MODULE-5

FORCED VIBRATION (SINGLE DEGREE OF FREEDOM SYSTEM)

In this chapter, the steady state response of harmonically excited single degree of freedom systems will be discussed. Simpler phasor diagram method will be used to obtain the steady state response. Response due to rotating unbalance, whirling of shafts, vibration isolations will also be discussed.

Steady state response due to Harmonic Oscillation:

Consider a spring-mass-damper system as shown in figure 1. The equation of motion of this system subjected to a harmonic forcing $F\sin\omega t$ can be given by

$$m\ddot{x} + kx + c\dot{x} = F\sin\omega t \tag{1}$$

where, m, k and c is the mass, spring stiffness and damping coefficient of the system.

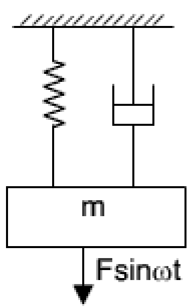
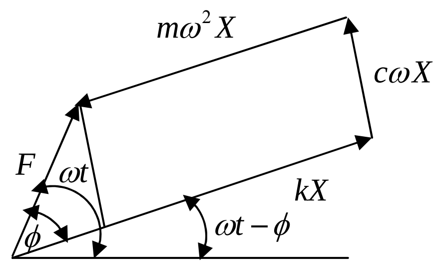


Figure 1 Harmonically excited system



Reference line

Figure 2: Force polygon

The steady state response of the system can be determined by solving equation (1) in may different ways. Here a simpler graphical method is used which will give physical understanding to this dynamic problem. From solution of differential equations it is known that the steady state solution (particular integral) will be in the form

$$x = X\sin(\omega t - \varphi) \tag{2}$$

As each term of equation (1) represents a forcing term viz., first term represent the inertia force, second term the spring force, third term the damping force and term in the right hand side is the applied force, one may draw a close polygon as shown in figure 2 considering the equilibrium of the system under the action of these forces. Considering equation (2),

- spring force $= kX\sin(\omega t - \phi)$
- damping force $= c\omega X\cos(\omega t - \phi)$

- inertia force = $-m\omega^2 X \sin(\omega t - \phi)$

From Figure 2

$$X = \frac{F}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

$$\phi = \tan^{-1}\frac{c\omega}{k - m\omega^2}$$

$$X = \frac{F/k}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}} \qquad (3)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$c_c = 2m\omega_n$ = Critical damping

$\zeta = \frac{c}{c_c}$ = damping factor or damping ratio

$$\frac{c\omega}{k} = \frac{c}{c_c} * \frac{c_c}{k} = 2\zeta\frac{\omega}{\omega_n}$$

$$\frac{c_c}{k} = \frac{2m\omega_n}{k} = \frac{2\omega_n}{\omega_n^2} = \frac{2}{\omega_n}$$

$$\Rightarrow \quad \frac{Xk}{F} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} \qquad (4)$$

$$\tan\phi = \frac{\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \qquad (5)$$

As the ratio F/k is the static deflection (*Xo*)f the spring, *Xk/F* =X/X_0 is known as the magnification factor or amplitude ratio of the system. Figure 3 shows the magnification ~ frequency ratio and phase angle (ϕ)~ frequency ratio plot. It is clear that for undamped system the magnification factor tends to infinity when the frequency of external excitation equals natural frequency of the system. But for underdamped systems the maximum amplitude of excitation has a definite value and it occurs at a frequency $\frac{\omega}{\omega_n} < 1$. For frequency of external excitation very less than the natural frequency of the system, with increase in frequency ratio, the dynamic deflection (*X*) dominates the static deflection (X_0), the magnification factor increases till it reaches a maximum value at resonant frequency after which the magnification factor decreases and for very high value of frequency ratio (say $\frac{\omega}{\omega_n} > 2$, the vibration is very much attenuated. One may observe that with increase in damping ratio, the resonant response amplitude decreases.

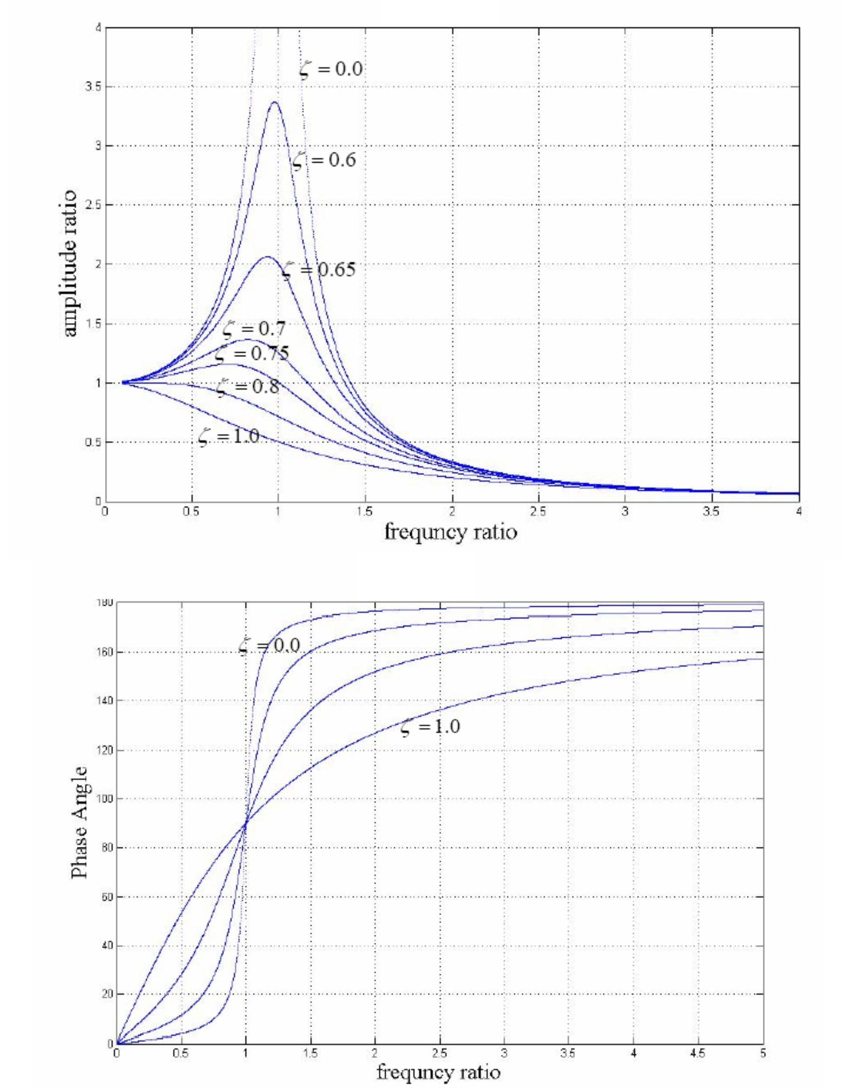


Figure 3: Magnification factor ~ frequency ratio and phase angle ~frequency ratio for different damping ratio.

So for a underdamped system the total response of the system which is the combination of transient response and steady state response can be given by

$$x(t) = x_1 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1\right) + \frac{F_0}{k}\frac{\sin(\omega t - \phi)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad (6)$$

It may be noted that as $t \to \infty$, the first part of equation (6) tends to zero and second part remains. From phase angle ~frequency ratio plot it is clear that, for very low value of frequency ratio, phase angle tends to zero and at resonant frequency it is 90^0 and for very high value of frequency ratio it is 180^0.

Example 1: Find the resonant frequency ratio (value of frequency ratio for which the steady state response will be maximum) for a spring-mass-damper system.

Solution: The steady state solution for a single degree of freedom system can be given by

$$X = \frac{F}{\sqrt{\left(\mathrm{K} - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

$$= \frac{F/K}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{F/K}{\sqrt{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}} \quad \text{where, } r = \frac{\omega}{\omega_n}.$$

X will be maximum if the denominator is minimum.

Hence $\frac{d}{dr}\left(\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2\right) = 0$

Or, $\frac{d}{dr}\left(1 + r^4 - 2r^2 + 4\zeta^2 r^2\right) = 4r^3 - 4r + 8\zeta^2 r = 0$

Hence, r = 0 or, $r^2 = 1 - 2\zeta^2$ or, $\underline{\underline{r = \sqrt{1 - 2\zeta^2}}}$ (7)

For $r = \sqrt{1 - 2\zeta^2}$, $X_{\max} = \frac{F/K}{\sqrt{\left(1 - 1 + 2\zeta^2\right)^2 + 4\zeta^2\left(1 - 2\zeta^2\right)}}$

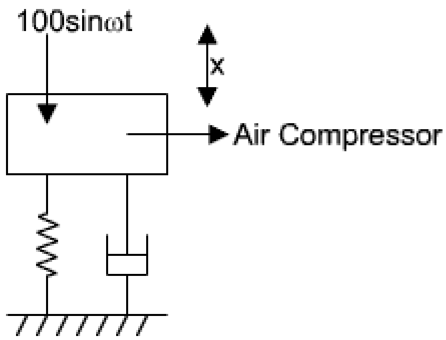
$$= \frac{F/K}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}} = \frac{F/K}{\sqrt{4\zeta^2 - 4\zeta^4}}$$

$$= \frac{F/K}{2\zeta\sqrt{1 - \zeta^2}} \quad (8)$$

So the peak magnification factor $== \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$ which occur at a frequency ratio of $r = \sqrt{1 - 2\zeta^2}$. Hence for underdamped system, it occurs when the external excitation frequency is slightly less than the natural frequency.

Example 2: An air compressor of mass 100 kg mounted on an elastic foundation. It has been observed that, when a harmonic force of amplitude 100N is applied to the compressor, the maximum steady state displacement of 5 mm occurred at a frequency of 300 rpm. Determine the equivalent stiffness and damping constants of the foundation.

Sol: The air compressor can be represented as a spring mass damper system as shown in figure below.



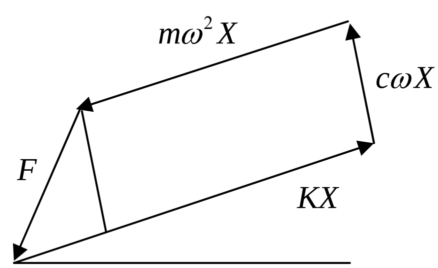
X = Steady state displacement = 5 mm
F = Forcing amplitude = 100 N

ω_{max} = frequency for max displacement = $\dfrac{2\pi \times 300}{60} = 10\pi$ rad/s.

We have to determine K_{eq} and $C_{eq.}$

The system can be modeled as a single dof system as shown in the above figure and the steady state solution can be given by

$$X = \frac{F}{\sqrt{\left(K - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$



$$= \frac{F/K}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{F/K}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}}$$

X will be maximum if the denominator is minimum.

Hence $\dfrac{d}{dr}\left(\left(1-r^2\right)^2 + \left(2\zeta r\right)^2\right) = 0$

Or, $\dfrac{d}{dr}\left(1 + r^4 - 2r^2 + 4\zeta^2 r^2\right) = 4r^3 - 4r + 8\zeta^2 r = 0$

Or, r = 0 or, $r^2 = 1 - 2\zeta^2$ or, $\underline{\underline{r = \sqrt{1 - 2\zeta^2}}}$

For $r = \sqrt{1 - 2\zeta^2}$, $X_{max} = \dfrac{F/K}{\sqrt{\left(1 - 1 + 2\zeta^2\right)^2 + 4\zeta^2\left(1 - 2\zeta^2\right)}}$

$$= \frac{F/K}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}} = \frac{F/K}{\sqrt{4\zeta^2 - 4\zeta^4}}$$

$$= \frac{F/K}{2\zeta\sqrt{1 - \zeta^2}}$$

So given

$X_{max} = 5 \times 10^{-3} = \dfrac{F/K}{2\zeta\sqrt{1 - \zeta^2}}$ $\qquad \omega_n^2 = \mathrm{K}/M = \mathrm{K}/100$

Also, $r = \dfrac{\omega}{\omega_n} = \dfrac{10\pi}{\sqrt{\mathrm{K}/100}} = \sqrt{1 - 2\zeta^2}$

$\Rightarrow 5 \times 10^{-3} = \dfrac{100}{\mathrm{K}\,2\zeta\sqrt{1 - \zeta^2}}$

and $\dfrac{10000\pi^2}{\mathrm{K}} = \sqrt{1 - 2\zeta^2}$

or, $\mathrm{K} = \dfrac{100}{5 \times 10^{-3} 2\zeta\sqrt{1 - \zeta^2}} = \dfrac{10000\pi^2}{\sqrt{1 - 2\zeta^2}}$

or, $\dfrac{1}{\zeta^2\left(1 - \zeta^2\right)} = \dfrac{\pi^4}{\left(1 - 2\zeta^2\right)^2}$

or, $1 + 4\zeta^4 - 4\zeta^2 - \pi^4\left(\zeta^2 - \zeta^4\right) = 0$

or, $\left(4 + \pi^4\right)\zeta^4 - \left(4 + \pi^4\right)\zeta^2 + 1 = 0$

or, $101.4091\zeta^4 - 101.4091\zeta^2 + 1 = 0$

or, $\zeta^2 = \dfrac{101.4091 \pm \sqrt{\left(101.4091\right)^2 - 4\left(101.4091\right)\left(1\right)}}{2 \times 101.4091}$

= 9.9603 x 10^{-3} or ζ =0.0998

$\mathrm{K} = \dfrac{100 \times 10^{-3}}{10\zeta\sqrt{1 - \zeta^2}} = \dfrac{10^4\pi^2}{1 - 2\zeta^2} = \dfrac{10^4\pi^2}{1 - 2\left(0.0998\right)^2}$

=100.7 x 10^3 N/m. = 100.7 KN/m.

$C = 2 \times 0.0998 \times 100 \times \sqrt{\dfrac{100.7 \times 10^3}{100}}$ where $\dfrac{C}{m} = 2\zeta\omega_n$

= **633.396 N.S/m Ans.**

<u>Rotating Unbalance</u>

One may find many rotating systems in industrial applications. The unbalanced force in such a system can be represented by an eccentric mass *m* with eccentricity *e*, which is rotating with angular velocity ω as shown in Figure 4.

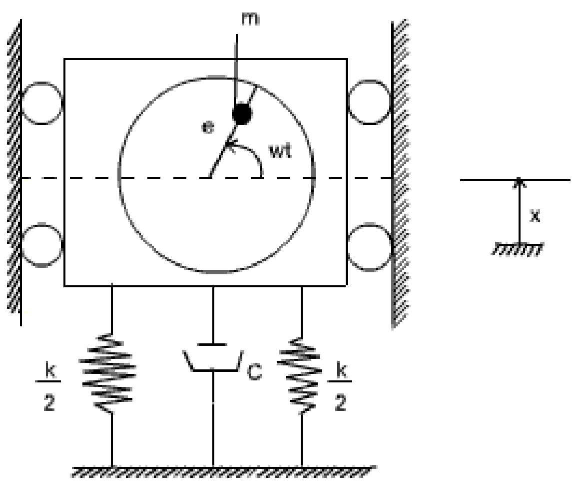


Figure 4. Vibrating system with rotating unbalance.

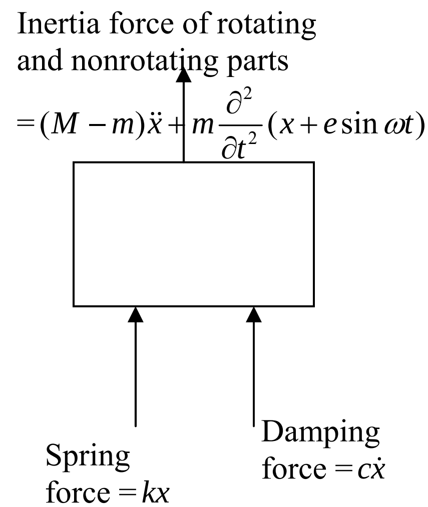


Figure 5. Freebody diagram of the system

Let x be the displacement of the nonrotating mass (*M-m*) from the static equilibrium position, then the displacement of the rotating mass *m* is $x + e\sin\omega t$.

From the freebody diagram of the system shown in figure 5, the equation of motion is

$$(M-m)\ddot{x}+m\frac{\partial^2}{\partial t^2}(x+e\sin\omega t)+kx+c\dot{x}=0=0 \tag{9}$$

or $M\ddot{x}+kx+c\dot{x}=me\omega^2\sin\omega t$ (10)

This equation is same as equation (1) where *F* is replaced by $me\omega^2$. So from the force polygon as shown in figure 6

$$me\omega^2=\sqrt{\{(-M\omega^2+k)^2+c\omega^2\}X^2} \tag{11}$$

$$\text{or, } X=\frac{me\omega^2}{\sqrt{(k-M\omega^2)^2+(c\omega)^2}} \tag{12}$$

$$\text{or, } \frac{X}{e}=\frac{m\omega/M}{\sqrt{(\frac{k}{M}-\omega^2)^2+(\frac{c}{M}\omega)^2}} \tag{13}$$

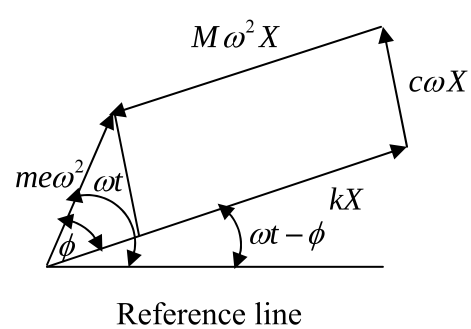


Figure 6: Force polygon

or, $$\frac{X}{e}\frac{M}{m}=\frac{\omega/\omega_n}{\sqrt{(1-(\frac{\omega}{\omega_n})^2)^2+(\frac{2\zeta\omega}{\omega_n})^2}} \tag{14}$$

and $$\tan\phi=\frac{2\zeta\frac{\omega}{\omega_n}}{1-(\frac{\omega}{\omega_n})^2} \tag{15}$$

So the complete solution becomes

$$x(t)=x_1e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\,\omega_n t+\phi_1)+\frac{me\omega^2}{\sqrt{(k-M\omega^2)^2+(c\omega^{2)}}}\sin(\omega t-\phi)$$

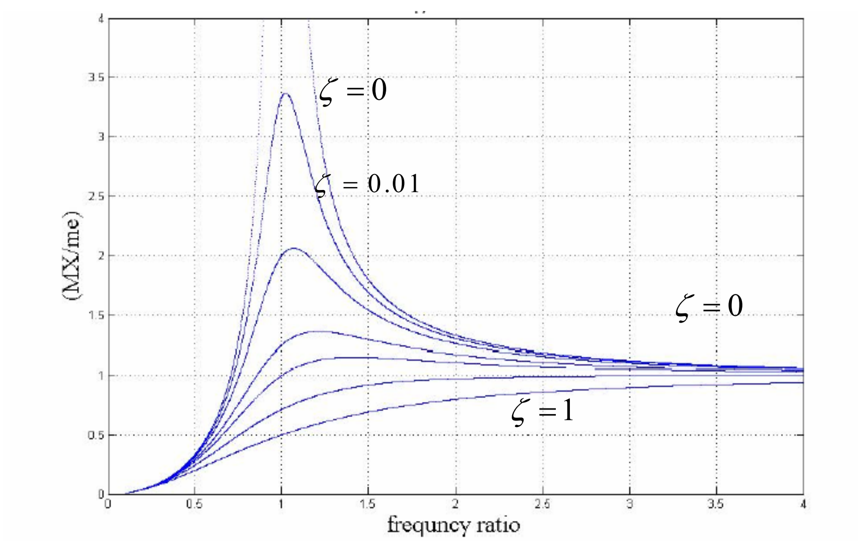


Figure 7: $\frac{MX}{me}\sim\frac{\omega}{\omega_n}$ plot for system with rotating unbalance

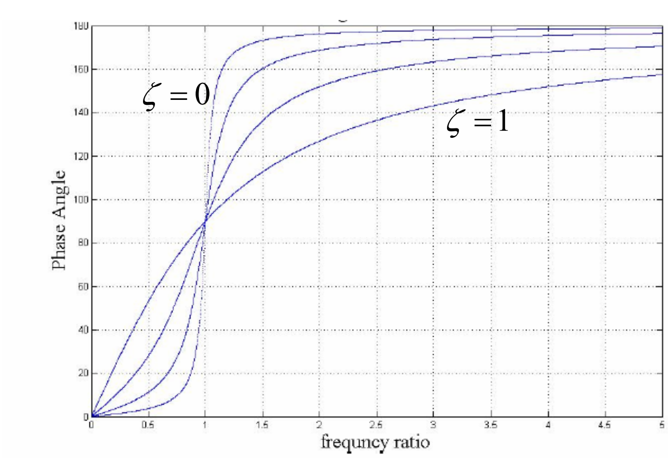


Figure 8 Phase angle ~ frequency ratio plot for system with rotating unbalance

It may be noted from figure 8 that, for a system with very very low damping, it is very unsafe to run the machine near the natural frequency ratio greater than 2, the system vibration reduces to $X = me/M$ and phase angle tends to 180^0.

Whirling of shaft:

Whirling is defined as the rotation of the plane made by the bent shaft and the line of the centre of the bearing. It occurs due to a number of factors, some of which may include (i) eccentricity, (ii) unbalanced mass, (iii) gyroscopic forces, (iv) fluid friction in bearing, viscous damping.

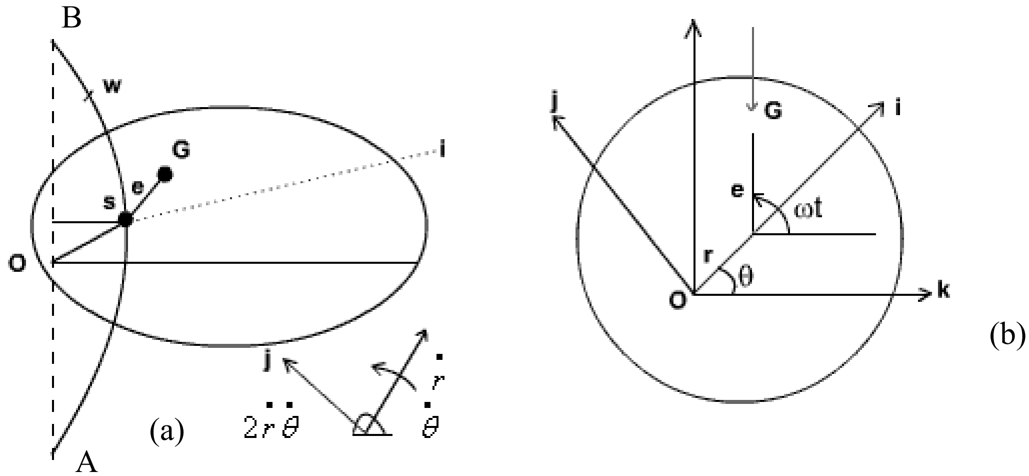


Figure 9: Whirling of shaft

Consider a shaft AB on which a disc is mounted at **s**. **G** is the mass center of the disc, which is at a distance ***e*** from **s.** As mass center of the disc is not on the shaft center, when the shaft rotates, it will be subjected to a centrifugal force. This force will try to bend the shaft. Now the shafts neutral axis, which is represented by line ASB, is different from the line joining the bearing centers AOB. The rotation of the plane containing the line joining bearing centers and the bend shaft (in this case it is AOBsA) is called the whirling of the shaft.

Considering unit vectors **i**, **j**, k as shown in the above figure 9(b), the acceleration of point G can be given by

$a_G = a_S + a_{G/S}$

$$=\left[\ddot{r}-r\dot{\theta}^2-e\omega^2\cos(\omega t-\theta)\right]\boldsymbol{i}+\left[\boldsymbol{r}\ddot{\theta}-e\omega^2\sin(\omega t-\theta)+2\dot{r}\dot{\theta}\right]\boldsymbol{j} \tag{16}$$

which is acting along radial direction k, which will give rise to restoring torque, assuming a viscous damping for a to be acting at S. The EOM in radial direction

$$m\left[\ddot{r}-r\dot{\theta}^2\ -e\omega^2\cos(\omega t-\theta)\right]+kr+C\dot{r}=0 \tag{17}$$

$$m\left[r\,\theta+2\dot{r}\dot{\theta}-e\ddot{\omega}^2\sin(\omega t-\theta)\right]+cr\dot{\theta}=0 \tag{18}$$

$$\ddot{r}+\frac{c}{m}\dot{r}+\left(\frac{k}{m}-\dot{\theta}^2\right)=e\omega^2\cos(\omega t-\theta) \tag{19}$$

$$r\,\ddot{\theta}+\left(\frac{c}{m}r+2\dot{r}\right)\dot{\theta}=e\omega^2\sin(\omega t-\theta) \tag{20}$$

Considering the synchronous whirl case, i.e., $\dot{\theta}=\omega$

So, $\theta=(\omega t-\phi)$ (21)

ϕ is a phase angle between e and r.

Taking $\ddot{\theta}=\ddot{r}=\dot{r}=0$, from equation (19)

$$(\frac{k}{m}-\omega^2)=e\omega^2\cos\phi \tag{22}$$

$$\&\quad \frac{\mathrm{c}}{\mathrm{m}}r\omega=e\omega^2\sin\phi \tag{23}$$

$$\Rightarrow \tan\phi=\frac{\frac{c}{m}\omega}{(\frac{k}{m}-\omega^2)}=\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \tag{24}$$

$\omega_n=\sqrt{\frac{k}{m}}$ and $\zeta=\frac{c}{c_e}$

$$\cos\phi = \frac{\frac{c}{m}\omega}{\sqrt{\left(\frac{k}{m}-\omega^2\right)^2+\left(\frac{c}{m}\omega\right)^2}} \tag{25}$$

Now equation (22) reduces to

$$\left(\frac{\mathrm{k}}{\mathrm{m}}-\omega^2\right)r = e\omega^2\,\frac{\left(\frac{k}{m}-\omega^2\right)}{\sqrt{\left(\frac{k}{m}-\omega^2\right)^2+\left(\frac{c}{m}\omega\right)^2}} = \frac{e\omega^2}{\sqrt{(\omega_n{}^2-\omega^2)+(2\zeta\frac{\omega}{\omega_n})^2}} \tag{26}$$

$$r = \frac{me\omega^2}{\sqrt{(k-m\omega^2)^2+\left(c\omega^2\right)^2}} \tag{27}$$

$$\text{or,}\frac{r}{e} = \frac{\omega^2/\omega_n{}^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \tag{28}$$

$$\tan\phi = \frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \tag{29}$$

The eccentricity line e=SG leads the displacement line r = OS by phase angle ϕ which depends on the amount of damping and the rotation speed ratio ω/ω_n. When the rotational speed equals to the natural frequency or critical speed, the amplitude is restrained by damping only. From equation (29 at very high speed $\omega >> \omega_n, \phi \to 180^0$ and the center of mass G tends to approach the fixed point O and the shaft center S rotates about it in a circle of radius e.

Support Motion:

Many machine components or instruments are subjected to forces from the support. For example while moving in a vehicle, the ground undulation will cause vibration, which will be transmitted, to the passenger. Such a system can be modeled by a spring-mass damper system as shown in figure 10. Here the support motion is considered in the form of $y = Y \sin \omega t$, which is transmitted to mass *m*, by spring (stiffness k) and damper (damping coefficient c). Let x be the vibration of mass about its equilibrium position.

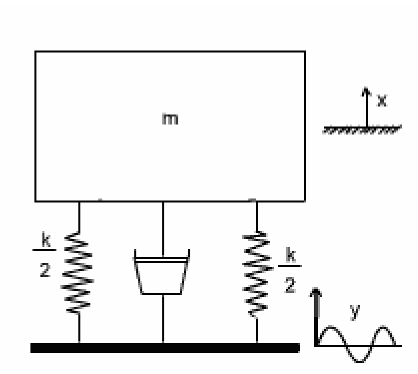


Figure 10: A system subjected to support motion

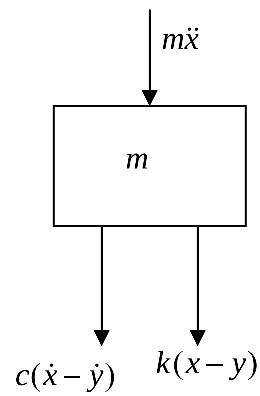


Figure 11: Freebody diagram

Now to derive the equation of motion, from the freebody diagram of the mass as shown figure 11

$$m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y}) \tag{30}$$

let $z=x-y$ (31)

$$m\ddot{z} + kz + c\dot{z} = -m\ddot{y} = m\omega^2 y \sin \tag{32}$$

$$m\ddot{z} + kz + c\dot{z} = m\omega^2 y \sin \omega t \tag{33}$$

As equation (33) is similar to equation (1), solution of equation (33) can be written as

$$z = Z \sin(\omega t - \phi) \tag{34}$$

$$Z = \frac{m\omega^2 y}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \text{ and } \tan\phi = \frac{c\omega}{k - m\omega^2} \tag{35}$$

If the absolute motion x of the mass is required, we can solve for $x = z + y$. Using the exponential form of harmonic motion

$$y = Ye^{i\omega t} \tag{36}$$

$$z = Ze^{i(\omega t-\phi)} = (Ze^{-i\phi})e^{i\omega t} \tag{37}$$

$$x = Xe^{i(\omega t-\psi)} = (Xe^{-i\psi})e^{i\omega t} \tag{38}$$

Substituting equation (38) in (30) one obtains

$$\left\{m(Ze^{-i\phi})\omega^2 + k(Ze^{-i\phi}) + ci\omega(Ze^{-i\phi})\right\}e^{i\omega t} = m\omega^2 Ye^{i\omega t} \tag{39}$$

$$Ze^{-i\phi}(k - m\omega^2 + ic\omega) = m\omega^2 Y \tag{40}$$

$$Ze^{-i\phi} = \frac{m\omega^2 Y}{k - m\omega^2 + ic\omega} \tag{41}$$

$$x = (Ze^{-i\phi} + Y)e^{i\omega t} \tag{42}$$

$$x = \left(\frac{k - m\omega^2 + ic\omega + m\omega^2}{k - m\omega^2 + ic\omega}\right)Ye^{i\omega t}$$

$$= X(\cos\psi - i\sin\psi)e^{i\omega t} \tag{43}$$

The steady state amplitude and Phase from this equation are

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} \tag{44}$$

$$\tan\psi = \frac{mc\omega^3}{k(k - m\omega^2) + (c\omega)^2} \tag{45}$$

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{1 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \tag{46}$$

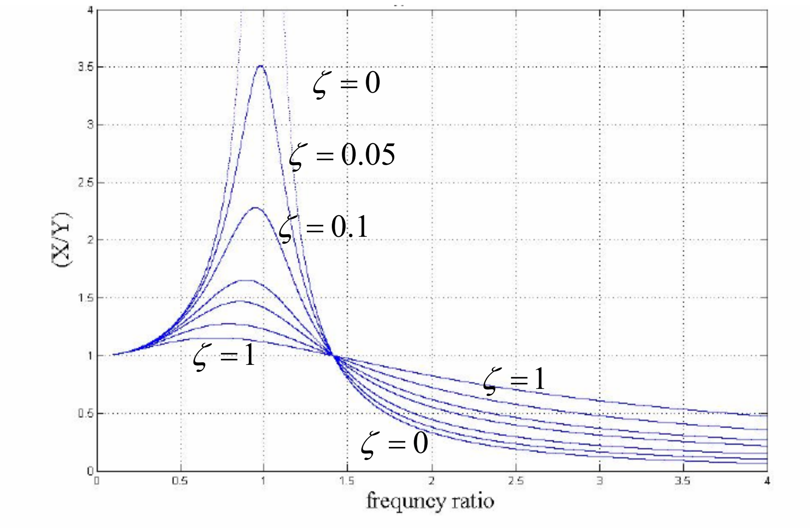


Figure 12: Amplitude ratio ~ frequency ratio plot for system with support motion

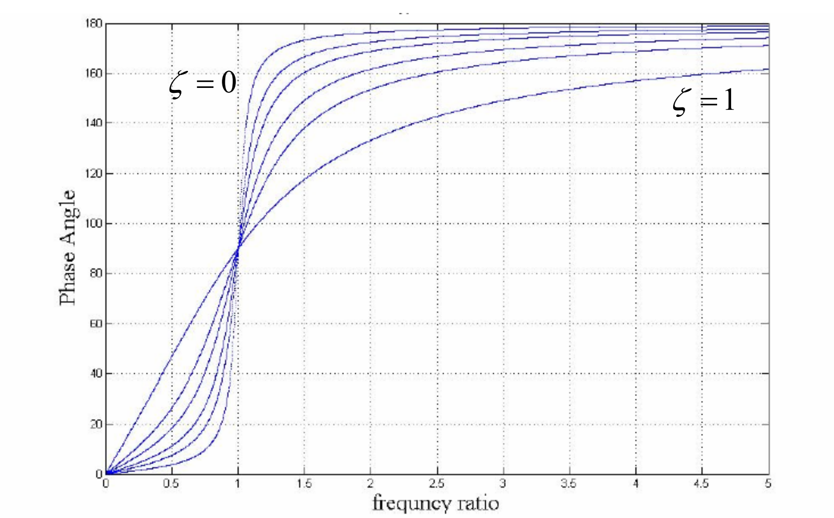


Figure 13: Phase angle ~ frequency ratio plot for system with support motion

From figure 12, it is clear that when the frequency of support motion nearly equal to the natural frequency of the system, resonance occurs in the system. This resonant amplitude decreases with increase in damping ratio for $\frac{\omega}{\omega_n} < \sqrt{2}$. At $\frac{\omega}{\omega_n} = \sqrt{2}$, irrespective of damping factor, the mass vibrate with an amplitude equal to that of the support and for $\frac{\omega}{\omega_n} > \sqrt{2}$, amplitude ratio becomes less than 1, indicating that the mass will vibrate with an amplitude less than the support motion.

But with increase in damping, in this case, the amplitude of vibration of the mass will increase. So in order to reduce the vibration of the mass, one should operate the system at a frequency very much greater than $\sqrt{2}$ times the natural frequency of the system. This is the principle of vibration isolation.

Vibration Isolation:

In many industrial applications, one may find the vibrating machine transmit forces to ground which in turn vibrate the neighbouring machines. So in that contest it is necessary to calculate how much force is transmitted to ground from the machine or from the ground to the machine.

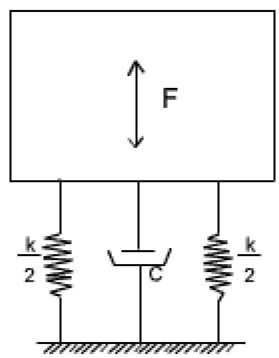


Figure 13 : A vibrating system

Figure 13 shows a system subjected to a force $F = F_0 \sin \omega t$ and vibrating with $x = X \sin(\omega t - \phi)$. This force will be transmitted to the ground only by the spring and damper.

Force transmitted to the ground

$$F_t = \sqrt{(KX)^2 + (c\omega X)^2} \quad = KX\sqrt{1+\left(\frac{2\zeta\omega}{\omega_n}\right)^2} \tag{47}$$

It is known from equation (3) that for a disturbing force $F = F_0 \sin \omega t$, the amplitude of resulting oscillation

$$X = \frac{F_0 / K}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + (\frac{2\zeta\omega}{\omega_n})^2}} \tag{48}$$

Substituting equation (48) in (47) and defining the transmissibility TR as the ratio of the force transmitted Force to the disturbing force one obtains

$$\left|\frac{F_t}{F_0}\right| = \sqrt{\frac{1+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \qquad (49)$$

Comparing equation (49) with equation (46) for support motion, it can be noted that

$$\text{TR}=\left|\frac{F_t}{F_0}\right|=\left|\frac{X}{Y}\right| \qquad (50)$$

When damping is negligible

$$TR=\frac{1}{\left(\frac{\omega}{\omega_n}\right)^2-1}, \qquad (51)$$

$\frac{\omega}{\omega_n}$ to be used always greater than $\sqrt{2}$

Replacing $\omega_n^2 = g/\Delta$

$$TR=\frac{1}{(2\pi f)^2\frac{\Delta}{g}-1} \qquad (52)$$

To reduce the amplitude *X* of the isolated mass m without changing TR, *m* is often mounted on a large mass M. The stiffness *K* must then be increased to keep ratio *K/(m+M)* constant. The amplitude X is, however reduced, because *K* appears in the denominator of the expression

$$X=\frac{F_0/K}{\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2+(\frac{2\zeta\omega}{\omega_n})^2}} \qquad (53)$$

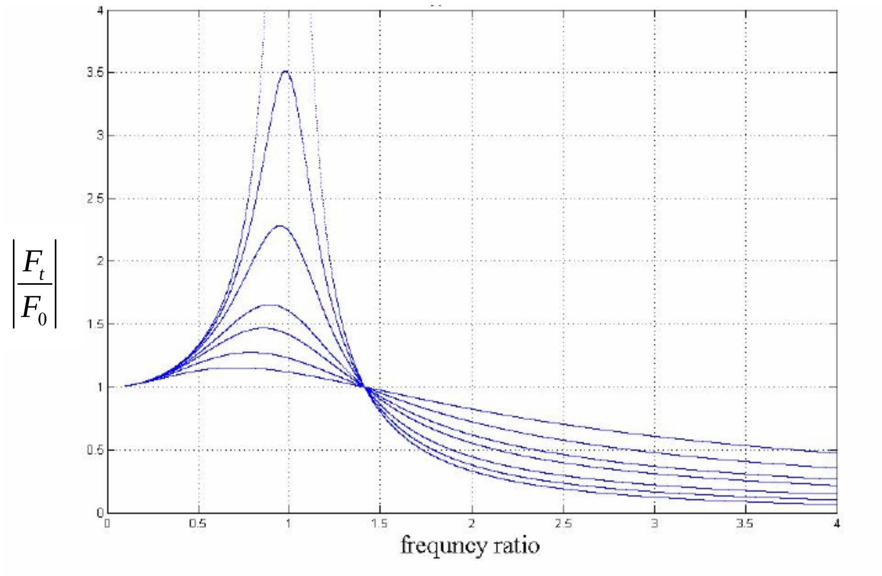


Figure 14: Transmissibility ~frequency ratio plot

Figure 14 shows the variation transmissibility with frequency ratio and it can be noted that vibration will be isolated when the system operates at a frequency ratio higher than $\sqrt{2}$.

Equivalent Viscous Damping:

In the previous sections, it is assumed that the energy dissipation takes place due to viscous type of damping where the damping force is proportional to velocity. But there are systems where the damping takes place in many other ways. For example, one may take surface to surface contact in vibrating systems and take Coulomb friction into account. Also in many cases energy is dissipated in joints also, which is a form of structural damping. In these cases one may still use the derived equations by considering an equivalent viscous damping. This can be achieved by equating the energy dissipated in the original and the equivalent system.

The primary influence of damping on the oscillatory systems is that of limiting the amplitude at resonance. Damping has little influence on the response in the frequency regions away from resonance. In case of viscous damping, the amplitude at resonance is

$$X = \frac{F_0}{c\omega_n} = \frac{F_0}{2\zeta k} \tag{54}$$

For other type of damping, no such simple expression exists. It is possible to however, to approximate the resonant amplitude by substituting an equivalent damping *Ceq* in the foregoing equation.

The equivalent damping *Ceq* is found by equating the energy dissipated by the viscous damping to that of the nonviscous damping with assumed harmonic motion.

$$\pi C_{eq}\omega X^2 = W_d \tag{55}$$

Where W_d must be evaluated from the particular type of damping.

Structural Damping:

When materials are cyclically stressed, energy is dissipated internally within the material itself. Experiments by several investigators indicate that for most structural metals such as steel and aluminum, the energy dissipated per cycle is independent of the frequency over a wide frequency range and proportional to the square of the amplitude of vibration. Internal damping fitting this classification is called solid damping or structural damping. With the energy dissipation per cycle proportional to the square of the vibration amplitude, the loss coefficient is a constant and the shape of the hysteresis curve remains unchanged with amplitude and independent of the strain rate. Energy dissipated by structural damping can be written as

$$W_d = \alpha X^2 \tag{56}$$

Where α is a constant with units of force displacement.

By the concept of equivalent viscous damping

$$W_d = \alpha X^2 = \pi c_{eq}\omega X^2 \qquad \text{or, } c_{eq} = \frac{\alpha}{\pi\omega} \tag{57}$$

Coulomb Damping:

Coulomb damping is mechanical damping that absorbs energy by sliding friction, as opposed to viscous damping, which absorbs energy in fluid, or viscous, friction. Sliding friction is a constant value regardless of displacement or velocity. Damping of large complex structures with non-welded joints, such as airplane wings, exhibit coulomb damping.

Work done per cycle by the Coulomb force F_d

$$W_d = 4F_d X \tag{58}$$

For calculating equivalent viscous damping

$$\pi C_{eq}\omega X^2 = 4F_d X \tag{59}$$

From the above equation equivalent viscous damping is found

$$c_{eq} = \frac{4F_d}{\pi\omega X} \tag{60}$$

Summary

Some important features of steady state response for harmonically excited systems are as follows-

- The steady state response is always of the form $x(t) = X\sin(\omega t - \phi)$. Where it is having same frequency as of forcing. X is amplitude of the response, which is strongly dependent on the frequency of excitation, and on the properties of the spring—mass system.
- There is a phase lag ϕ between the forcing and the system response, which depends on the frequency of excitation and the properties of the spring-mass system.
- The steady state response of a forced, damped, spring mass system is independent of initial conditions

In this chapter response due to rotating unbalance, support motion, whirling of shaft and equivalent damping are also discussed.

Exercise Problems

1. An underdamped shock absorber is to be designed for a motor cycle of mass 200Kg. When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in fig(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 s and amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e $x_{1.5}$=x_1/4). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.
2. Develop equation of motion for a spring mass system with Coulomb damping.
3. An electronic instrument of mass m= 8kg is placed on four elastic support pad of special rubber. The force displacement curve of each pad is given by F = $(5x+1000x^2)*10^3$. Determine the spring constant between the instrument and ground in the vertical direction.
4. A machine of 100kg mass is supported on springs of total stiffness 700 KN/m and has an unbalanced rotating element, which results in a disturbing force of 350N at a speed of 3000 rev/min. Assuming a damping factor of $\xi = 0.20$, determine (a) its amplitude of motion due to the unbalance, (b) the transmissibility, and (c) the transmitted force.
5. Find the steady state response of the spring mass damper system to a force $F = 5\sin 4t + 10\cos 4t$.
6. If the steady state response of a linear system to a force of $F = 5\sin 2t$ is $4\sin(2t + 0.25)$, what will be the response if a force of $F = 10\sin 2t$ will act on it..

Computer Assignment

1. Develop a general-purpose program, to find the free vibration response of a viscously damped system. Use the program to find response of a system with m= 450 Kg, K= 26519.2, c= 1000.0, x_0 =0.539657, v_0(initial velocity)=1.0.
2. Find the free vibration response of a critically damped and over damped system with the above mentioned values of m and k.
3. Plot magnification factor vs. frequency ratio and |X/Y| or $|F_t/F_0|$ for different values of $\frac{\omega}{\omega_n}$