Module 4:

Relations *contd.*:

- Properties of Relations,
- Computer Recognition – Zero-One Matrices and Directed Graphs,
- Partial Orders – Hasse Diagrams,
- Equivalence Relations and Partitions.

Definition and Properties

A binary relation R from set x to y (written as xRy or R(x,y)) is a subset of the Cartesian product x × y. If the ordered pair of G is reversed, the relation also changes.

Generally an n-ary relation R between sets A_1, ... , and A_n is a subset of the n-ary product $A_1 \times \ldots \times A_n$. The minimum cardinality of a relation R is Zero and maximum is n^2 in this case.

A binary relation R on a single set A is a subset of A × A.

For two distinct sets, A and B, having cardinalities *m* and *n* respectively, the maximum cardinality of a relation R from A to B is *mn*.

Domain and Range

If there are two sets A and B, and relation R have order pair (x, y), then −

- The **domain** of R is the set { x | (x, y) � R for some y in B }
- The **range** of R is the set { y | (x, y) � R for some x in A }

Examples

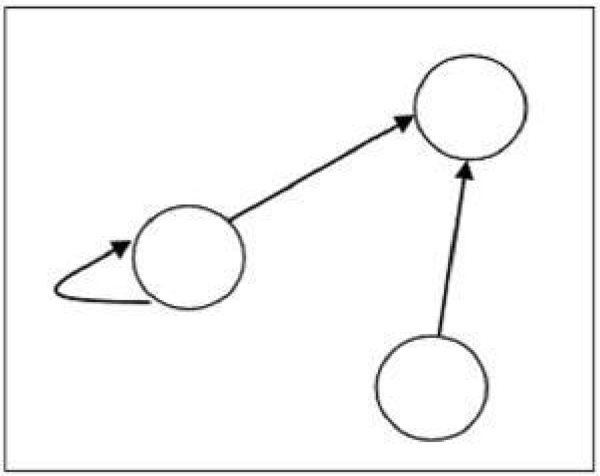
Let, A = {1, 2, 9} and B = {1, 3, 7}

- Case 1 − If relation R is ‘equal to’ then R = {(1, 1), (3, 3)}
- Case 2 − If relation R is ‘less than’ then R = {(1, 3), (1, 7), (2, 3), (2, 7)}
- Case 3 − If relation R is ‘greater than’ then R = {(2, 1), (9, 1), (9, 3), (9, 7)}

A relation can be represented using a directed graph.

The number of vertices in the graph is equal to the number of elements in the set from which the relation has been defined. For each ordered pair (x, y) in the relation R, there will be a directed edge from the vertex ‘x’ to vertex ‘y’. If there is an ordered pair (x, x), there will be self- loop on vertex ‘x’.

Suppose, there is a relation R = {(1, 1), (1,2), (3, 2)} on set S = {1, 2, 3}, it can be represented by the following graph −



Types of Relations

- The **Empty Relation** between sets X and Y, or on E, is the empty set �
- The **Full Relation** between sets X and Y is the set X × Y
- The **Identity Relation** on set X is the set {(x, x) | x � X}
- The Inverse Relation R' of a relation R is defined as − R’ = {(b, a) | (a, b) � R} **Example** − If R = {(1, 2), (2, 3)} then R’ will be {(2, 1), (3, 2)}
- A relation R on set A is called **Reflexive** if �a�A is related to a (aRa holds). **Example** − The relation R = {(a, a), (b, b)} on set X = {a, b} is reflexive
- A relation R on set A is called **Irreflexive** if no a � A is related to a (aRa does not hold). **Example** − The relation R = {(a, b), (b, a)} on set X = {a, b} is irreflexive

- A relation R on set A is called **Symmetric** if xRy implies yRx, �x�A and �y�A.

  Example − The relation R = {(1, 2), (2, 1), (3, 2), (2, 3)} on set A = {1, 2, 3} is symmetric.

- A relation R on set A is called **Anti-Symmetric** if xRy and yRx implies x = y �x � A and �y � A.

  Example − The relation R = {(1, 2), (3, 2)} on set A = {1, 2, 3} is antisymmetric.

- A relation R on set A is called **Transitive** if xRy and yRz implies xRz, �x,y,z � A.

  Example − The relation R = {(1, 2), (2, 3), (1, 3)} on set A = {1, 2, 3} is transitive

- A relation is an **Equivalence Relation** if it is reflexive, symmetric, and transitive.

  Example − The relation R = {(1, 1), (2, 2), (3, 3), (1, 2), (2,1), (2,3), (3,2), (1,3), (3,1)} on set A = {1, 2, 3} is an equivalence relation since it is reflexive, symmetric, and transitive.

COMPUTER RECOGNITION-ZERO-ONE MATRICES AND DIRECTED GRAPHS

In this section, we will discuss two alternative methods for representing relation, one method used (ZERO-ONE) matrices, the other method uses directed graphs. These methods are recognised in Computer Science.

(i) Method (Using Zero-one Matrices)

Suppose A and B are both finite sets and R is a relation from A to B, then R may be represented as a matrix called the relation matrix of R.

Definition: (Relation Matrix)

If $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are finite sets containing m and n elements respectively and R is a relation from A to B, then we can represent the relation R by an $m \times n$ matrix, called relation matrix, denoted by

$$M_R = [m_{ij}]_{m \times n};$$

where
$$m_{ij} = \begin{cases} 1, \text{ if } (a_i\ b_j) \in R \\ 0, \text{ if } (a_i\ b_j) \notin R \end{cases}$$

where m_{ij} is the element in the i^{th} now j^{th} column. The matrix representing R has a '1' as its $i \equiv j$ entry when a_i is related to b_j, and a '0' in this position if a_i is not related to b_j.

(Such a representation depends on the orderings used for A and B).

Definition:

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ (zero-one) matrices. Then the join of A are B is the zero-one matrix $(i,j)^{th}$ entry $a_{ij} \vee b_{ij}$.

The join of A and B is denoted by $A \vee B$. The meet of A and B is the zero matrix with $(i,j)^{th}$ entry.

$a_{ij} \wedge b_{ij}$ The meet of A and B is denoted by $A \wedge B$.

For example,

If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Then the joint of A and B is

$$(A \vee B) = \begin{bmatrix} 1\vee 0 & 0\vee 1 & 1\vee 0 \\ 0\vee 1 & 1\vee 1 & 0\vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and meet of A and B is $(A \wedge B) = \begin{bmatrix} 1\wedge 0 & 0\wedge 1 & 1\wedge 0 \\ 0\wedge 1 & 1\wedge 1 & 0\wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Note: A relation matrix reflects some of the properties of a relation:

(a) The matrix of a reflexive relation has a '1' on all its principal diagonal elements.

(b) R is symmetric iff $m_{ij} = 1$ whenever $m_{ji} = 1$, This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently R is symmetric if and only if $m_{ij} = m_{ji}$ for all pairs of integers i and j with $i = 1$ to n and $j = 1$ to n. R is symmetric iff $M_R = (M_R)^T$

(c) R is antisymmetric of $m_{ij} = 1$ with $i \neq j$, then $m_{ij} = 0$.
In other words, either $m_{ij} = 0$ or $m_{ji} = 0$
when $i \neq j$

Working Rule

To write the relation matrix for a given relation:

"From a rectangular array whose rows are labelled by the elements of A and whose columns are labelled by the elements of B. Then put the integer '1' in each position of the array where $a \in A$ is related to $b \in B$ i.e., when $(a, b) \in R$ and put 0 in the remaining positions i.e., where $(a, b) \notin R$. This final array, is the matrix M_R of the relation R".

7. Let $A = \{1, 2, 3\}$ and $R = \left\{(x, y) / x < y\right\}$ find M_R.

Solution:

Given: $R = \{(1, 2), (1, 3), (2, 3)\}$

The table and corresponding relation matrix for the relation R are given below:

	1	2	3
1	0	1	1
2	0	0	1
3	0	0	1

(a)

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

8. Let $A = \{1, 2, 3, 4\}$ $B = \{x, y, z\}$
and $R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$
Define matrix representation of R.

Solution:

Given: $R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$

The table and corresponding relation matrix for the R given below:

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0
4	1	0	1

(a)

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

5.5 HASSE DIAGRAM

A Hasse diagram is a pictorial representation of a finite partial order on a set. In this representation, the objects i.e., the elements are shown as vertices (or dots).

Two related vertices in the Hasse diagram of a partial order are connected by a line if and only if they are related.

Example 1 Let $A = \{3, 4, 12, 24, 48, 72\}$ and the relation $\leq$ be such that $a \leq b$ if a divides b. The Hasse diagram of $(A, \leq)$ is shown in Fig. 5.1.

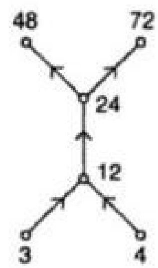


Fig. 5.1

12.2.2 Equivalence Relations and Partitions

Equivalence relations merit additional exposition. One notable application of equivalence relations occurs when the chain of familiar number systems, $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$, is carefully constructed. In particular, when moving from the integers, $\mathbb{Z}$, to the set of rational numbers, $\mathbb{Q}$, it is common to define the elements of $\mathbb{Q}$ by starting with ordered pairs of integers such that the second coordinate is not 0. The first coordinate represents the numerator and the second coordinate represents the denominator. One complication is that the ordered pairs (1, 2) and (2, 4) really represent the same rational number (the fraction $\frac{1}{2}$). Equivalence relations provide a mechanism for combining the infinitely many representations for the same number into a single element of the new set, $\mathbb{Q}$. Additional details will be presented later in this section.

A previous example from a Quick Check will motivate the major new idea.

EXAMPLE 12.9 **Congruence Classes mod 5**

Quick Check 12.3 on page 739 introduced the equivalence relation

$$\mathcal{R}_4 = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid (x \bmod 5) = (y \bmod 5)\}$$

It is easy to see that if $(x, y) \in \mathcal{R}_4$, then $x - y = 5n$, for some integer, n. Definition 3.19 on page 98 provides an alternate description: $(x \bmod 5) = (y \bmod 5)$ if and only if $x \equiv y \pmod 5$.

There is a natural partitioning[15] of ℕ into the *congruence classes* mod 5:

Remainder 0 {0, 5, 10, 15, 20, 25, ...}
Remainder 1 {1, 6, 11, 16, 21, 26, ...}
Remainder 2 {2, 7, 12, 17, 22, 27, ...}
Remainder 3 {3, 8, 13, 18, 23, 28, ...}
Remainder 4 {4, 9, 14, 19, 24, 29, ...}

Every pair of natural numbers from the same congruence class is in the relation, $\mathcal{R}_4$. Any two numbers from different congruence classes are not in the relation. ■

The phenomenon of being able to partition the elements of a set into natural subsets defined by an equivalence relation is not unique to congruences in ℕ. It happens with every equivalence relation.

DEFINITION 12.17 ***Equivalence Class***

Let $\mathcal{R}$ be an equivalence relation on a set, $\mathcal{A}$, and let $x \in \mathcal{A}$. The *equivalence class* of x is denoted by $[x]$ and is defined as

$$[x] = \{a \in \mathcal{A} \mid (x, a) \in \mathcal{R}\}$$

The set $\{[x] \mid x \in \mathcal{A}\}$ is referred to as the set of equivalence classes *induced by* R on A. The element, x, that appears in the notation "$[x]$" is called a *class representative*.

✔ **Quick Check 12.5**

Let $\mathcal{A}$ be the set of all students who reside on campus at a particular college or university. Let $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ be the relation defined by $(x, y) \in \mathcal{R}$ if and only if x and y live in the same residence hall (dorm).

1. Show that $\mathcal{R}$ is an equivalence relation.
2. Determine the equivalence classes of $\mathcal{R}$. ☑

PROPOSITION 12.18 ***Equivalence Classes Are Disjoint***

Let $\mathcal{R}$ be an equivalence relation on a set, $\mathcal{A}$. If x and y are two elements in $\mathcal{A}$, then either $[x] = [y]$ or else $[x] \cap [y] = \emptyset$.

Proof:

***Case 1:* $(x, y) \in \mathcal{R}$**

If $(x, y) \in \mathcal{R}$, then $y \in [x]$ (by Definition 12.17). Since $\mathcal{R}$ is symmetric, it is also true that $(y, x) \in \mathcal{R}$, so $x \in [y]$. $\mathcal{R}$ is reflexive, so $x \in [x]$. Thus, $[x] \cap [y] \neq \emptyset$.

Suppose that $a \in [x]$. Then $(x, a) \in \mathcal{R}$. Since $\mathcal{R}$ is symmetric, $(a, x) \in \mathcal{R}$ is also true. But then the transitivity of $\mathcal{R}$, combined with $(a, x) \in \mathcal{R}$ and $(x, y) \in \mathcal{R}$ implies that $(a, y) \in \mathcal{R}$ and (by symmetry) $(y, a) \in \mathcal{R}$. Consequently, $a \in [y]$ and $[x] \subseteq [y]$. A similar argument shows that $[y] \subseteq [x]$. The conclusion is that $[x] = [y]$ whenever $x\mathcal{R}y$.

***Case 2:* $(x, y) \notin \mathcal{R}$**

Suppose that $(x, y) \notin \mathcal{R}$ but that $a \in [x]$ and $a \in [y]$. Then $(x, a) \in \mathcal{R}$ and $(y, a) \in \mathcal{R}$. Using the symmetry and transitivity of $\mathcal{R}$, it is then true that $(x, y) \in \mathcal{R}$, a contradiction. The contradiction arose by assuming that $[x]$ and $[y]$ had a common element. The conclusion is that $[x] \cap [y] = \emptyset$. □

[15]See Defintion 2.13 on page 24.

Since an equivalence relation is reflexive, $(x, x) \in \mathcal{R}$ for every $x \in \mathcal{A}$. Consequently, $x \in [x]$. The partitioning phenomenon that was observed in Example 12.9 is fully developed in the next theorem.

One consequence of Proposition 12.18 is that any one of the elements in an equivalence class may be unambiguously used as the class representative.

THEOREM 12.19 ***Equivalence Relations and Partitions***

Let $\mathcal{A}$ be a set.

- If $\mathcal{R}$ is an equivalence relation on $\mathcal{A}$, then the equivalence classes of $\mathcal{R}$ form a partition of $\mathcal{A}$.
- Every partition of $\mathcal{A}$ determines an equivalence relation on $\mathcal{A}$.

Proof:

Equivalence Relation Implies Partition

Let $\mathcal{R}$ be an equivalence relation on $\mathcal{A}$. Proposition 12.18 implies that the equivalence classes induced by $\mathcal{R}$ are disjoint. Since every element, x, in $\mathcal{A}$ is in an equivalence class, $[x]$, every element of $\mathcal{A}$ is in some equivalence class. Therefore, the equivalence classes form a partition of $\mathcal{A}$; they are disjoint and their union is all of $\mathcal{A}$.

Partition Implies Equivalence Relation

Let $P_{\mathcal{A}} = \{\mathcal{A}_i \subseteq \mathcal{A} \mid i \in \Upsilon\}$ for some set of indices, Υ.[16] Assume also that $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ if $i \neq j$ and $\mathcal{A} = \cup_{i \in \Upsilon} A_i$. $P_{\mathcal{A}}$ is a partition of $\mathcal{A}$.

Define a relation, $\mathcal{R}$, on $\mathcal{A}$ by

$$(x, y) \in \mathcal{R} \text{ if and only if } x \in \mathcal{A}_i \text{ and } y \in \mathcal{A}_i, \text{ for a common } i \in \Upsilon$$

It is an easy exercise to show that $\mathcal{R}$ is an equivalence relation whose equivalence classes are the members of $P_{\mathcal{A}}$. □

It is now time to provide the missing details for deriving the rational numbers from the integers.