Module 2

Compound Stresses

Objectives:

Derive the equations for principal stress and maximum in-plane shear stress and calculate their magnitude and direction. Draw Mohr circle for plane stress system and interpret this circle.

Learning Structure

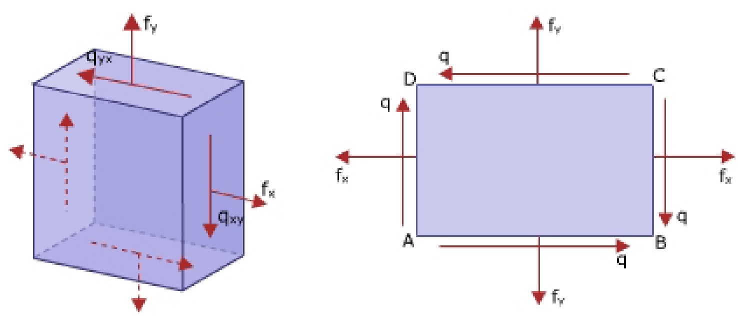
- 2.1 Introduction
- 2.2 Plane Stress Or 2–D Stress System Or Biaxial Stress System
- 2.3 Expressions For Normal And Tangential Components Of Stress On A Given Plane
- 2.4 Mohr's Circle
- 2.5 Problems
- 2.6 Thick Cylinders
- 2.7 Thin Cylinders
- Outcomes
- Further Reading

2.1 Introduction

Structural members are subjected to various kinds of loads. This results in combination of different stresses which changes from point to point. When an element (considered at any point) in a body is subjected to a combination of normal stresses (tensile and/or compressive) and shear stresses over its various planes, the stress system is known as compound stress system. In a compound stress system, the magnitude of normal stress may be maximum o n some plane and minimum on some plane, when compared with those acting on the element. Similarly, the magnitude of shear stresses may also be maximum on two planes when compared with those acting on the element. Hence, for the considered compound stress system it is important to find the magnitudes of maximum and minimum normal stresses, maximum shear stresses and the inclination of planes on which they act.

2.2 PLANE STRESS OR 2–D STRESS SYSTEM OR BIAXIAL STRESS SYSTEM

Generally a body is subjected to 3–D state of stress system with both normal and shear stresses acting in all the three directions. However, for convenience, in most problems, variation of stresses along a particular direction can be neglected and the remaining stresses are assumed to act in a plane. Such a system is called 2–D stress system and the body is called plane stress body.



In a general two dimensional stress system, a body consists of two normal stresses (fx and fy), which are mutually perpendicular to each other, with a state of shear (q) as shown in figure. Further, since planes AD and BC carry normal stress fx they are called planes of fx. These planes are parallel to Y–axis. Similarly, planes AB and CD represent planes of fy, which are parallel to X–axis.

2.2.1 PRINCIPAL STRESSES AND PRINCIPAL PLANES

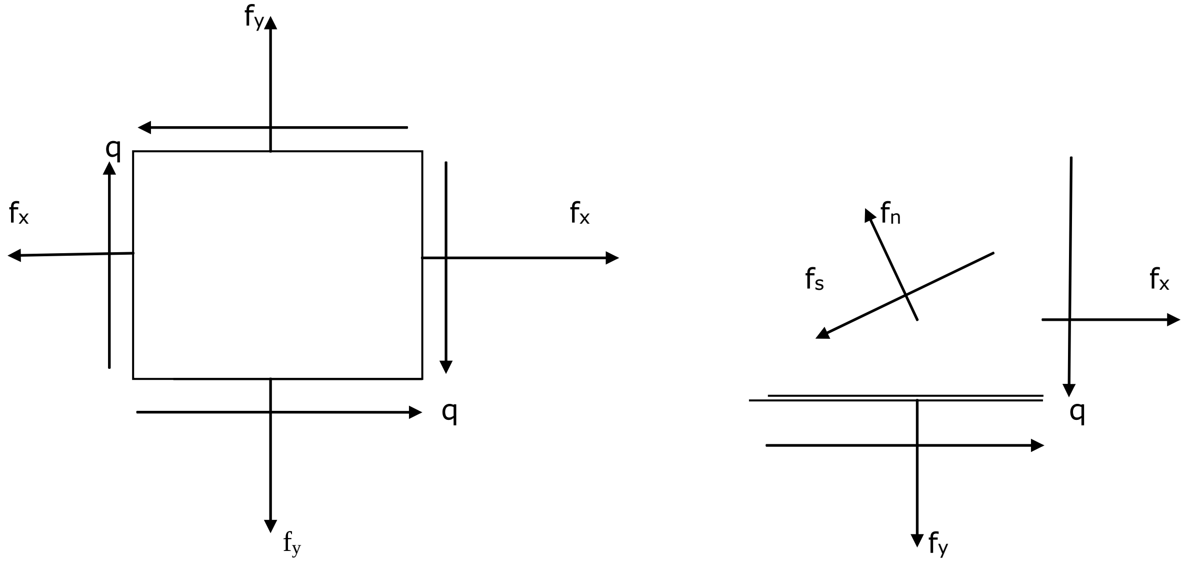
For a given compound stress system, there exists a maximum normal stress and a minimum normal stress which are called the Principal stresses. The planes on which these Principal stresses act are called Principal planes. In a general 2-D stress system, there are two Principal planes which are always mutually perpendicular to each other. Principal planes are free from shear stresses. In other words Principal planes carry only normal stresses.

2.2.2 MAXIMUM SHEAR STRESSES ANDITS PLANES

For a given 2–D stress system, there will be two maximum shear stresses (of equal magnitude) which act on two planes. These planes are called planes of maximum shear. These planes are mutually perpendicular. Further, these planes may or may not carry normal stress. The planes of maximum shear are always inclined at 450 with Principal planes.

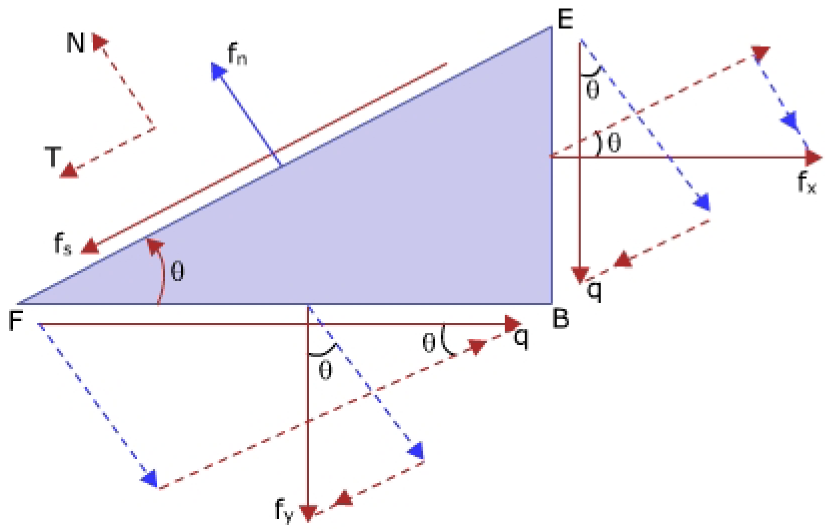
2.3 EXPRESSIONS FOR NORMAL AND TANGENTIAL COMPONENTS OF STRESS ON A GIVEN PLANE

Consider a rectangular element ABCD of unit thickness subjected to a general 2-D stress system as shown in figure. Let f n and f s represent the normal and tangential components of resultant stress 'R' on any plane EF which is inclined at an angle '?' measured counter clockwise with respect to the plane of f y or X–axis.



To derive expression for f_n

Consider the Free Body Diagram of portion FBE as shown in figure.



Applying equilibrium along N-direction, we have

$$\Sigma F_N = 0 \quad [\ \nwarrow\ +ve]$$

$$f_n(EF.1) - f_x(BE.1)\sin\theta - q(BE.1)\cos\theta - f_y(BF.1)\cos\theta - q(BF.1)\sin\theta = 0$$

$$f_n = f_x \frac{BE}{EF}\sin\theta + q\frac{BE}{EF}\cos\theta + f_y\frac{BF}{EF}\cos\theta + q\frac{BF}{EF}\sin\theta$$

Since $\frac{BE}{EF} = \sin\theta$ and $\frac{BF}{EF} = \cos\theta$

$$\therefore f_n = f_x\sin^2\theta + 2q\sin\theta\cos\theta + f_y\cos^2\theta$$

But $\cos 2\theta = 2\cos^2\theta - 1$ Hence $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

Also $\cos 2\theta = 1 - 2\sin^2\theta$ Hence $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

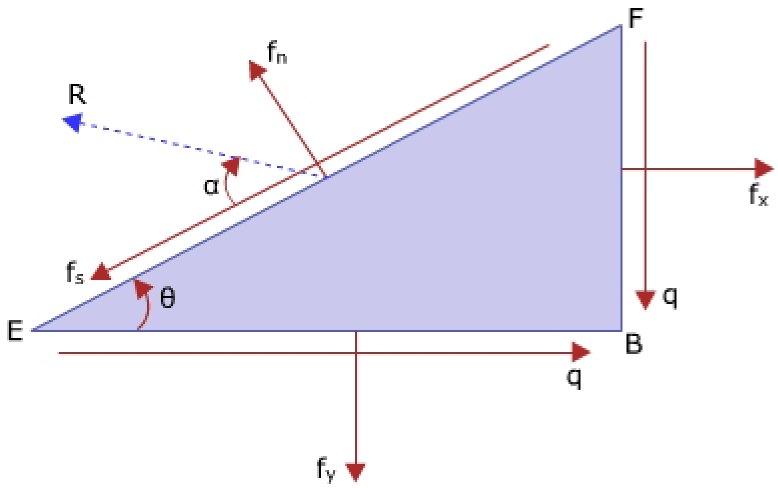
$$f_x = f_x\frac{1}{2}(1-\cos 2\theta) + f_y\frac{1}{2}(1+\cos 2\theta) + q\sin 2\theta$$

$$f_n = \left(\frac{f_x + f_y}{2}\right) - \left(\frac{f_x - f_y}{2}\right)\cos 2\theta + q\sin 2\theta \qquad \text{-------- (1)}$$

Equation (1) is the desired expression for normal component of stress on a given plane, inclined at an angle ' θ ' measured counter clockwise with respect to the plane of f_Y or X–axis

To derive expression for f_s

Consider the Free Body Diagram of portion FBE shown in figure above. For equilibrium along T direction, we have



$$\Sigma F_T = 0 \; [\swarrow + ve]$$

$$f_s(EF.1) - f_x(BE.1)\cos\theta + q(BE.1)\sin\theta + f_y(BF.1)\sin\theta - q(BF.1)\cos\theta = 0$$

$$f_s = f_x \frac{BE}{EF}\cos\theta - q\frac{BE}{EF}\sin\theta - f_y\frac{BF}{EF}\sin\theta + q\frac{BF}{EF}\cos\theta$$

Since $\frac{BE}{EF} = \sin\theta \qquad \frac{BF}{EF} = \cos\theta$

$$\therefore f_s = f_x \sin\theta\cos\theta - q\sin^2\theta - f_y\cos\theta\sin\theta + q\cos^2\theta$$

$$\therefore f_s = (f_x - f_y)\sin\theta\cos\theta + q(\cos^2\theta - \sin^2\theta)$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$f_s = \left(\frac{fx - fy}{2}\right)\sin 2\theta + q\cos 2\theta \qquad \text{--------} \; (2)$$

Equation (2) is the desired expression for tangential component of stress on a given plane, inclined at an angle ' θ ' measured counter clockwise with respect to the plane of fy or X–axis.

Note:

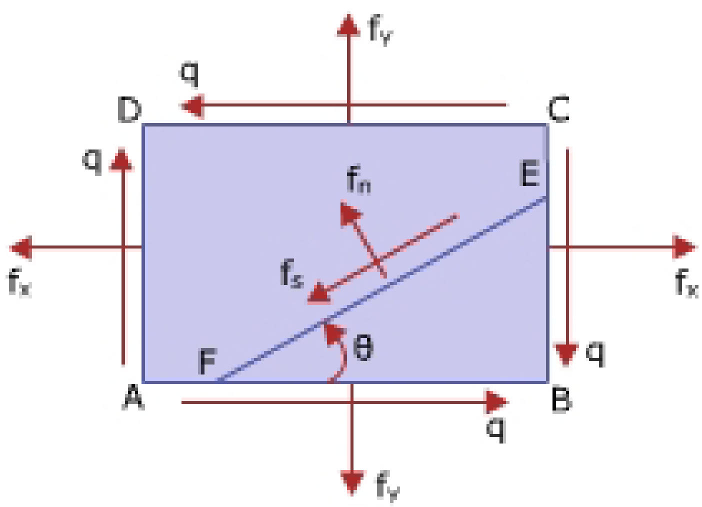
The resultant stress 'R', and its inclination ' α ' on the given plane EF which is inclined at an angle ' θ ' measured counter clockwise with respect to the plane of fy or X–axis, can be determined from the normal (fn) and tangential (fs) components obtained from eqns. (1) and (2).

$$R = \sqrt{f_n^2 + f_s^2}$$

$$\alpha = \tan^{-1}\left(\frac{f_n}{f_s}\right)$$

2.3.1 Expresions for Principal stresses and Principal planes

Consider a rectangular element ABCD of unit thickness subjected to general 2-D stress system as shown in figure. Let f_n and f_s represent the normal and tangential components of stress on any plane EF which is inclined at an angle ' θ ' measured counter clockwise with respect to the plane of f_y or X–axis



The expression for normal component of stress $f_n$on any given plane EF is given by

$$f_n = \frac{f_x + f_y}{2} - \frac{f_x - f_y}{2}\cos 2\theta + q\sin 2\theta \qquad \text{-------- (1)}$$

To find values of θ at which f n is maximum or minimum, the necessary condition is

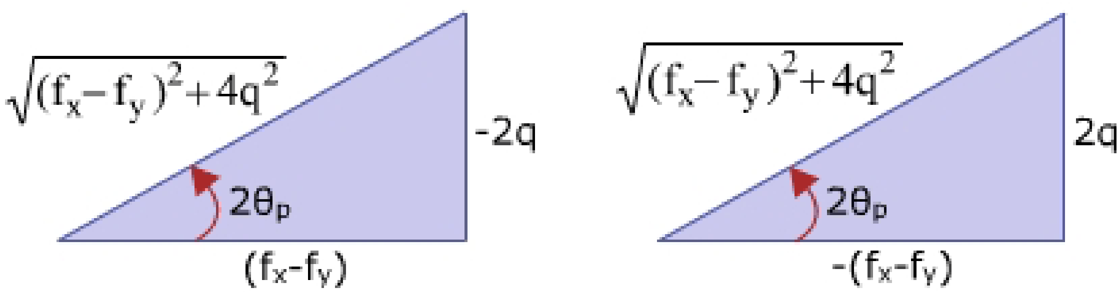
$$\frac{df_n}{d\theta} = 0$$

From eqn. (1) $$-\left(\frac{f_x - f_y}{2}\right)(-2\sin 2\theta) + 2q\cos 2\theta = 0$$

$$\therefore \tan 2\theta_p = -\frac{2q}{f_x - f_y} \qquad \text{-------} \quad (2)$$

Inclination of principal planes can be obtained from eqn. (2). It gives two values of θ differing by 90°. Hence, Principal planes are mutually perpendicular. Here, the two principal planes are designated as θ_{p1} and θ_{p2}.

Graphical representation of eqn. (2) leads to the following



From the above figures,

$$\sin 2\theta_p = \pm \frac{2q}{\sqrt{(f_x - f_y)^2 + 4q^2}} \qquad \cos 2\theta_p = \pm \frac{(f_x - f_y)}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

Substituting in eqn.(1)

$$f_n = \frac{f_x + f_y}{2} \pm \frac{f_x - f_y}{2}\,\frac{fx - fy}{\sqrt{(fx - fy)^2 + 4q^2}} \pm q\,\frac{2q}{\sqrt{(fx - fy)^2 + 4q^2}}$$

On simplification,

$$f_{n1,2} = \frac{fx + fy}{2} \pm \frac{1}{2}\sqrt{(fx - fy)^2 + 4q^2} \qquad \text{---------} \; (3)$$

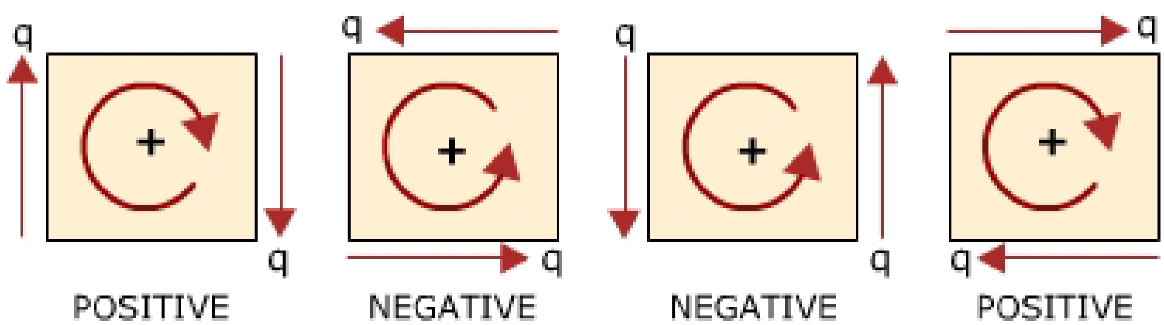
Equation (3) is the desired expression for Principal stresses. Here, the Principal stresses are represented by f_{n1} and f_{n2}.

2.4 Mohr's Circle

The formulae developed so far (to find f_n, f_s, $f_{n\text{-}max}$, $f_{n\text{-}min}$, θ_{p1} , θ_{p2} , $f_{s\ max}$, θ_{s1} , θ_{s2}) may be used for any case of plane stress. A visual interpretation of these relations, devised by the German Engineer Christian Otto Mohr in 1882, eliminates the necessity of remembering them. In this interpretation a circle is used; accordingly, the construction is called **Mohr's Circle.** If this construction is plotted to scale the results can be obtained graphically; usually, however, only a rough sketch is drawn and results are obtained from it analytically.

Rules for applying Mohr's Circle to compound stresses

1. The normal stresses f_x and f_y are plotted along X-axis. Tensile stresses are treated as positive and compressive stresses are treated as negative.
2. The shear stress q is plotted along Y-axis. It is consider positive when its moment about the center of the element is clockwise and negative when its moment about the center of the element is anti-clockwise.

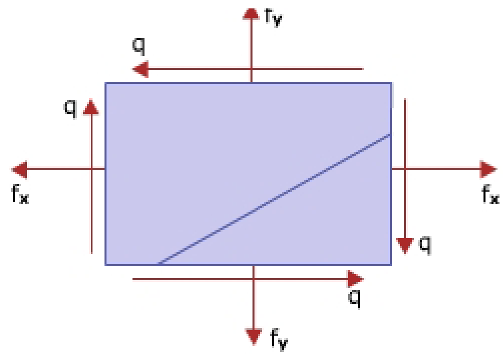


3. Positive angles in the circle are obtained when measured in counter clockwise sense. Further, an angle of '2θ' in the circle corresponds to an angle θ in the element.

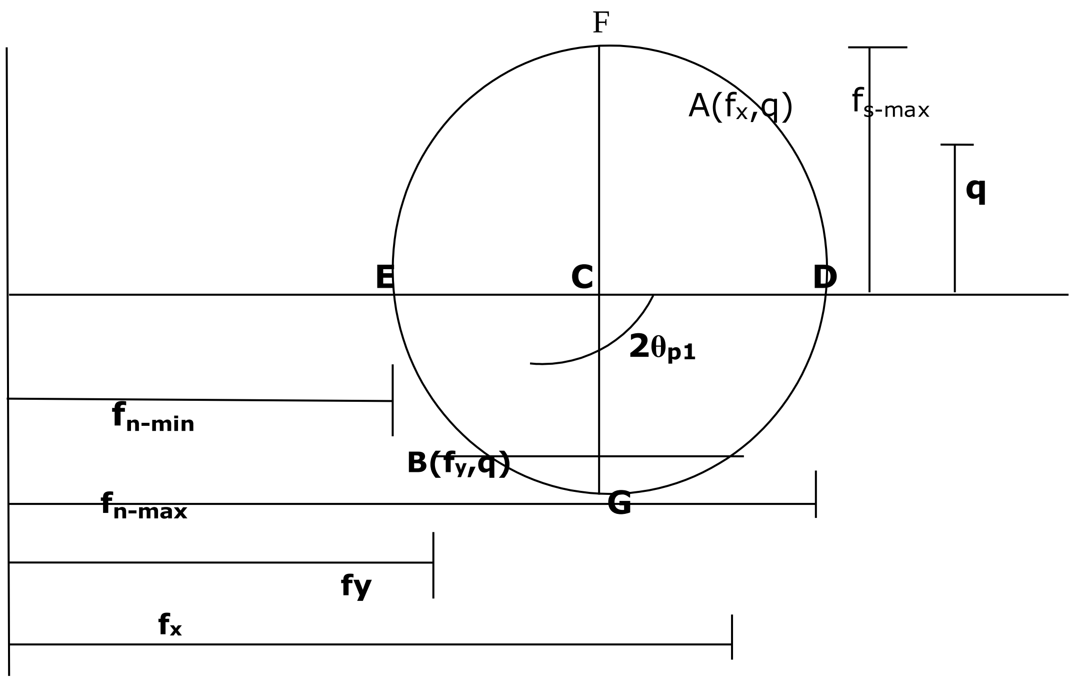
4. A plane in the given element corresponds to a point on the Mohr's circle. Further, the coordinates of the point on the Mohr's circle represent the stresses acting on the plane

Procedure to construct Mohr's circle

Consider an element subjected to normal stresses f_x and f_y accompanied by shear stress q as shown in figure. Let f_x be greater than f_y.



1. In the rectangular coordinate system, locate point A which will be should be a point on the circle representing the stress condition on the plane f_x of the element. The coordinates of point A are (f_x , q).
2. Similarly locate point B, representing stress conditions on plane fy of the element. The coordinates of point B are (f_y – q).
3. Join AB to cut X-axis at point C. Point C corresponds to the center of Mohr's circle.
4. With C as center and CA as radius, draw a circle.



Fig

From figure, it can be seen that OD and OE represent maximum and minimum normal stresses which are nothing but principal stresses. The coordinates of points D and E give the stress condition on principal planes. It can be seen that the value of shear stress is '0' on these two planes. Further, angles BCD = $2\theta_{p1}$ and BCE $=2\theta_{p2}$ (measured counter clockwise) give inclinations of the principal planes with respect to plane of f_y or X-axis. It is seen that $2\theta_{p1} \sim 2\theta_{p2} = 180^0$.

Hence, $\theta_{p1} \sim \theta_{p2} = 90^0$.

It can be observed that shear stress reach maximum values on planes corresponding two points F and G on the Mohr's circle. The coordinates of points F and G represents the stress conditions on the planes carrying maximum shear stress. The ordinate CF and CG represent the maximum shear stresses. The angles BCG = $2\theta_{s1}$ and BCF $=2\theta_{s2}$ (measured counter clockwise) give inclinations of planes carrying maximum shear stress with respect to plane of f_y or X-axis. It is seen that $2\theta_{s1} \sim 2\theta_{s2} = 180^0$.

Hence, $\theta_{s1} \sim \theta_{s2} = 90^0$.

Also it is seen that $2\theta_{p1} \sim 2\theta_{s1} \sim 2\theta_{p2} \sim 2\theta_{s2} = 90^0$. Hence, $\theta_{p1} \sim \theta_{s1} \sim \theta_{p2} \sim \theta_{s2} = 45^0$.

To find the normal and tangential stresses on a plane inclined at θ to the plane of f_y , first locate point M on the circle such that angle BCM = 2θ(measured counter clockwise) as shown in figure. The coordinates of point M represents normal and shear stresses on that plane. From figure, ON is the normal stress and MN is the shear stress.

2.5 Problems:

1. **In a 2-D stress system compressive stresses of magnitudes 100 MPa and 150 MPa act in two perpendicular directions. Shear stresses on these planes have magnitude of 80 MPa. Use Mohr's circle to find,**

(i) Principal stresses and their planes
(ii) Maximum shears stress and their planes and
(iii) Normal and shear stresses on a plane inclined at 45^0 to 150 MPa stress.

Given, $f_x = -150$ MPa
$f_y = -100$ MPa
$q = 80$ MPa

If Mohr's circle is drawn to scale, all the quantities can be obtained graphically. However, the present example has been solved analytically using Mohr's circle.

Construct Mohr's circle with earlier fig

From figure

$$OC = \frac{f_x + f_y}{2} = -125\,MPa$$

To find Radius of Circle

$$CH = \frac{f_x - f_y}{2} = 25\,MPa$$

$$CA = \sqrt{CH^2 + HA^2} = 83.82$$

$$\therefore \text{ Radius} = CD = CE = CF = CG = CA = 83.82 \text{ Units}$$

To find Principal Stress and Principal Planes

f_{n-max} = OC + CD

= – 125 – 83.82

= – 208.82 MPa

$f_{n\ min}$ = OC – CE

= – 125 –(–83.82)

= – 41.18 MPa

$$\alpha = \tan^{-1}\left(\frac{AH}{MC}\right) = 72^0.65$$

But $2\theta_{P1} = \angle ACH = \alpha = 72^0.65$

Hence, $\theta_{P1} = 36^0.32$

Further, $2\theta_{P2} = \angle ACE = 180 + \alpha = 252^0.65$

Hence, $\theta_{P2} = 126^0.32$

2.6 Thick Cylinders

2.6.1 Difference in treatment between thin and thick cylinders - basic assumptions:

The theoretical treatment of thin cylinders assumes that the hoop stress is constant across the thickness of the cylinder wall (Fig. 6.1), and also that there is no
pressure gradient across the wall. Neither of these assumptions can be used for thick

cylinders for which the variation of hoop and radial stresses is shown in (Fig. 6.2), their values being given by the Lame equations: -

$$\sigma_H = A + \frac{B}{r^2}$$

$$\sigma_r = A - \frac{B}{r^2}$$

Where: -

σ_H= Hoop stress ($\frac{N}{m^2} = Pa$).

σ_r= Radial stress ($\frac{N}{m^2} = Pa$).

r= Radius (m). A and B are Constants.

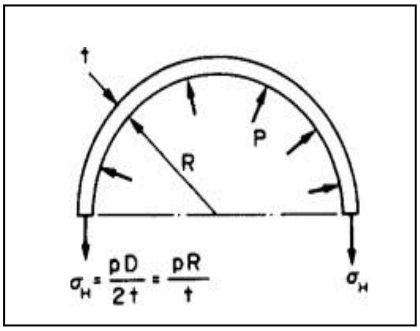


Figure 6.1: - Thin cylinder subjected to internal pressure.

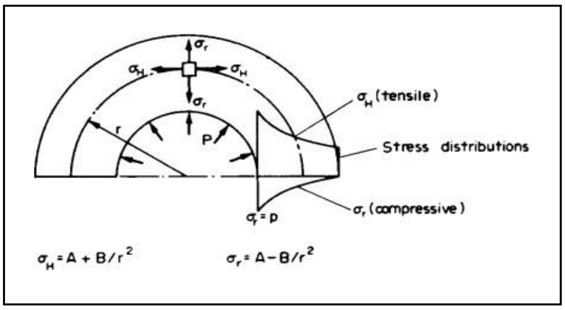


Figure - Thick cylinder subjected to internal pressure.

2.6.2 Thick cylinder- internal pressure only: -

Consider now the thick cylinder shown in (Fig. 6.3) subjected to an internal pressure P, the external pressure being zero.

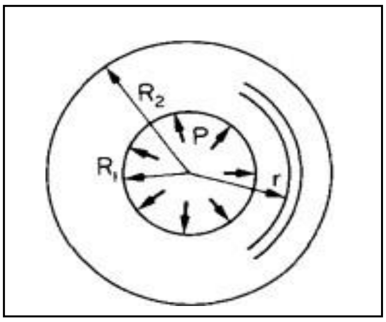


Figure: - Cylinder cross section.

The two known conditions of stress which enable the Lame constants A and B to be determined are:

At $r = R_1$, $\sigma_r = -P$ and at $r = R_2$, $\sigma_r = 0$

Note: -The internal pressure is considered as a negative radial stress since it will produce a radial compression (i.e. thinning) of the cylinder walls and the normal stress convention takes compression as negative.

Substituting the above conditions in eqn. (.2),

$$\sigma_r = A - \frac{B}{r^2}$$

$$-P = A - \frac{B}{R_1{}^2} \text{ and } 0 = A - \frac{B}{R_2{}^2}$$

Then $A = \frac{PR_1{}^2}{(R_2{}^2 - R_1{}^2)}$ and $B = \frac{PR_1{}^2 R_2{}^2}{(R_2{}^2 - R_1{}^2)}$

Substituting A and B in equations 6.1 and 6.2,

$$\boxed{\sigma_r = \frac{PR_1{}^2}{(R_2{}^2 - R_1{}^2)}\left[1 - \frac{R_2{}^2}{r^2}\right]}$$

$$\boxed{\sigma_H = \frac{PR_1{}^2}{(R_2{}^2 - R_1{}^2)}\left[1 + \frac{R_2{}^2}{r^2}\right]}$$

2.6.3 Longitudinal stress: -

Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure P_1 and an external pressure P_2, (Fig).

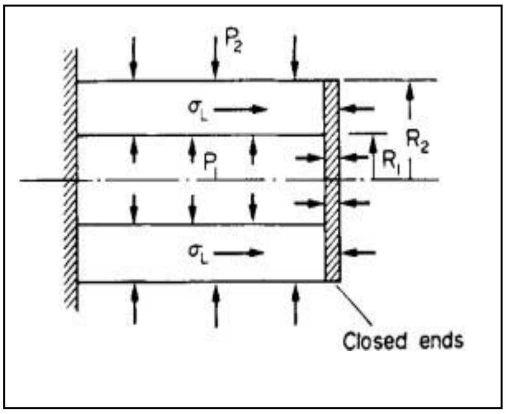


Figure: - Cylinder longitudinal section.

For horizontal equilibrium:

$$P_1{}^2 * \pi R_1{}^2 - P_2{}^2 * \pi R_2{}^2 = \sigma_L * \pi\left[R_2{}^2 - R_1{}^2\right]$$

Where σ_L is the longitudinal stress set up in the cylinder walls,

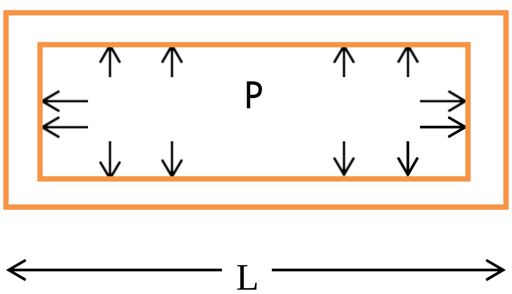
= A, constant of the Lame equations. ….6.6

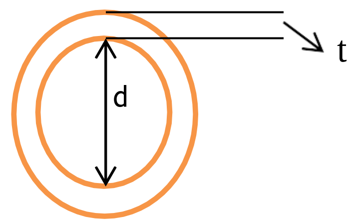
2.7 Thin Cylinders

2.7.1 Introduction

When the thickness of the wall of the cylinder is less than $\frac{1}{1}$ to $\frac{1}{1}$ of the diameter of cylinder then the cylinder is considered as **thin cylinder**.

Otherwise it is termed as thick cylinder.





L=Length of the
cylinder d= Diameter of
cylinder
t = thickness of cylinder
P= Internal Pressure due to fluid

Generally, cylinders are employed for transporting or storing fluids i.e. liquids and gases. Examples-: LPG cylinders, boilers, storage tanks etc.

Due to the fluids inside a cylinder, these are subjected to fluid pressure or internal pressure (Say P). Hence at any point on the wall of the cylinder, three types of stresses are developed in three perpendicular directions. These are:-

1. Circumferential Stress or Hoop Stress (σ_h)
2. Longitudinal Stress (σ_L)
3. Radial Stress (σ_r)

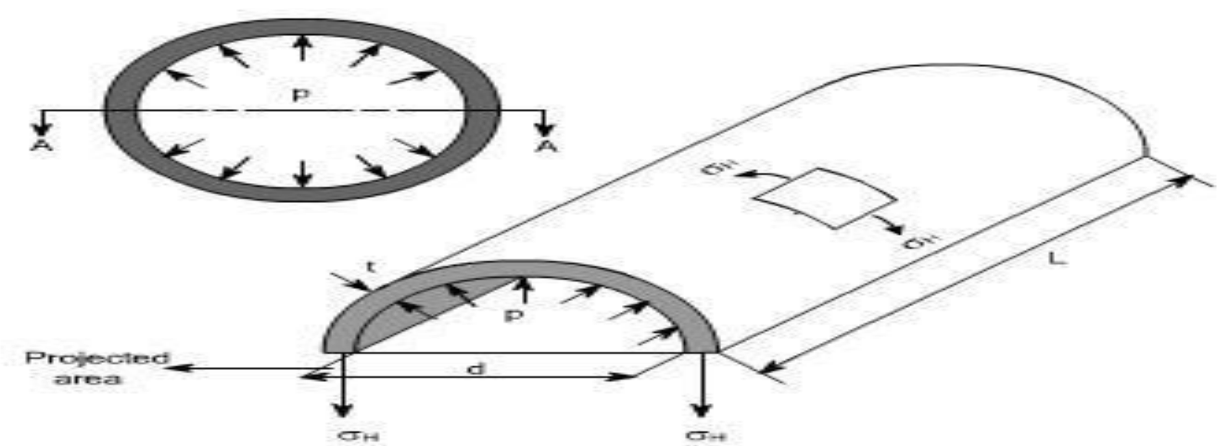
2.7.2 Assumptions in Thin Cylinders

1. It is assumed that the stresses are uniformly distributed throughout the thickness of the wall.

2. As the magnitude of radial stresses is very small in thin cylinders, they are neglected while analyzing thin cylinders i.e. $\sigma_r = 0$

2.7.3 Stresses in Thin Cylinder

1. Circumferential Stress (σ_h):- This stress is directed along the tangent to the circumference of the cylinder. This stress is tensile in nature. This stress tends to increase the diameter.



The bursting in the cylinder will takes place if the force due to internal fluid pressure(P) acting vertically upwards and downwards becomes more than the resisting force due to

circumferential stress(σ_h) developed in the cylinder.

Total diametrical Bursting force= P * Projected area of the curved surface

= P * d * L

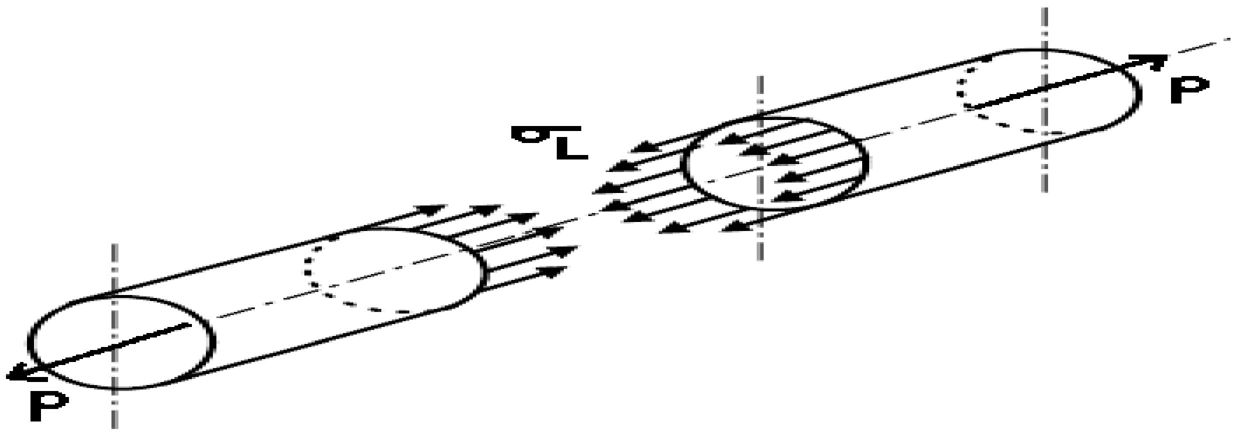
Resisting force due to circumferential stress= $2 * \sigma_h * t * L$

Under equilibrium, Resisting force = Total diametrical Bursting force

$$2 * \sigma_h * t * L = P * d * L$$

Circumferential stress, $\sigma_h = \frac{Pd}{2t}$

2. Longitudinal Stress (σ_L)- This stress is directed along the length of the cylinder. This stress is also tensile in nature. This stress tends to increase the length.



Total longitudinal bursting force (on the ends of cylinder) $= P * \frac{\Pi}{4} * d^2$

Area of crossection where longitudinal stress is developed= $\Pi * d * t$

Resisting force due to longitudinal stress = $\sigma_L * \Pi * d * t$

Under equilibrium, Resisting force = Total longitudinal Bursting force

$$\sigma_L * \Pi * d * t = P * \frac{\Pi}{4} * d^2$$

$$\text{Longitudinal stress, } \sigma_L = \frac{Pd}{4t}$$

Note:- Due to the presence of longitudinal stress and hoop stress, there is shear stress developed in the cylinder. Maximum in-plane shear stress is given by

$$(\tau_{max})_{inplane} = \frac{\sigma_h - \sigma_L}{2} = \frac{Pd}{8t}$$

2.7.4 Strains in Thin Cylinder

1. Strain in longitudinal direction , $\varepsilon_L = \frac{\sigma_L}{E} - \mu \frac{\sigma_h}{E}$

Longitudinal strain = $\varepsilon_L = \frac{Pd}{4tE}(1-2\mu)$

2. Strain in circumferential direction, $\varepsilon_h = \frac{\sigma_h}{E} - \mu \frac{\sigma_L}{E}$

Circumferential strain = $\varepsilon_h = \frac{Pd}{4TE}(2-\mu)$

3. Volumetric strain = $\varepsilon_V = \frac{Pd}{4tE}(5-4\mu)$

Where μ = Poisson's ratio

E= Modulus of Elasticity

2.7.5 For Objective Questions

1. (a) Major principal stress= Hoop stress or circumferential stress (σ_h)

   (b)Minor principal stress= Longitudinal stress (σ_L)

2. If σ_t is the permissible stress for the cylinder material, then major principal stress (σ_h) should be less than or equal to σ_t.

$$\sigma_h \leq \sigma_t$$

$$\frac{Pd}{2t} \leq \sigma_t$$

$$t \geq \frac{Pd}{2\sigma_t}$$

3. In order to produce pure shear state of stress in thin walled cylinders,

$$\sigma_h = -(\sigma_L)$$

4. Maximum shear stress in the wall of the cylinder (**not in-plane shear stress**) is given by :

$$\tau_{max} = \frac{\sigma_h}{2} = \frac{Pd}{4t}$$

5. In case of thin spherical shell, longitudinal stress and circumferential stress are equal and given by

$$\sigma_L = \sigma_h = \frac{Pd}{4t} \text{ (tensile)}$$

$$(\tau_{max})_{inplane} = \frac{\sigma_h - \sigma_L}{2} = 0$$