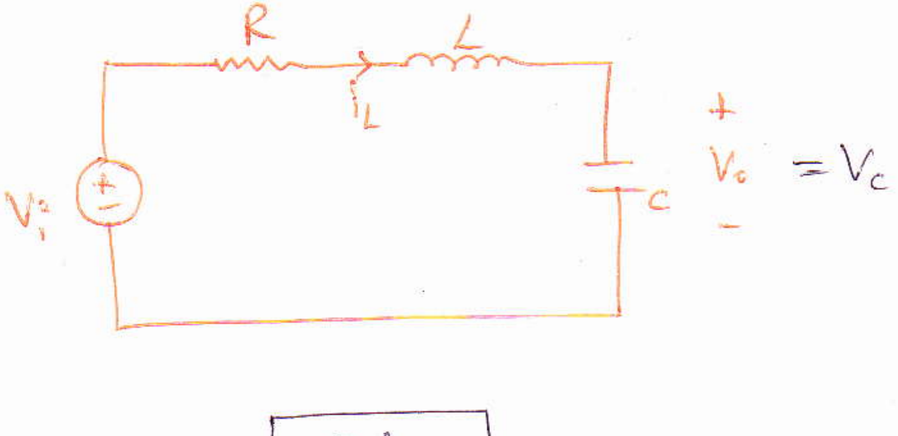
Eg:-

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix}_{3\times3} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \overset{\text{state variable}}{\begin{bmatrix} & B & \end{bmatrix}} \begin{bmatrix} u(t) \end{bmatrix}$$

Output Equation

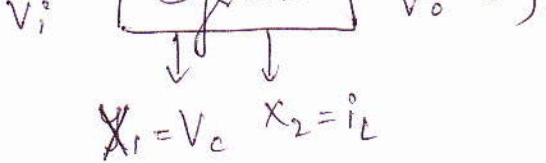
$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} & c & \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix}$$

1) Develop a state model for the electrical circuit given below



$V_o = V_c$

Soln:-



from the circuit

$$y = V_0 = V_c = X_1 \longrightarrow (1)$$

$$y = C[x(t)] + D[u(t)]$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + [0][u]$$

State Equation

$$\dot{x}(t) = A x(t) + B u(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = [\ ]_{2\times 2} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + [\ ] u(t)$$

$\dot{X}_1(t)$ $\qquad$ $\dot{X}_2(t)$

$\frac{d V_c}{dt}$ $\qquad$ $\frac{d i_L}{dt}$

Current across capacitor

$$i_L = C \frac{d V_c}{dt}$$

$$\frac{d V_c}{dt} = \frac{i_L}{C}$$

$$\dot{X}_1 = \frac{1}{C} X_2 \longrightarrow (1)$$

$$V_L = L \frac{d i_L}{dt}$$

$$\frac{d i_L}{dt} = \frac{V_L}{L}$$

$$L \frac{d i_c}{dt} = V_L \longrightarrow (2)$$

$$V_i = V_R + V_L + V_c$$

$$V_i = I_L R + V_L + X_1$$

$$U = X_2 R + V_L + X_1$$

$$V_L = U - X_2R - X_1 \longrightarrow ③$$

By Subst ③ in ②

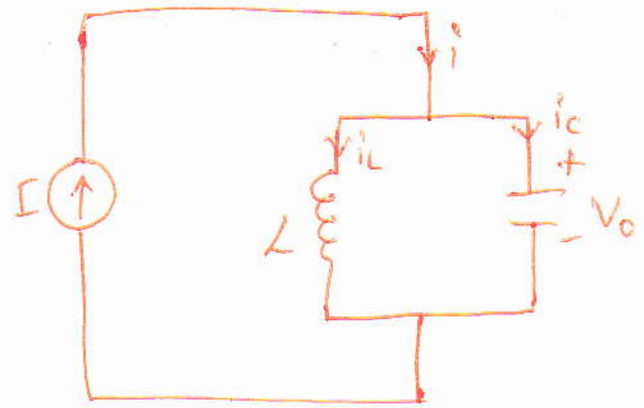
$$L\frac{di_L}{dt} = U - X_2R - X_1$$

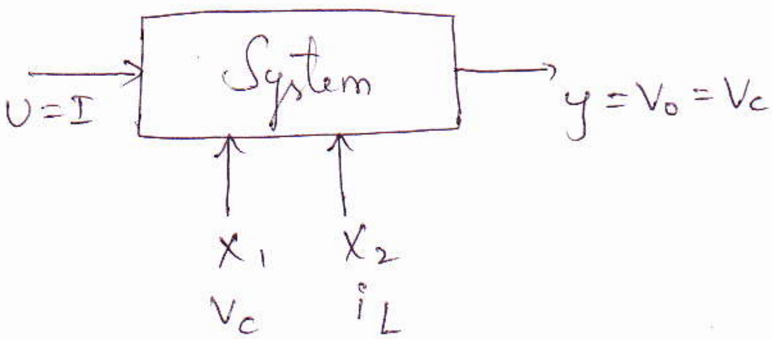
$$\frac{di_L}{dt} = \frac{U}{L} - \frac{X_2R}{L} - \frac{X_1}{L}$$

$$\dot{X}_2 = \frac{1}{L}U - \frac{R}{L}X_2 - \frac{1}{L}X_1 \longrightarrow ④$$

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\end{bmatrix} = \begin{bmatrix}0 & 1/C\\ -1/L & -R/L\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix} + \begin{bmatrix}0 & \frac{1}{L}\end{bmatrix}u$$

2)





3)

$$\dot{X}_1 = \frac{dV_c}{d} \qquad \dot{X}_2 = \frac{di_L}{dt}$$

$$i = i_L + i_C$$

$$i = i_L + C\frac{dV_c}{dt}$$

$$C\frac{dV_c}{dt} = i - i_L$$

$$\frac{dV_c}{dt} = \frac{i}{c} - \frac{i_L}{c}$$

$$\dot{x}_1 = \frac{u}{c} - \frac{x_2}{c}$$

$$\boxed{\dot{x}_1 = \frac{1}{c}u - \frac{1}{c}x_2} \longrightarrow 1$$

$$V_c = V_L$$

$$x_1 = L\frac{di_L}{dt}$$

$$\frac{di_L}{dt} = \frac{x_1}{L}$$

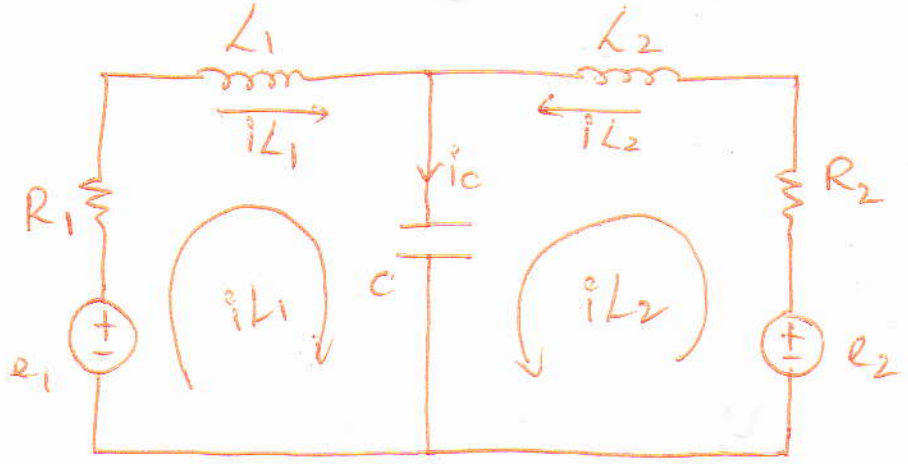
$$\boxed{\dot{x}_2 = \frac{1}{L}x_1} \longrightarrow 2$$

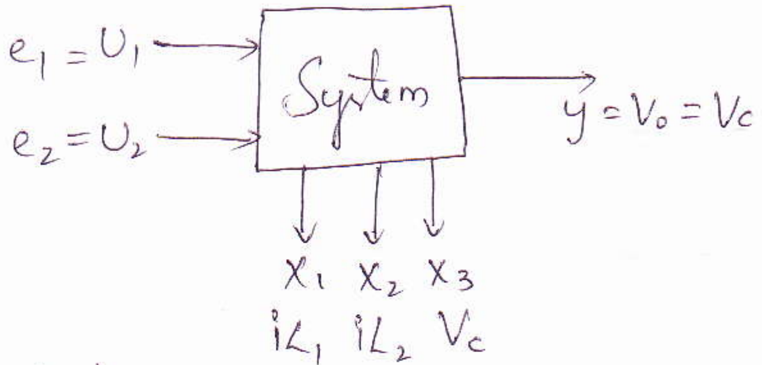
$$\boxed{y = V_o = V_c = x_1}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{c} \\ \frac{1}{c} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{c} & 0 \end{bmatrix} [u]$$

$$[y] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]\,u$$

3)





KVL at loop iL_1

$$-R_1 iL_1 - L_1 \frac{di_L}{dt} - V_c + e_1 = 0$$

$$R_1 iL_1 + L_1 \frac{d i_L}{dt} + V_c - e_1 = 0$$

$$L_1 \frac{d}{dt} i_L = e_1 - R_1 iL_1 - V_c$$

$$\frac{di_L}{dt} = \frac{e_1}{L_1} - \frac{R_1 iL_1}{L_1} - \frac{V_c}{L_1}$$

$$\dot{X}_1 = \frac{U_1}{L_1} - \frac{R_1}{L_1} X_1 - \frac{X_3}{L_1}$$

$$\dot{X}_1 = \frac{1}{L_1} U_1 - \frac{R_1}{L_1} X_1 - \frac{1}{L_1} X_3 \longrightarrow (1)$$

KVL at loop iL_2

$$+e_2 - R_2 iL_2 - L_2 \frac{d}{dt} iL_2 - V_c = 0$$

$$L_2 \frac{d}{dt} iL_2 = e_2 - R_2 iL_2 - V_c$$

$$\frac{d}{dt} iL_2 = \frac{e_2}{L_2} - \frac{R_2 iL_2}{L_2} - \frac{V_c}{L_2}$$

$$\dot{X}_2 = \frac{U_2}{L_2} - \frac{R_2 X_2}{L_2} - \frac{X_3}{L_2}$$

$$\dot{X}_2 = \frac{1}{L_2} U_2 - \frac{R_2}{L_2} X_2 - \frac{1}{L_2} X_3 \longrightarrow (2)$$

$$i_c = i_{L_1} + i_{L_2}$$

$$C\frac{dV_c}{dt} = X_1 + X_2$$

$$\frac{dV_c}{dt} = \frac{X_1}{C} + \frac{X_2}{C}$$

$$\dot{X}_3 = \frac{1}{C}X_1 + \frac{1}{C}X_2 \longrightarrow (3)$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \\ 0 & \frac{1}{L_2} & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$[y] \quad \boxed{y = V_o = V_c = X_3} \longrightarrow (4)$$

$$[y] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + [0] \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

---

4) Obtain a state model for the system described by the differential Eqn given below.

$$\frac{d^3y}{dt^3} + \frac{6d^2y}{dt^2} + \frac{11dy}{dt} + 6y = 5u_1 + 10u_2$$

Soln:-

$$y \longrightarrow x_1 \longrightarrow (1)$$

$$\dot{y} \longrightarrow \dot{x}_1 \longrightarrow x_2 \longrightarrow (2)$$

$$\ddot{y} \longrightarrow \ddot{x}_1 \longrightarrow \dot{x}_2 \longrightarrow x_3 \longrightarrow (3)$$

$$\dddot{y} \longrightarrow \dddot{x}_1 \longrightarrow \ddot{x}_2 \longrightarrow \dot{x}_3$$

Given $\frac{d^3y}{dt^3} + \frac{6d^2y}{dt^2} + \frac{11dy}{dt} + 6y = 5u_1 + 10u_2$

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 5u_1 + 10u_2$$

$$\dot{X}_3 + 6X_3 + 11x_2 + 6x_1 = 5u_1 + 10u_2$$

$$\dot{x}_3 = 5u_1 + 10u_2 - 6x_3 - 11x_2 - 6x_1$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$[y] = [1 \quad 0 \quad 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\cancel{4)\ \frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 2y(t) = 5u(t)}$$

5) $$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 2y(t) = 5u(t)$$

$$\Rightarrow \quad y \rightarrow x_1 \longrightarrow (1)$$

$$\dot{y} \longrightarrow \dot{x}_1 \longrightarrow x_2 \longrightarrow (2)$$

$$\ddot{y} \longrightarrow \ddot{x}_1 \longrightarrow \dot{x}_2 \longrightarrow x_3 \longrightarrow (3)$$

$$\dddot{y} \longrightarrow \dddot{x}_1 \longrightarrow \ddot{x}_2 \longrightarrow \dot{x}_3$$

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 2y(t) = 5u(t)$$

$$\dddot{y} + 4\ddot{y} + 7\dot{y} + 2y = 5u(t)$$

$$\dot{x}_3 + 4x_3 + 7x_2 + 2x_1 = 5u(t)$$

$$\dot{x}_3 = 5u(t) - 4x_3 - 7x_2 - 2x_1 \longrightarrow (4)$$

from Eqn (2), (3) and (4)

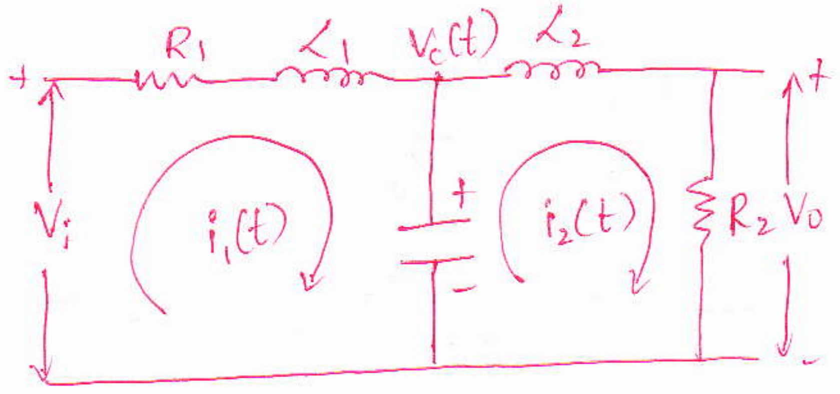
State Eqn

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + [0 \quad 0 \quad 5][U(t)]$$

Output Eqn

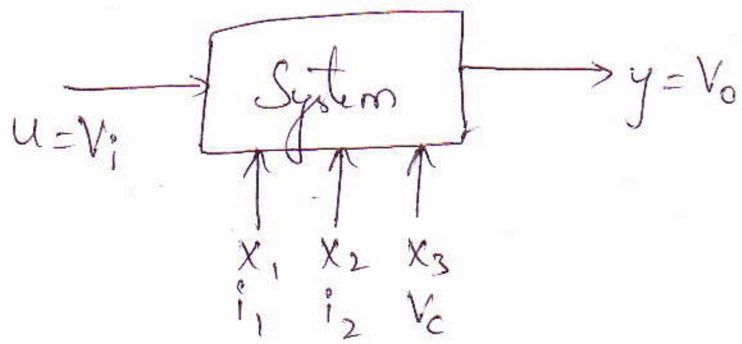
$$[y] = [1 \ 0 \ 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

6)



Obtain the state model for the electrical circuit given in figure. choose the state variables as $i_1(t)$, $i_2(t)$ and $V_c(t)$

Solu:-



By applying KVL at loop i_1

$$-i_1R_1 - L_1\frac{di_1}{dt} - V_c + V_i = 0$$

$$L_1\frac{di_1}{dt} = -i_1R_1 - V_c + V_i$$

$$\frac{di_1}{dt} = -\frac{i_1R_1}{L_1} - \frac{V_c}{L_1} + \frac{V_i}{L_1}$$

$$\boxed{\dot{x}_1 = -\frac{R_1}{L_1}x_1 - \frac{1}{L_1}x_3 + \frac{1}{L_1}u} \longrightarrow (1)$$

By applying KVL at loop i_2

$$-L_2 \frac{d i_2}{dt} - R_2 i_2 + V_c = 0$$

$$L_2 \frac{d i_2}{dt} = -R_2 i_2 + V_c$$

$$\frac{d i_2}{dt} = \frac{-R_2 i_2}{L_2} + \frac{V_c}{L_2}$$

$$\boxed{\dot{x}_2 = \frac{-R_2}{L_2} x_2 + \frac{1}{L_2} x_3} \longrightarrow (2)$$

$$I_c = I_1 - I_2$$

$$C \frac{dV_c}{dt} = x_1 - x_2$$

$$\frac{dV_c}{dt} = \frac{1}{C} x_1 - \frac{1}{C} x_2$$

$$\boxed{\dot{x}_3 = \frac{1}{C} x_1 - \frac{1}{C} x_2} \longrightarrow (3)$$

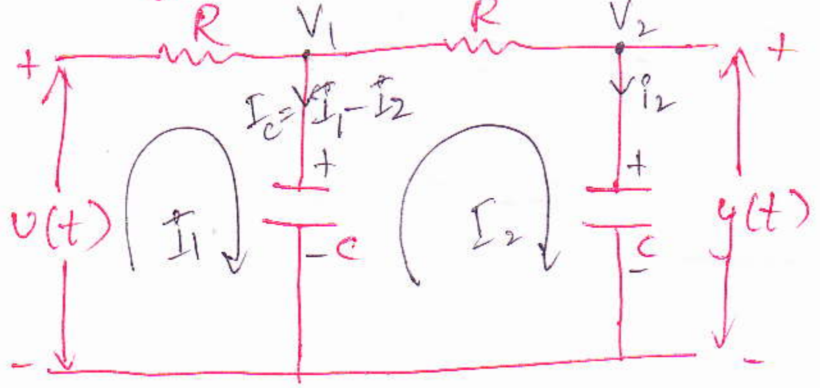
$$y = V_0 = I_2 R_2$$

$$\boxed{y = x_2 R_2}$$

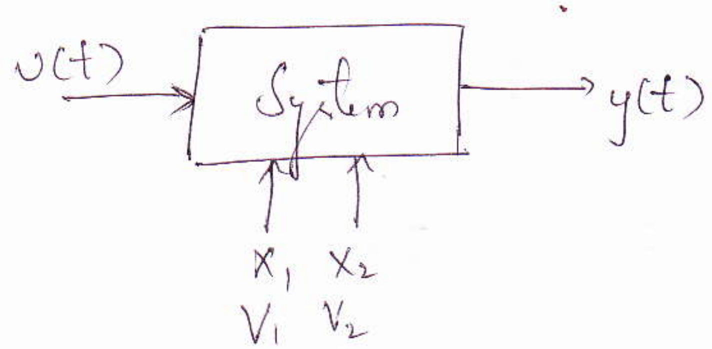
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

7) Obtain the appropriate state mode for a system represented by an electrical ~~system~~ circuit given below.



Soln:-



KVL to loop I_1

$$-R\dot{I}_1 - V_1 + u(t) = 0$$

$$R\dot{I}_1 = -V_1 + u(t)$$

$$\dot{I}_1 = \frac{-V_1 + u(t)}{R} \Rightarrow \boxed{\dot{i}_1 = \frac{u}{R} - \frac{x_1}{R}} \rightarrow (1)$$

KVL to loop I_2

$$-R\dot{I}_2 - V_2 + V_1 = 0$$

$$R\dot{I}_2 = V_1 - V_2$$

$$\dot{I}_2 = \frac{V_1 - V_2}{R} \rightarrow \boxed{\dot{i}_2 = \frac{x_1}{R} - \frac{x_2}{R}} \rightarrow (2)$$

$$\cancel{\dot{I}_c = \dot{I}_1 - \dot{I}_2}$$

$$\cancel{= \left(\frac{V_1 - u}{R}\right) - \left(\frac{V_1 + V_2}{R}\right)}$$

$$i_c = i_1 - i_2$$

$$C\frac{dV_1}{dt} = \frac{u}{R} - \frac{x_1}{R} - \left(\frac{x_1}{R} - \frac{x_2}{R}\right)$$

$$= \frac{u}{R} - \frac{x_1}{R} - \frac{x_1}{R} + \frac{x_2}{R}$$

$$\boxed{\dot{x}_1 = \frac{u}{RC} - \frac{2x_1}{RC} + \frac{x_2}{RC}} \longrightarrow (3)$$

$$i_2 = C\frac{d}{dt}V_2$$

$$C\frac{dV_2}{dt} = \frac{x_1}{R} - \frac{x_2}{R}$$

$$\frac{dV_2}{dt} = \frac{1}{RC}x_1 - \frac{1}{RC}x_2$$

$$\boxed{\dot{x}_2 = \frac{1}{RC}x_1 - \frac{1}{RC}x_2} \longrightarrow (4)$$

$$\boxed{y = V_0 = V_2 = x_2}$$

State eqn

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} & 0 \end{bmatrix} u$$

Output Eqn

$$[y] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

Transfer function for State Model

From State Eqn

$$\dot{x}(t) = A x(t) + B u(t)$$

$$s X(s) = A X(s) + B u(s) \longrightarrow (1)$$

From output Eqn

$$y(t) = C x(t) + D u(t)$$

$$Y(s) = C X(s) + D u(s) \longrightarrow (2)$$

From (1)

$$S X(s) - A X(s) = B u(s)$$

$$S I X(s) - A X(s) = B u(s)$$

$$X(s)[SI - A] = B u(s)$$

$$X(s) = \frac{B u(s)}{SI - A}$$

$$X(s) = \frac{1}{SI - A} B u(s)$$

$$\boxed{X(s) = [SI - A]^{-1} B u(s)} \longrightarrow (3)$$

By Subst (3) in (2)

$$Y(s) = C[SI - A]^{-1} B u(s) + D u(s)$$

$$Y(s) = u(s)\left[C[SI - A]^{-1} B + D\right]$$

$$\boxed{\frac{Y(s)}{u(s)} = C[SI - A]^{-1} B + D}$$

where $[SI - A]^{-1} = \dfrac{\text{Adj}[SI - A]}{|SI - A|}$

$$\boxed{\frac{Y(s)}{U(s)} = C\,\frac{\text{adj}[SI-A]}{|SI-A|}\,B + D}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{C\,\text{adj}[SI-A]\,B + D|SI-A|}{|SI-A|}}$$

1) find the transfer function for the system having state model given belao.

$$\dot{X} = \begin{bmatrix} 0 & +1 \\ -2 & -3 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} X$$

Soln:- $\dfrac{Y(s)}{U(s)} = C[SI-A]^{-1} B + D$

$$\frac{Y(s)}{U(s)} = C[SI-A]^{-1} B$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$[SI-A] = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$[SI-A]^{-1} = \frac{\text{adj}[SI-A]}{|SI-A|}$$

$$|sI-A| = s(s+3)+2$$

$$\text{adj}[sI-A] = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2+3s+2}$$

$$\frac{Y(s)}{U(s)} = [1 \quad 0] \frac{\begin{bmatrix} s+3 & 1 \\ 2 & s \end{bmatrix}}{s^2+3s+2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{[1 \quad 0]\begin{bmatrix} s+3 \\ -2 \end{bmatrix}}{s^2+3s+2}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}}$$

Formula for state transition matrix $[\phi(t)]$

$$STM = \phi(t) = \mathcal{L}^{-1}[sI-A]^{-1}$$

2) Find the state-transition matrix for the matrix given below $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Soln:- $STM = \phi(t) = e^{At} = \mathcal{L}^{-1}[sI-A]^{-1}$

$$sI-A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$sI-A = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{\operatorname{adj}[sI-A]}{|sI-A|}$$

$$\operatorname{adj}[sI-A] = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$\begin{aligned} |sI-A| &= (s-1)(s-1) \\ &= s^2 - s - s + 1 \\ &= s^2 - 2s + 1 \end{aligned}$$

$$[sI-A]^{-1} = \frac{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}}{(s-1)(s-1)}$$

$$= \begin{bmatrix} \frac{s-1}{(s-1)(s-1)} & \frac{0}{(s-1)(s-1)} \\ \frac{1}{(s-1)(s-1)} & \frac{s-1}{(s-1)(s-1)} \end{bmatrix}$$

$$[sI-A]^{-1} = \begin{bmatrix} \frac{1}{(s-1)} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)} \end{bmatrix}$$

$$\phi(t) = e^{At} = \mathcal{L}^{-1}[sI-A]^{-1}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{(s-1)} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{(s-1)} \end{bmatrix}$$

$$\phi(t) = e^{At} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$\phi^{-1}(t) = \frac{\operatorname{adj}[\phi(t)]}{|\phi(t)|}$$

$$adj[\phi(t)] = \begin{bmatrix} e^{t} & 0 \\ -te^{t} & e^{t} \end{bmatrix}$$

$$|\phi(t)| = e^{2t}$$

$$\phi^{'}(t) = \frac{\begin{bmatrix} e^{t} & 0 \\ -te^{t} & e^{t} \end{bmatrix}}{e^{2t}}$$

$$\phi^{-1}(t) = \begin{bmatrix} \frac{e^{t}}{e^{2t}} & \frac{0}{e^{2t}} \\ \frac{-te^{t}}{e^{2t}} & \frac{e^{t}}{e^{2t}} \end{bmatrix}$$

3) Find the state transition matrix and inverse transition matrix given below $A = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$

Solu:- $STM = \phi(t) = \mathcal{L}^{-1}[SI - A]^{-1}$

$$SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$$

$$SI - A = \begin{bmatrix} S & 1 \\ -2 & S+3 \end{bmatrix}$$

$$[SI - A]^{-1} = \frac{adj[SI - A]}{|SI - A|}$$

$$adj[SI - A] = \begin{bmatrix} S+3 & -1 \\ 2 & S \end{bmatrix}$$

$$|SI - A| = \begin{vmatrix} S & 1 \\ -2 & S+3 \end{vmatrix} = S(S+3) + 2$$

$$= S^2 + 3S + 2$$

$$[SI - A]^{-1} = \frac{\begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}}{s^2 + 3s + 2}$$

$$= \frac{\begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}}{(s+1)(s+2)}$$

$$= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

~~Taking~~ By partial fractions

$$= \begin{bmatrix} \frac{A}{s+1} + \frac{B}{s+2} & \frac{A}{s+1} + \frac{B}{s+2} \\ \frac{A}{(s+1)} + \frac{B}{s+2} & \frac{A}{s+1} + \frac{B}{s+2} \end{bmatrix}$$

$$s+3 = A(s+2) + B(s+1)$$

put $s = -2$

$$1 = A(0) + (-B)$$

$$\boxed{B = -1}$$

put $s = -1$

$$2 = 1A + B(0)$$

$$\boxed{A = 2}$$

$$-1 = A(s+2) + B(s+1)$$

put $s = -2$

$$-1 = A(0) + (-B)$$

$$\boxed{B = 1}$$

put $s = -1$

$$-1 = A(1)$$

$$\boxed{A = -1}$$

$2 = A(s+2) + B(s+1)$

put $s=-2$

$2 = A(0) + B(-1)$

$\boxed{B=-2}$

put $s=-1$

$2 = A(1) + B(0)$

$\boxed{A=2}$

$s = A(s+2) + B(s+1)$

put $s=-2$

$-2 = A(0) + B(-1)$

$\boxed{B=2}$

put $s=-1$

$-1 = A(1) + B(0)$

$\boxed{A=-1}$

$$= \begin{bmatrix} \frac{2}{s+1} + \frac{-1}{s+2} & \frac{-1}{s+1} + \frac{1}{s+2} \\ \frac{2}{s+1} + \frac{-2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at} \quad ; \quad \mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$\Rightarrow \mathcal{L}^{-1} \begin{bmatrix} \frac{2}{s+1} + \frac{-1}{s+2} & \frac{-1}{s+1} + \frac{1}{s+2} \\ \frac{2}{s+1} + \frac{-2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\Rightarrow \mathcal{L}^{-1}[sI-A]^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} = \phi(t)$$

$$\phi^{-1}(t) = \frac{\operatorname{adj} \phi(t)}{|\phi(t)|}$$

$$\operatorname{adj}\phi(t) = \begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}$$

$$|\phi(t)| = \begin{vmatrix} 2e^{-t}-e^{-2t} & -e^{-t}+e^{-2t} \\ 2e^{-t}-2e^{-2t} & -e^{-t}+2e^{-2t} \end{vmatrix}$$

$$= (2e^{-t}-e^{-2t})(-e^{-t}+2e^{-2t}) - (-e^{-t}+e^{-2t})(2e^{-t}-2e^{-2t})$$

$$= -2e^{-2t}+4e^{-3t}+e^{-3t}-2e^{-4t} - (-2e^{-2t}+2e^{-3t}+2e^{-3t}-2e^{-4t})$$

$$= \cancel{-2e^{-2t}}+4e^{-3t}+e^{-3t}\cancel{-2e^{-4t}}\cancel{+2e^{-2t}}-2e^{-3t}-2e^{-3t}\cancel{+2e^{-4t}}$$

$$= 2e^{-3t}-e^{-3t} = e^{-3t}$$

$$\phi^{-1}(t) = \frac{\begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}}{e^{-3t}}$$

$$\phi^{-1}(t) = \begin{bmatrix} \dfrac{-e^{-t}+2e^{-2t}}{e^{-3t}} & \dfrac{e^{-t}-e^{-2t}}{e^{-3t}} \\[2ex] \dfrac{-2e^{-t}+2e^{-2t}}{e^{-3t}} & \dfrac{2e^{-t}-e^{-2t}}{e^{-3t}} \end{bmatrix}$$

Module 1

1) Define Control System. With an example explain types of control System
2) Compare open loop and closed loop control system
3) Define the following (i) Mass
(ii) Dashpot
(iii) Spring
4) Mention the properties of transfer function
5) Explain Meissan's Gain formula.
6) From the closed loop system prove that

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+GH} \quad (\text{or}) \quad \frac{G(s)}{1-GH}$$

Module 2

1) With a neat diagram explain diffeent type of

7) Ex
Mod
1) Ex
2) M
3) W
Mod
1) D
(i) G
freque
2) W
3) W
Wit
(v) R
4) W
(i) L
(iii)

test signal.

2) Derive an expression for first order system

3) Derive the output response for 2nd order system for underdamped case.

4) With a neat diagram explain time domain specification

5) Derive the following (i) Rise time
(ii) Peak time
(iii) Settling time
(iv) Maximum peak overshoot

6) Show that $\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$

7) Explain the error constants

Module 3

1) Explain Routh's stability Criteria

2) Mention the steps of Root locus

3) Write a short note on P-D controller, P-I controller and PID controller

Module 4

1) Define the following
(i) Gain Margin (ii) Phase margin (iii) Gain cross over frequency (iv) Phase cross over frequency.

2) What are the steps involved in Nyquist plot

3) What are the steps involved in polar plot

With the neat diagram explain the following
(v) Resonant frequency (vi) Bandwidth (vii) Cut off rate

4) Write a short note on
(i) Lag compensator (ii) Lead compensator
(iii) Lag - lead compensator

~~List the properties of state-transition matrices~~
~~Derive the transfer function of state model.~~

Module - 5

1) With a neat block diagram explain digital control system
2) with a basic structure explain finite pulse width samples
3) Explain the concept of Alasing
4) With a neat structure explain the signal reconstruction or data reconstruction
(Zero order hold, First order hold)
5) Define the following
(i) state variable (ii) state space (iii) state vector
(iv) state equation (v) Output equation
(vi) state model (vii) state transition matrix
6) List the properties of state transition matrices
7) Derive the transfer function of state model.