MODULE 2
STRESSES IN SOIL

Structure

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2.0 Objectives
- To determine magnitude of stresses in soils
- To apply various theories and methods to determine stresses in soils
- To understand contact pressure and pressure distribution in soils
- To Estimate settlement in soils

2.1 Introduction

Stresses in soil are caused due to

a) Self weight of soil

b) Structural loads, applied at or below the surface

The estimation of vertical stresses at any point in a soil mass due to external loading is essential to the prediction of settlements of buildings, bridges and pressure.

The stresses induced in a soil due to applied loads depend upon its Stress – Strain characteristics. The stress strain behaviour of soils is extremely complex and it depend upon a large number of factors, such as drainage conditions, water content, void ratio, rate of loading, the load level, and the stress path. However simplifying assumptions are generally made in the analysis of soil behaviour to obtain stresses. It is generally assumed

that the soil mass is homogeneous and isotropic. The stress strain relationship is assumed to be linear. The theory of elasticity is used to determine the stresses in the soil mass. Though it involves considerable simplification of real soil behaviour and the stresses computed are approximate, the results are good enough for soil problems usually encountered in the practice.
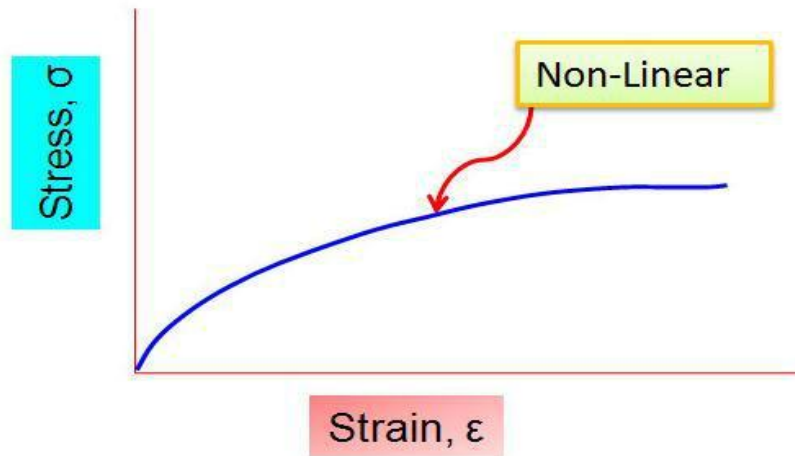

Fig.2.1 Typical Stress-strain curve for soils

2.2 Boussinesq's Theory / Solution

Boussinesq (1885) has given the solution for the stresses caused by the application of the point load at the surface of a elastic medium with the aid of the mathematical theory of elasticity.

3.3.1 Assumptions

1) The soil medium is an elastic continuum having a constant value of modulus of elasticity (E). i.e. it obeys Hooke"s law

2) The soil is homogeneous, i.e. it has identical elastic properties at all points in identical directions.

3) The soil is isotropic, i.e. it has identical elastic properties in all direction at a point.

4) The soil mass is semi-infinite, i.e. it extends to infinity in the downward directions and lateral directions. In other words, it is limited on its top but a horizontal plane and extends to infinity in all other directions.

5) The self weight of the soil is ignored.

6) The soil is initially unstressed i.e. it is free from residual stresses before the application of the load.

7) The top surface of the medium is free of shear stresses and is subjected to only the point load at a specified location.

8) Continuity of stress is considered to exist in the medium.

9) The stresses are distributed symmetrically with respect to z axis (vertical axis).

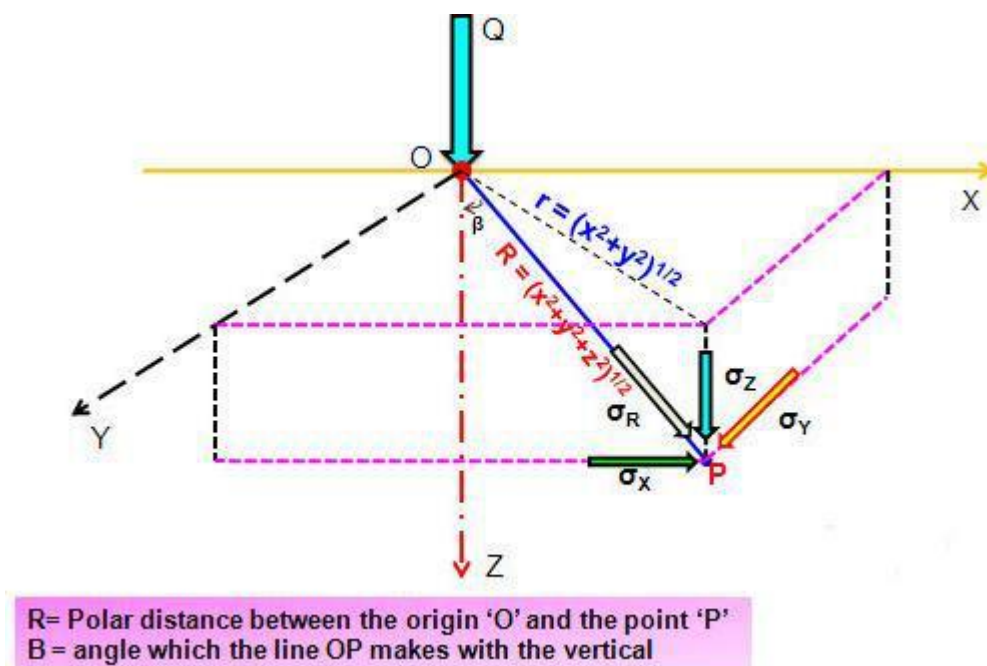**2.3 Vertical stresses due to a concentrated load**

R= Polar distance between the origin 'O' and the point 'P'
B = angle which the line OP makes with the vertical

Fig.2.2 Derivation of vertical stress at a point due to point load acting on the

ground surface

Figure 3.2 shows a horizontal surface of an elastic continuum subject to a point load Q at a point O. Using logarithmic stress function for solution of elasticity problem, Bossinesq proved that the polar stress σ_R at point P is given by

$$\sigma_R = \frac{3}{2\pi}\left(\frac{Q\ cos\beta}{R^2}\right) \quad \text{--------(3.1)}$$

where R = Polar distance between the origin O and point t

β = angle which the line OP makes with the vertical

or $R = \sqrt{r^2 + Z^2}$

Where $r^2 = x^2 + y^2$

$Sin\beta = \frac{r}{R}$ and $cos\beta = \frac{Z}{R}$

Now, the vertical stress(σ_z) at Point P is given by

$\sigma_z = \sigma_R \cos^2\beta$ ------------(3.2)

$$= \frac{3}{2\pi}(\frac{Q\ cos\beta}{R^2})\ cos^2\beta$$

$$= \frac{3Q}{2\pi}(\frac{cos^3\beta}{R^2})$$

$$= \frac{3Q}{2\pi}(\frac{(\frac{Z}{R})^3}{R^2})$$

$$\boxed{\sigma_z = \frac{3Q}{2\pi}(\frac{Z^3}{R^5})}$$ ------------------------(3.3)

$$= \frac{3Q}{2\pi}(\frac{Z^2}{Z^2}\frac{Z^3}{R^5})$$

$$= \frac{3Q}{2\pi}(\frac{Z^5}{Z^2\ R^5})$$

$$\sigma_z = \frac{3Q}{2\pi}\frac{1}{Z^2}(\frac{Z^5}{(r^2+Z^2)^{\frac{5}{2}}})$$

$$\sigma_z \frac{3Q}{2\pi}\frac{1}{Z^2}\left(\frac{1}{(1+(\frac{r}{z})^2)^{\frac{5}{2}}}\right)$$ ------------------ (3.4)

or, $\sigma_z = \frac{I_B\ Q}{Z^2}$ ------------------------ (3.5)

Where $I_B = \frac{3}{2\pi}(\frac{1}{(1+(\frac{r}{z})^2)^{\frac{5}{2}}})$ ------------------(3.6)

The coefficient I_B is known as the Boussinesq influence for the vertical stress. The value of I_B can be determined for the given value of r/Z from Eq. 3.6.

$$\sigma_x = \frac{Q}{2\pi}\left[\left(\frac{3x^2z}{R^5} - (1-2v)\left(\frac{x^2-y^2}{R\,r^2(R+Z)} + \frac{y^2z}{R^3r^2}\right)\right)\right] \text{----------- (3.7)}$$

$$\sigma_y = \frac{Q}{2\pi}\left[\left(\frac{3y^2z}{R^5} - (1-2v)\left(\frac{y^2-x^2}{R\,r^2(R+Z)} + \frac{x^2z}{R^3r^2}\right)\right)\right] \text{-----------(3.8)}$$

If v = 0.5

$$\boxed{\sigma_x = \frac{3Q}{2\pi}\frac{x^2z}{R^5}}$$ $$\boxed{\sigma_y = \frac{3Q}{2\pi}\frac{y^2z}{R^5}}$$

$$\sigma_y = \frac{Q}{2\pi}\left[\left(\frac{3y^2z}{R^5} - (1-2v)\left(\frac{y^2-x^2}{R\,r^2(R+Z)} + \frac{x^2z}{R^3r^2}\right)\right)\right] \text{-----------(3.8)}$$

If v = 0.5

$$\boxed{\sigma_x = \frac{3Q}{2\pi}\frac{x^2z}{R^5}}$$ $$\boxed{\sigma_y = \frac{3Q}{2\pi}\frac{y^2z}{R^5}}$$

The following points are to be noted while using Eq. (3.5)

1) The vertical stress does not depend upon the modulus of elasticity (E) and the poisson ratio (v). But the solution has been derived assuming that the soil is linearly elastic.

2) The Intensity of vertical stress just below the load point is given by

   $$\sigma_z = 0.4775\frac{Q}{z^2}$$ ($\because$ when, r/z = 0, I_B = 0.4775)

3) At the surface (z = 0), the vertical stress just below the load is theoretically infinite. However, in an actual case, the soil under the load yields due to very high stresses. The load point spreads over a small but finite area and, therefore, only finite stresses develop.

4) The vertical stress (σ_z) decreases rapidly with an increase in r/z ratio. Theoretically, the vertical stress would be zero only at an infinite distance from the koad point. Actually, at r/z = 5, or more, the vertical stress becomes extremely small and is neglected.

5) In actual practice, foundation loads are not applied directly on the ground surface. However, it has been established that the Boussinesq" s solution can be applied conservatively to field problems concerning loads at shallow depths, provided the distance „z" is measured from the point of application of the load.

6) Boussinesq" s solution can even be used for negative (upward) loads. For example if the vertical stress decrease due to an excavation is required, the negative load is equal to the weight of the soil removed. However, as the soil is not fully elastic the stresses determined are (necessarily) approximate.

7) The field measurements indicate that the actual stresses are generally smaller than the theoretical values given by Boussinesq" s solution especially at shallow depths. Thus, the Boussinesq" s solution gives conservative values and is commonly used in soil engineering problems.

Limitations

1) The solution was derived assuming the soil as an elastic medium, but the soil doesnot behave as an elastic material.

2) When the stress decrease occurs in soil, the relation between the stress and the strain is not linear as assumed, therefore, the solution is not strictly applicable.

3) In deep sand deposits, the modulus of elasticity increases with an increase in depth and therefore, the Boussinesq"s solution will not give satisfactory results.

4) The point loads applied below ground surface causes somewhat smaller stresses than are caused by surface loads, and, therefore, the solution is not strictly applicable.

Numerical Example

1) Find intensity of vertical pressure at a point 3 m directly below 25 kN point load acting on a horizontal ground surface. What will be the vertical pressure at a point 2m horizontally away from the axis of loading and at same depth of 3 m? Use Boussinesq"s equation.

   Solution:Case (i)

   Here, $Q = 25$ kN

   $z = 3$ m

   $r = 0$

   $$\sigma_z = \frac{3}{2\pi} \frac{Q}{z^2} \left[\frac{1}{1 + \left(\frac{r}{z}\right)^2} \right]^{\frac{5}{2}}$$

   $= 1.33 \ kN/m^2$

Case (ii)

Here,

r = 2 m

$\sigma_z = 0.53 \, kN/m^2$

2.4 Stress Isobar or pressure bulb

An „isobar" is a stress contour or a curve which connects all points below the ground surface at which the vertical pressure is the same. An isobar is a spatial curved surface and resembles a bulb in shape. The stress isobar is also called „pressure bulb".

Any number of pressure bulb may be drawn for any applied load, since each one corresponds to an arbitrarily chosen value of stress. The isobar of a particular intensity can be obtained by:

$$\sigma_z = \frac{I_B \, Q}{z^2}$$

An isobar consisting of a system of isobars appears somewhat as shown in Fig.
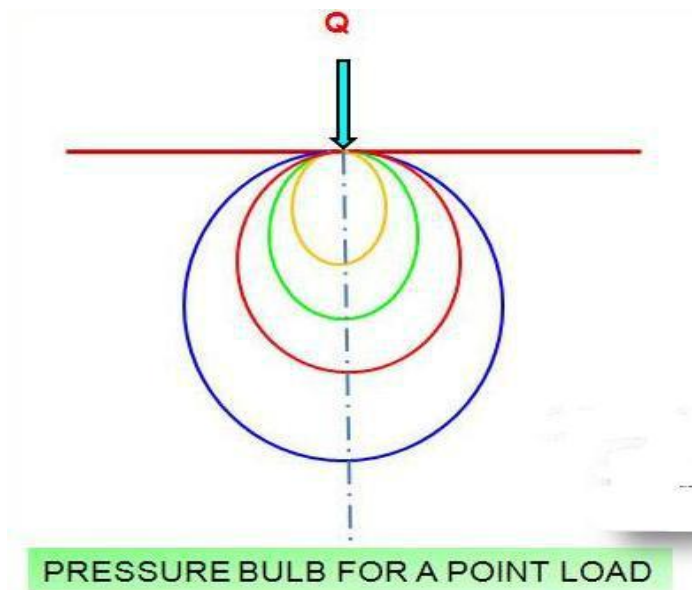

Fig. 2.3 Pressure bulb for a point load

2.5.1 Procedure

The procedure for plotting an isobar is as follows:

Let it be required to plot an isobar for which $\sigma_z = 0.1Q$ per unit area (10 % isobar)

$$0.1Q = \frac{I_B \, Q}{Z^2}$$

$$I_B = 0.1Z^2$$

$$\sigma_Z = \frac{I_B \, Q}{Z^2}$$

We know that

Assuming various values of z, the corresponding I_B values are computed. For the values of I_B, the corresponding r/z values are determined and hence the values of r are obtained.

An isobar is symmetrical about the load axis, the other half can be drawn from symmetry.
When r = 0, I_B = 0.4775; the isobar crosses the line of action of the load at a depth of

$$Z = \sqrt{\frac{I_B}{0.1}} = \sqrt{\frac{0.4775}{0.1}} = \sqrt{4.775} = 2.185 \text{ units}$$

The shape of an isobar approaches a lemniscates curve (not circle) as shown in Fig.3.3

The calculations are best performed in the form of a table as given below:

When r = 0
$$I_B = \frac{3}{2\pi} \left(\frac{1}{(1+(\frac{r}{z})^2)^{\frac{5}{2}}} \right)$$

$$= \frac{3}{2\pi}$$
$$= 0.4775$$

Table 3.1 Data for isobar of $_z$ = 0.1Q per unit area

Depth Z (units)	Influence coefficient)*	r/Z	r (units)	z
0.5	0.0250	1.501	0.750	0.1Q
1.0	0.1000	0.932	0.932	0.1Q
1.5	0.2550	0.593	0.890	0.1Q
2.0	0.4000	0.271	0.542	0.1Q
2.185	0.4775	0	0	0.1Q

2.5 Pressure distribution

It is possible to calculate the following pressure distribution by Eq.3.5 of Boussinesq and present them graphically

2.5.1 Vertical pressure distribution on horizontal plane

The vertical stress on horizontal plane at depth „z‟ is given by

$$\sigma_z = \frac{I_B \, Q}{Z^2}$$

Z being a specified depth

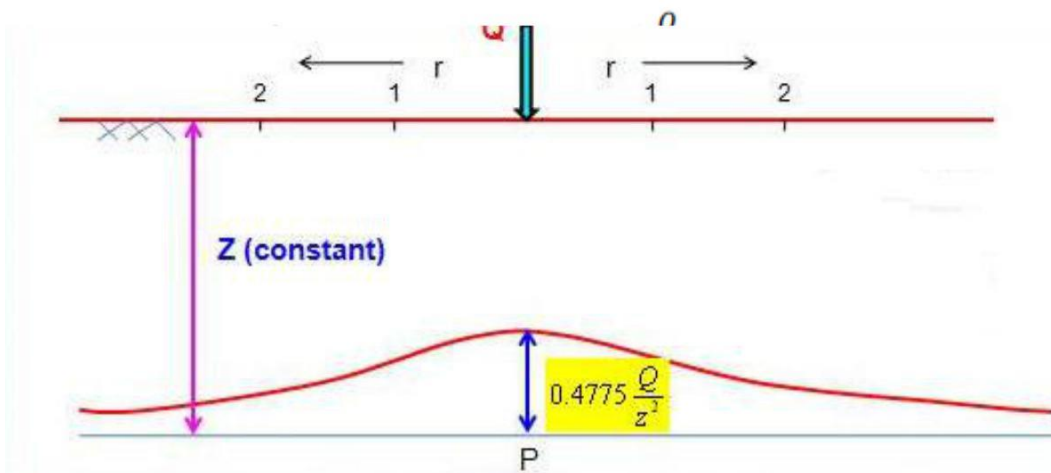For several assumed values of r, r/Z is calculated and the influence factor I_B, is found for each, the value of is then computed.

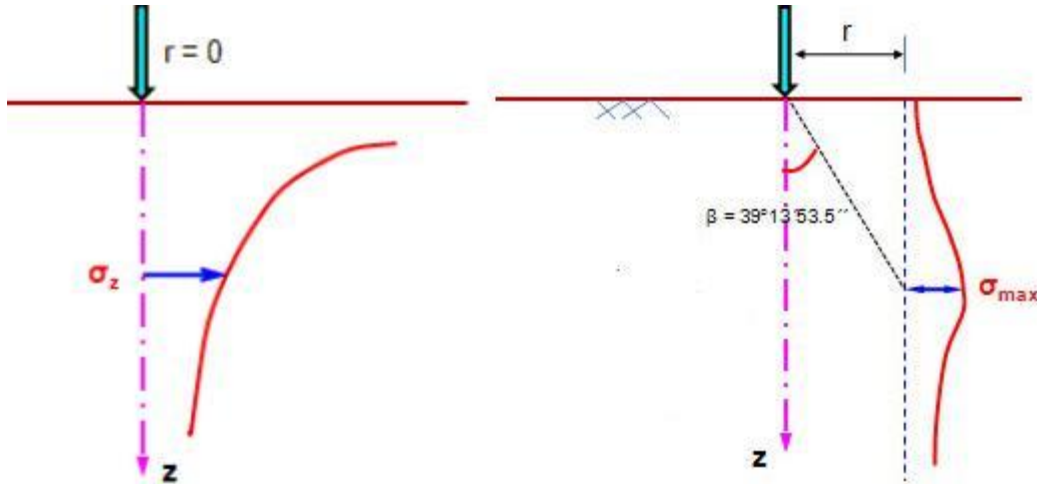**Fig.2.4 Vertical pressure distribution on horizontal plane**

Theoretically, the stress approaches zero at infinity. However, practically it reaches a negligible value at a short finite distance. The maximum pressure ordinate is relatively high at shallow elevations and it decreases with increase depth. In other words, the bell shaped fig flattens out with increasing depth. The vertical stress distribution diagram on a horizontal plane can also be obtained graphically if the isobars of different intensities are available. The horizontal plane is drawn on the isobar diagram. The points of intersection of the horizontal plane with the isobar of a particular intensity give that vertical stress.

2.5.2 Vertical stress distribution on a vertical plane

The vertical stress distribution on a vertical plane at a radial distance of „r‟ can be obtained from

$$\sigma_Z = \frac{I_B\,Q}{Z^2}$$

The variation of vertical stress with depth at a constant radial distance from the axis of the load is as shown in Fig.3.5

a) **directly under the point load**   **b) at a distance from point load**

Fig.2.5 Vertical pressure distribution on vertical plane

In this case radial distance „r‟ is constant and the depth „z‟ changes. As z increases, r/z decreases, for a constant value of „r‟. As r/z decreases +, value in the equation for $_z$ increases, but, since E is involved in the denominator of the expression for $_z$, its value first increases with depth, attains a maximum value, and then decreases with further increase in depth.

The maximum vertical stress occur at r/Z=0.817.this corresponds to the point of intersection of the vertical plane with the line drawn at $39^0 13\text{''}53.5\text{''}$ to the vertical axis of the load.

2.6 Vertical stress due to a line load

Certain road and rail traffic loads, and loads from walls may be resolved into line load, which has length along a given line but not breadth (theoretical)

The vertical stress in a soil mass due to a vertical line load can be obtained using boussinesq"s solution. Let the vertical line load be of intensity q per unit length, along the y axis, acting on the surface of a semi-infinite soil mass as shown in Fig.
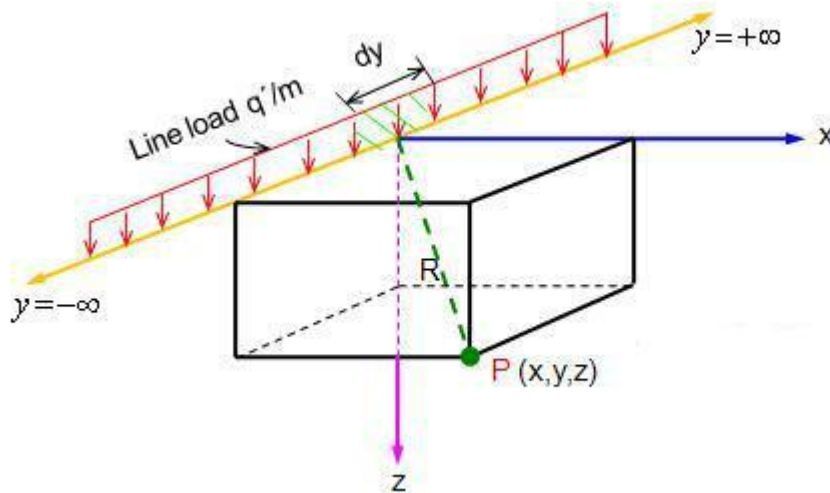

Fig.2.6 Vertical stress due to a line load

Let us consider the load acting on a small length dy. The load can be taken as a point load of q" dy. using boussinesq"s solution the vertical stress at P is given by

$$\Delta \sigma_Z = \frac{3(q'dy)}{2\pi} \frac{z^3}{(r^2+Z^2)^{\frac{5}{2}}} \qquad \text{-------------------------(3.9)}$$

$$\therefore \sigma_Z = \frac{3Q}{2\pi} \frac{1}{Z^2} \left(\frac{z^5}{(r^2+Z^2)^{\frac{5}{2}}} \right)$$

The vertical stress at P due to line load extending from -∞ to + ∞ is obtained by integration,

$$\sigma_Z = \frac{3q'Z^3}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{(r^2+Z^2)^{\frac{5}{2}}}$$

$$\sigma_Z = \frac{3q'Z^3}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{(x^2+y^2+Z^2)^{\frac{5}{2}}} \qquad \text{-----------(3.10)}$$

Substituting $x^2 + Z^2 = u^2$ in Eq. 3.10 we get

$$\sigma_Z = \frac{3q'Z^3}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{(u^2+y^2)^{\frac{5}{2}}} \qquad \text{----------------(3.11)}$$

Let $y = u \tan\theta$ $(dy = u \sec^2\theta.d\theta)$

When, $y = -\infty$, $\tan\theta = -\infty$, $\theta = -\pi/2$

$Y = +\infty$, $\tan\theta = \infty$, $\theta = \pi/2$

Eqn (3.11) can be written as

$$\sigma_Z = \frac{3q'Z^3}{2\pi} \int_0^{\pi/2} \frac{u\sec^2\theta\, d\theta}{(u^2+u^2\tan^2\theta)^{\frac{5}{2}}}$$

$$\sigma_Z = \frac{3q'Z^3}{2\pi} \int_0^{\pi/2} \frac{u\sec^2\theta\, d\theta}{u^5\sec^5\theta} \qquad \{\because (1+\tan^2\theta) = \sec^2\theta\}$$

$$\sigma_Z = \frac{3q'Z^3}{2\pi u^4} \int_0^{\pi/2} \cos^3\theta.d\theta \qquad \text{-------------------(3.12)}$$

Let $\sin\theta = t$

$\cos\theta.d\theta = dt$ Also $\cos^3\theta d\theta = \cos^2\theta.\cos\theta.d\theta$

When $\theta = 0$, $\sin\theta = 0$

$\theta = \pi/2$, $\sin\theta = 1$ $\cos^2\theta + \sin^2\theta = 1$

or $\cos^2\theta = (1-\sin^2\theta)$

$\therefore \cos^3\theta\, d\theta = (1 - \sin^2\theta) \cos\theta\, d\theta$

(3.12) becomes,

$$\sigma_Z = \frac{3q'z^3}{\pi u^4} \int_0^1 (1 - t^2).\,dt$$

$$= \frac{3q'z^3}{\pi u^4} \left(t - \frac{1}{3}t^3 \right) o^1$$

$$\sigma_Z = \frac{3q'z^3}{\pi u^4} \frac{2}{3}$$

$$= \frac{2q'z^3}{\pi u^4}$$

$$= \frac{2q'z^3}{\pi(x^2 + Z^2)^2}$$

$$\boldsymbol{\sigma_Z = \frac{2q'}{\pi Z}\left(\frac{1}{1 + (\frac{x}{z})^2} \right)^2} \quad \text{----------------------------------- (3.13)}$$

$$\sigma_Z = (I_B)_L \frac{q'}{Z}$$

$(I_B)_L$ = Boussinesq's influence factor for line load

$$= \frac{2}{\pi}\left(\frac{1}{1 + (\frac{x}{z})^2} \right)^2$$

When the point P lies vertically below the line load x = 0

$$\sigma_Z = \frac{2q'}{\pi Z}$$

Numerical Example

2) A line load of 100 kN/m run extends to a long distance. Determine the intensity of vertical stress at a point, 2m below the surface for the following two cases:

   i)   Directly under the line load, and

   ii)   At a distance of 2 m perpendicular to the line load.

Use Boussinesq's theory

Solution:

Data:

q =100 kN/m z = 2 m

Using Boussinesq's theory, the stress at any depth „z" for line load is given by:

$$\sigma_z = \frac{2q'}{\pi\,z}\left[\frac{1}{1+\left(\frac{x}{z}\right)^2}\right]^2$$

Case (i)

x = 0

∴ $\sigma_z = 31.83$ kN/m^2

Case (ii)

x = 2 m; z = 2; x/z = 1

∴ $\sigma_z = 7.96$ kN/m^2

2.7 Vertical stress due to a Strip load

The expression for vertical stress at any point „P" under a strip load can be derived from Eq. (2.13) of the line load. The expression will depend upon whether the point P lies below the centre of the strip load or away from the centre.

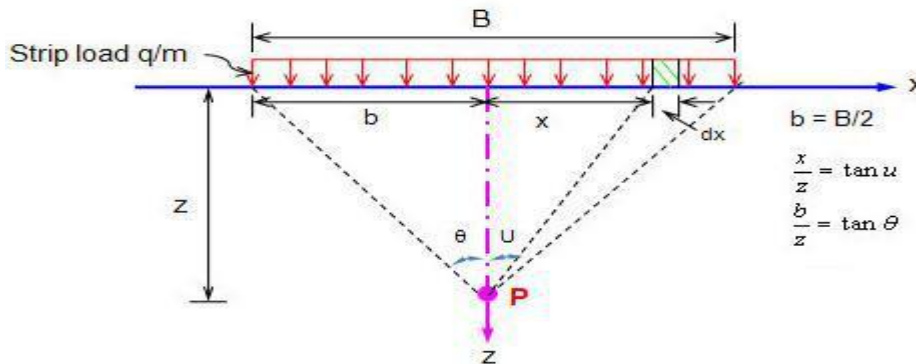### 2.7.1   Point P below the centre of the strip

Fig. 2.7 Vertical stress due to a strip load- point 'P' below the centre

Consider a strip load of width B (= 2b) and intensity „q‟ as shown in the Fig.3.7

Let us consider the load acting on a small elementary width dx at a distance x from the center of the load. This small load of (q dx) can be considered as a line load of intensity q.

From Eq. 2.13, we have

$$\Delta\sigma_z = \frac{2q'}{\pi z}\left(\frac{1}{1+(\frac{x}{z})^2}\right)^2 \quad \text{where } q' = q\,dx$$

The stress due to entire strip load is obtained as

$$\sigma_z = \frac{2q'}{\pi z}\int_{-b}^{+b}\left(\frac{1}{1+(\frac{x}{z})^2}\right)^2 . dx \qquad \text{-------------------(3.14)}$$

Let $\frac{x}{z} = tanu$, dx $= zsec^2u\,du$

$$\sigma_z = \frac{2q}{\pi Z}2\int_{0}^{\pi}\frac{z\,sec^2u}{(1+tan^2u)^2} . du$$

Where, $\theta = tan^{-1}\frac{b}{z}$

$$\sigma_z = \frac{4q}{\pi}\int_{0}^{\theta}cos^2u . du$$

$$= \frac{4q}{\pi}\int_{0}^{\theta}\frac{1+cos2u}{z} . du$$

$$= \frac{2q}{\pi}\left[u + \frac{sin2u}{2}\right]_{0}^{\theta}$$

$$= \frac{2q}{\pi}\left[\theta + \frac{sin2\theta}{2}\right]$$

$$\sigma_z = \frac{q}{\pi}[2\theta + sin2\theta] \qquad \text{------------------------ (3.15)}$$

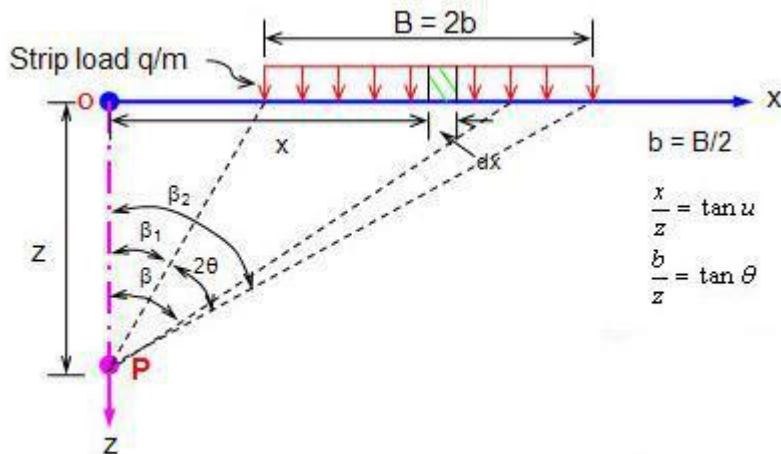**2.7.2 Point P not below the centre of the strip**

Fig.2.8 Vertical stress due to a strip load- point 'P' not at the centre

Fig. 3.8 shows the case when the point P is not below the centre of the strip. Let the extremities of the strip make angles of s and s with the vertical at P. Similar to the previous case, the load q dx acting on a small length dx can be considered as a line load. The Vertical stress at P given by Eq. 2.13 as

$$\Delta\sigma_Z = \frac{2(qdx)}{\pi Z}\left(\frac{1}{1+(\frac{x}{z})^2}\right)^2$$

$$Now\ substituting\ x = Z\tan\beta \quad dx = z\sec^2\beta d\beta$$

$$\Delta\sigma_Z = \frac{2q(Zsec^2\beta d\beta)}{\pi Z}\left(\frac{1}{1+tan^2\beta}\right)^2$$

$$\Delta\sigma_Z = \frac{2q}{\pi}cos^2\beta d\beta$$

$$\sigma_Z = \frac{2q}{\pi}\int_{\beta_{21}}^{\beta_2}\frac{1+cos2\beta}{2}d\beta$$

$$Or\ \sigma_Z = \frac{q}{\pi}\left[\beta + \frac{sin2\beta}{2}\right]_{\beta_1}^{\beta_2}$$

$$\sigma_Z = \frac{q}{\pi}[\beta + sin\beta\ cos\beta]_{\beta_1}^{\beta_2}$$

$$\sigma_Z = \frac{q}{\pi}((\beta_2 - \beta_1) + (sin\beta_2 cos\beta_2 - sin\beta_1 cos\beta_1)) \qquad \therefore \ (sin2\beta = 2sin\beta \ cos$$

$substituting \ \beta_2 - \beta_1 = 2\theta$

$$\boldsymbol{\sigma_Z = \frac{q}{\pi}(2\theta + sin\beta_2 cos\beta_2 - sin\beta_1 cos\beta_1)} \text{------------------ (3.16)}$$

If $\beta_2 + \beta_1 = 2\varphi$, it can be shown that

$sin\beta_2 cos\beta_2 - sin\beta_1 cos\beta_1 = sin2\theta \ cos2\emptyset$

$$\boldsymbol{\sigma_Z = \frac{q}{\pi}(2\theta + sin2\theta \ cos2\emptyset)} \text{------------------------- (3.17)}$$

Numerical Example

3) **A strip footing 2 m wide is loaded on the ground surface with a pressure of 150 kN/m^2. A 4 m thick soft clay layer exists at a depth of 10 m below the foundation. Find the average increase in vertical stress at the centre of clay layer below the centre line and at the edge of footing. Adopt Boussinesq's theory for strip load.**

Solution:
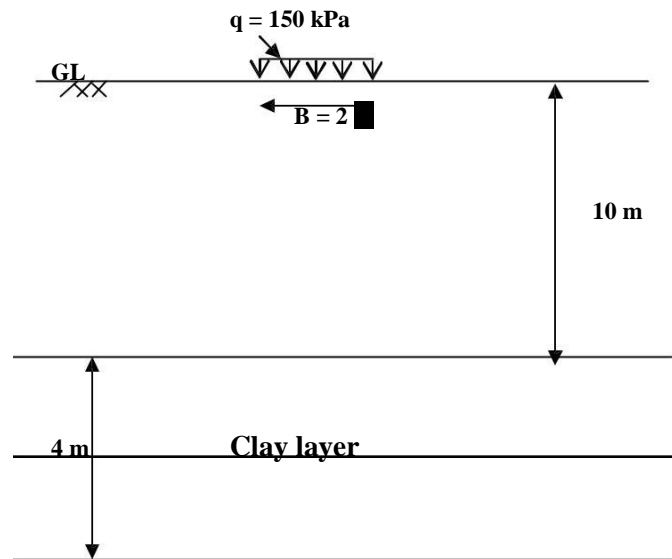
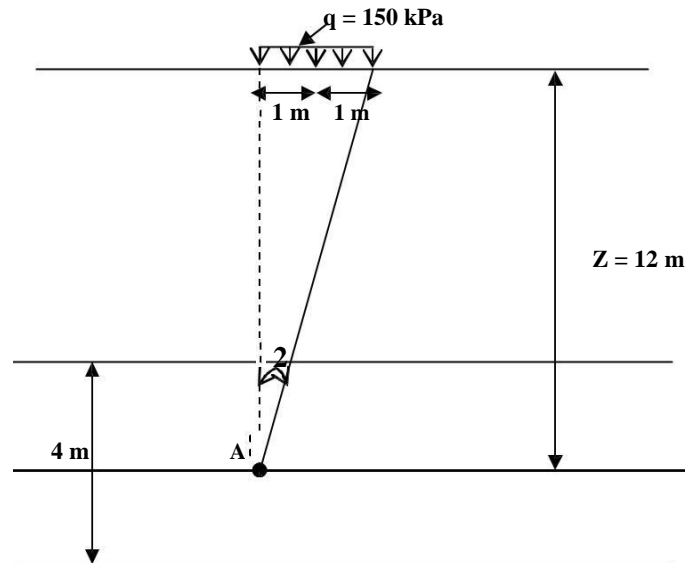**Case (i): Point at the centre of clay layer below the centre line of strip load**
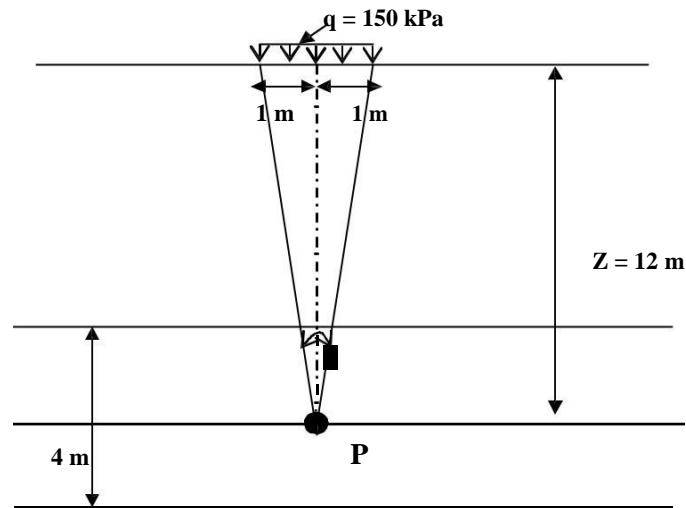
$\theta = \tan^{-1}(1/12) = 4°45\ 49 \quad = 0.083 \text{ rad}$

$\therefore 2\theta = 0.166 \text{ rad}$

$$\sigma_z = \frac{-q}{\pi}\left[2\theta + \sin 2\theta\ \right]$$

or $\sigma_z = 15.83 \text{ kN/m}^2$

Case (ii): Point at the centre of clay layer below the edge of footing (strip load)

$2\theta = \tan^{-1}(2/12) = 9°27\ 44 = 0.166 \text{ rad}$

$$\sigma_z = \frac{q}{\pi}\left[2\theta + \sin 2\theta\right] \quad = 15.72 \text{ kN/m}^2$$

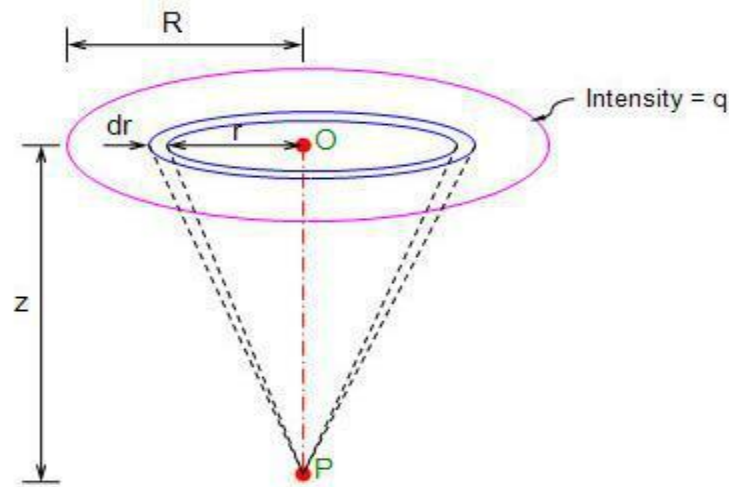### 2.8 Vertical Stress under a uniform loaded circular area

Fig.2.9 Vertical Stress under a uniform loaded circular area

This problem may arise in connection with settlement studies of structures on circular foundations, such as gasoline tanks, grain elevators, and storage bins.

The Boussinesq"s equation for vertical stress due to a point load can be extended to find the vertical stress at any point beneath the centre of a uniformly loaded circular area.

Let q = intensity of the load per unit area

And R = the radius of the loaded area

Let us consider an elementary ring of radius r and thickness „dr" of the loaded area.

The load on the elementary ring = q(2πr)dr

But we know that

$$\sigma_z = \frac{3Q}{2\pi}\frac{1}{Z^2}\left(\frac{1}{(1+(\frac{r}{z})^2)^{\frac{5}{2}}}\right)$$

$$\Delta\sigma_z = \frac{3(q2\pi rdr)}{2\pi}\frac{1}{Z^2}\left(\frac{1}{(1+(\frac{r}{z})^2)^{\frac{5}{2}}}\right)$$

$$\Delta\sigma_z = \frac{3qrdr}{(r^2+Z^2)^{\frac{5}{2}}}Z^3$$

The vertical stress due to full load is given by

$$\sigma_z = 3\, q\, z^3 \int_0^R \frac{r\, dr}{(r^2+z^2)^{\frac{5}{2}}}$$ ----------------(3.18)

$Let\ r^2 + Z^2 = u^2$ | when r = 0, $u = Z^2$

$2r\, dr = du$ | r = R, $u = R^2 + Z^2$

Eq. 3.18 becomes,

$$\sigma_z = 3\, q\, z^3 \int_{Z^2}^{R^2+Z^2} \frac{du}{2(u)^{\frac{5}{2}}}$$

$$\sigma_z = 3\, q\, z^3 \left(\frac{1}{\left(\frac{-3}{2}\right)}\right) \left[u^{\frac{-3}{2}}\right]_{Z^2}^{R^2+Z^2}$$

$$= -qz^3 \left[\frac{1}{(R^2+Z^2)^{3/2}} - \frac{1}{(Z^2)^{3/2}}\right]$$

$$= qz^3 \left[\frac{1}{(Z^2)^{3/2}} - \frac{1}{(R^2+Z^2)^{3/2}}\right]$$

$$\sigma_z = \left[1 - \left(\frac{1}{1+\left(\frac{R}{Z}\right)^2}\right)^{\frac{3}{2}}\right]$$ ------------------------ (3.19)

$$\sigma_z = I_c\, q$$ ------------------------(3.20)

Where, I_c is the influence coefficient for the circular area and is given by

$$I_c = \left[1 - \left(\frac{1}{1+\left(\frac{R}{Z}\right)^2}\right)^{\frac{3}{2}}\right]$$ ----------------------------(3.21)

Eq. 3.21 for the influence coefficient I_c can be written in terms of the angle 2θ subtended at point P by the load.

Now tanθ = R/Z $$I_c = \left[1 - \left(\frac{1}{1+(\tan)^2}\right)^{\frac{3}{2}}\right]$$

$$= (1-(cos^2\theta)^{3/2}$$

$$I_c = 1 - cos^3\theta$$ ----------- (3.22)

Numerical Example

4) **A circular area 6 m in diameter, carries uniformly distributed load of 10 kN/m^2. Determine the vertical stress at a depth of 2m, 4 m and 8 m. Plot the variation of vertical stress with depth. [VTU- July 2006 – 8 M]**

Solution:

According to Boussinesq's theory for a circular loaded area, the vertical pressure at any point below the centre of the loading is given by:
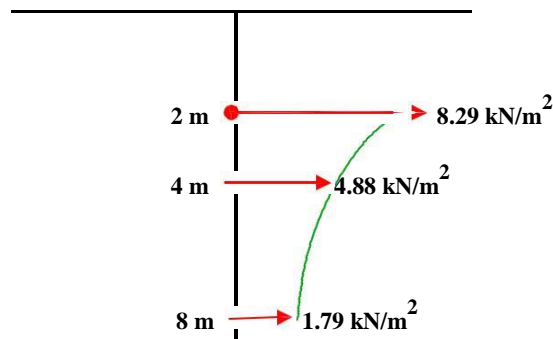
$$\sigma_z = q \left[1 - \frac{1}{1 + \left(\dfrac{R}{z}\right)^2} \right]^{\frac{3}{2}}$$

Here R = 3 m and q = 10 kN/m^2

At z = 2 m, σ_z = 8.29 kN/m^2

At z = 4 m, σ_z = 4.88 kN/m^2

At z = 8 m, σ_z = 1.79 kN/m^2

2.9 Uniform load on rectangular area

The more common shape of a loaded area in foundation engineering practice is a rectangle, especially in the case of buildings.
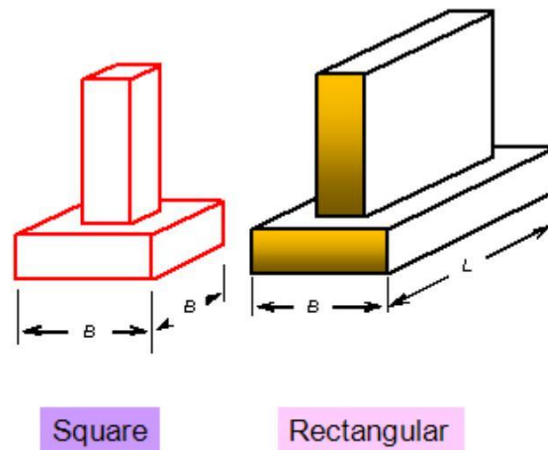
Fig. 2.10 Common shape of foundations

2.9.1 Vertical stress under corner of a uniformly loaded rectangular area

The vertical stresses under a corner of a rectangular area with a uniformly distributed load of intensity q can be obtained from Boussinesq‟s solution.
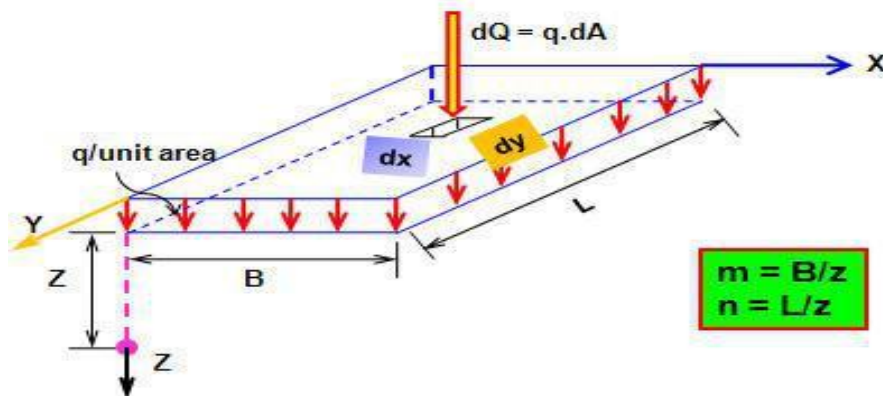

Fig. 2.11 Vertical stress under corner of a uniformly loaded rectangular area

$$\Delta\sigma_z = \frac{3(q \cdot dx \cdot dy) \times z^3}{2\pi} \frac{1}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

By integration,

$$\sigma_z = \frac{3qz^3}{2\pi} \int_0^L \int_0^B \frac{q \cdot dx \cdot dy}{\left(x^2 + y^2 + z^2\right)^{\frac{5}{2}}} \qquad \text{------------(3.23)}$$

Newmark (1935) has done the integration which is quite complicated and has expressed in the form:

$$\sigma_z = \frac{q}{4\pi}\left[\frac{2mn\sqrt{\left(m^2 + n^2 + 1\right)}}{m^2 + n^2 + 1 + m^2 n^2}\left(\frac{m^2 + n^2 + 2}{m^2 + n^2 + 1} \right) + \tan^{-1}\frac{2mn\sqrt{\left(m^2 + n^2 + 1\right)}}{m^2 + n^2 + 1 - m^2 n^2} \right] \quad \text{-----(3.24)}$$

Where m = (B/Z) and n = (L/Z)

or $\sigma_z = I_N\, q_N$ --------------------(3.25)

Where I_N is Newmark's influence coefficient

Note: The value of m and n can be interchanged without any effect on the value of ‹Œ

Vertical stress at any point under a rectangular area

The equation as developed above can also be used for finding the vertical stress at a point which is not loaded below the corner of loaded area. The rectangular area is subdivided into rectangles such that each rectangle has a corner at the point where the vertical stress is determined using the principle of superposition.

The following cases can occur:

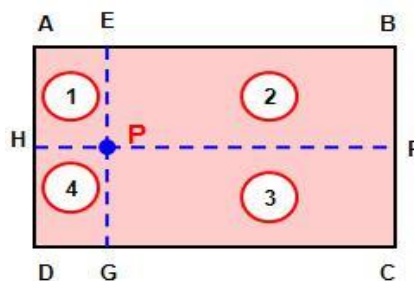**Point anywhere below the rectangular area**

Fig. 3.12 Point anywhere below the rectangular area

Fig. 2.12 shows the location of the point P below the rectangular area ABCD. The given rectangle is subdivided into four small rectangles (AEPH, EBFP, HPGD and PFCG) each having one corner at p. the vertical stress at P due to the given rectangular load is equal to that from the four small rectangles.

$\sigma_Z = q [I_{N1} + I_{N2} + I_{N3} + I_{N4}]$

Where, I_{N1}, I_{N2}, I_{N3}, and I_{N4}, are Newmark‟s influence factors for the four rectangles marked 1, 2, 3, 4.

For the special case, when the point „P‟ is at the centre of the rectangle ABCD (Fig.2.13), all the four small rectangles are equal, and the above equation becomes
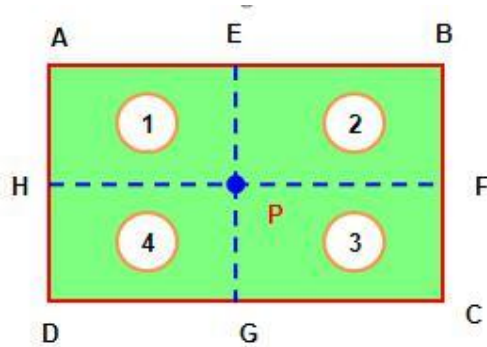

Fig. 2.13 Point at the centre of the rectangular area

$\sigma_Z = 4 I_N q$

Where I_N = influence factor for the small rectangle.

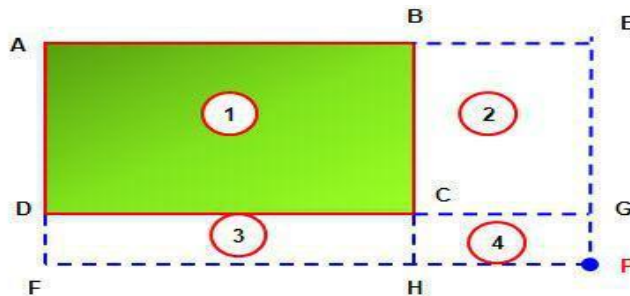**Point outside the loaded rectangular area**

Fig. 2.14 Point outside the loaded rectangular area

Fig. 2.14 shows the point P outside the loaded area ABCD. In this case, a large rectangle AEPF is drawn with its one corner at P.

Now rectangle ABCD = rectangle AEPF – rectangle BEPH – rectangle DGPF + rectangle CGPH

The last rectangle CGPH is given plus sign because this area has been deducted twice (once in rectangle BEPH and once in DGPF)

The stress at P due to a load on rectangle ABCD is given by

$$\sigma_Z = q[(I_N)_1 - (I_N)_2 - (I_N)_3 + (I_N)_4]$$

Where, $(I_N)_1$, $(I_N)_2$, $(I_N)_3$, & $(I_N)_4$ are the influence coefficient for the rectangles AEPF, BEPH, DGPF, CGPH respectively.

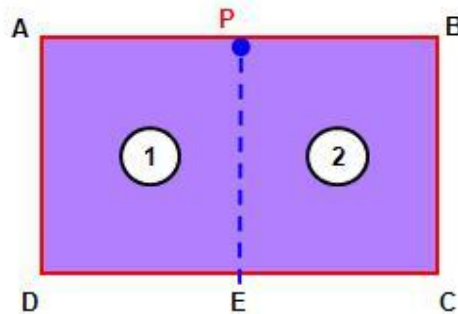**Point below the edge of the loaded rectangular area**

Fig. 2.15 Point below the edge of the loaded rectangular area

If the point P is below the edge of the loaded area ABCD (Fig. 2.15), the given rectangle is divided into two small rectangles APED and PBCE

$$\sigma_Z = q[(I_N)_1 + (I_N)_2]$$

Where, $(I_N)_1$ & $(I_N)_2$ are the influence coefficient for the rectangles 1 & 2.

2.10 Westergaard's Solution

Boussinesq's solution assumes that the soil deposit is isotropic. Actual sedimentary deposits are generally anisotropic. There are thin layers of sand embedded in homogeneous clay strata. Westergaard's solution assumes that there are thin sheets of rigid materials sandwiched in a homogeneous soil mass. These thin sheets are closely spaced and are of infinite rigidity and therefore prevent the medium from undergoing lateral strain. These permit only downward displacement of the soil mass as a whole without any lateral displacement. Therefore Westergaard's solution represents more closely the actual sedimentary deposits.

According to Westergaard the vertical stress at a point "P" at a depth "z" below the concentrated load Q is given by

$$\sigma_z = \frac{C/2\pi}{\left[c^2 + \frac{r^2}{z}\right]^{3/2}} \frac{Q}{z^2} \text{-----------------------------(3.26)}$$

Where, C depends upon the Poisson ratio (v) and is given by

$$C = \sqrt{\frac{(1-2v)}{(2-2v)}} \text{---------------------(3.27)}$$

For an elastic material, the value of "v' varies between 0 to 0.5 Since it

$$v = 0 \ \& \ C = 1/\sqrt{2}$$

is assumed that there is lateral restrained,

$$\sigma_z = \frac{1}{\pi\left[1 + 2\frac{r^2}{z}\right]^{3/2}} \frac{Q}{z^2} = I_w \frac{Q}{z^2} \text{----------------------(3.28)}$$

Where $I_w = \dfrac{1}{\pi\left[1 + 2\frac{r^2}{z}\right]^{3/2}}$ is known as Westergaard's influence factor

2.11 Newmark's Influence Chart

2.11.1 Introduction:

The vertical stress at any particular depth in the soil due to the action of vertical load on the surface of the ground was given and explained by the famous Boussinesq"s theory. This theory gave formulae to calculate vertical stresses at a point for different types of vertical loading, taking into consideration only a few well defined and standard shape of loading like a point loading, line loading, strip loading, rectangular loading and circular loading. When some complex shape of loading, like a plan of a structure was given, it became very cumbersome to calculate the vertical stress using these formulas. Hence, a need for more simpler and faster method of stress calculation was realized. Newmark formulated a new simple graphical method to calculate the vertical stress at any particular depth caused due to any shape of vertical uniformly distributed loading in the interior of an elastic, homogeneous and isotropic medium, which is bounded by horizontal planes (i.e. semi-infinite medium).

2.11.2 Theory behind construction of Newmark's Influence chart:

Newmark"s chart utilizes the equation given by Boussinesq for vertical stress caused due to uniformly distributed load on a circular area (vertical load) at any particular depth.

$$\sigma_z = I_c q \qquad\qquad (2.20)$$

Where σ_z = vertical stress at depth of

q = load intensity

I_c = Boussinesq"s influence value for circular loading.

The Boussinesq"s influence factor, I_c depends on the radius of circular loaded area, R and on the depth, z, at which the stress is required. It is given by:

$$I_c = 1 - \frac{1}{\left[1 + (R/z)^2\right]^{\frac{3}{2}}}$$

From this equation the ratio (r/z) is expressed as:

$$\frac{R}{z} = \sqrt{\left[1 - \frac{\sigma_z}{q}\right]^{-\frac{2}{3}} - 1} \qquad\qquad (2.29)$$

This ratio (r/z) represents the *relative size or relative radii* of circular loaded bearing area, which gives a particular value of vertical stress to applied load ration (σ_z/q). The ratio (σ_z/q) can vary from 0 to 1. By substituting various values of (σ_z/q) with any desired interval (i) say 0.1 or 0.05 etc. we can get various relative size ratio i.e. (r/z). Now let us assign a series of values for the ratio (σ_z/q), such as 0, 0.1, 0.2,…., 0.9, and 1. A corresponding set of values for the relative radii, (R/z) may be obtained. If a particular depth is specified, then a series of concentric circles can be drawn. Since the first circle has a zero radius and the eleventh has infinite radius. In practice, only nine circles are drawn. Each ring or annular space causes a stress of q/10 at a point beneath the center at the specified depth z, since the number of annular space (c) is ten. Thus, each loaded area enclosed between any two successive circles contributes the same influence on the vertical stress at the point we require. The typical tabulation of (R/z) and (σ_z/q) values are shown with an interval, i = 0.1.

Table 2.2:   Relative radii for Newmark's Influence chart

Sl. No. of circle	(σ_z/q)	Relative radii (R/z)
1	0.0	0.000
2	0.1	0.270
3	0.2	0.400
4	0.3	0.518
5	0.4	0.637
6	0.5	0.766
7	0.6	0.918
8	0.7	1.110
9	0.8	1.387
10	0.9	1.908
11	1.0	∞

From this table it can be seen that the widths of the annular slices or rings are greater the farther away they are from the center. The circle for an influence of 1.0 has an infinitely large radius. Now let us assume that a set of equally spaced rays, say „*s'* in number, is drawn

emanating from the center of the circles, thus dividing each annular area into *s* sectors, and the total number area into *cs* sectors. If the usual value of 20 is adopted for *s*, the total number of sectors in this case will be 10x20 or 200. Each sector will cause a vertical stress of $1/200^{th}$ of the total value at the center at the specified depth and is referred to as a „mesh‟ or an „influence unit‟. The value 1/200 or 0.005 is said to be the „influence value‟ or „influence factor‟ (I) for the chart. Each mesh may thus be understood to represent an influence area. The value of „c‟ and „s‟ are so selected that it gives the accuracy we desire for. Higher value of „c‟ and „s‟ gives more meshes and hence higher accuracy.

2.11.3 Construction of Newmark's Influence chart

For the specified depth z, say 10 m, the radii of the circles, R, are calculated from the relative radii of Table 1. (2.70m, 4.00m, 5.18m, …and so on). The circles are then drawn to a convenient scale (say, 1 cm = 2 m or 1:200). A suitable number of uniformly spaced rays (to get required influence value) are drawn, emanating from the center of the circles. The resulting diagram will appear as shown in Fig. 3.16. On the figure is drawn a line AB, representing the depth z to the scale used in drawing the circles. If the scale used is 1 cm = 2 m, then AB will be 5 cm. The influence value for this chart will be I = (1/c x s).

The same chart can be used for other values of the depth „z‟. The length AB is taken equal to the depth „z‟ of the given problem and to that scale the loaded diagram is plotted on a tracing sheet to be superimposed later on the Newmark‟s chart to obtain the vertical stress at the desired point.
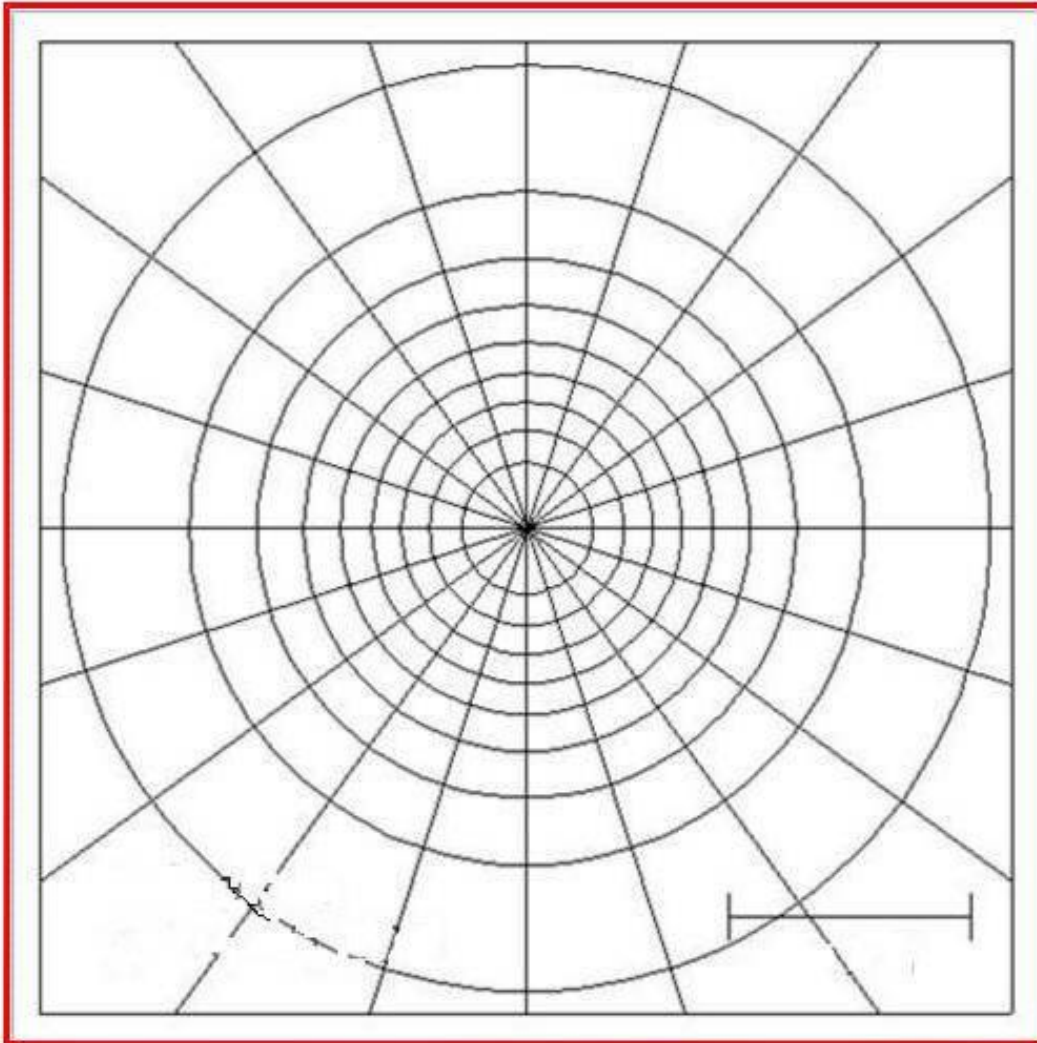
Fig. 2.16 Newmark's chart

2.11.4 Application of Newmark's Influence chart

Application of Newmark"s Influence chart in solving problems is quite easy and simple. The plan of the loaded area is first drawn on a tracing sheet to the same scale as the scale of the line segment AB on the chart representing the depth „z". The location of the point where the vertical stress is required is marked on the plan, say as „P". Now, the tracing sheet is placed over the chart, such that the point „P" comes exactly over the center of the chart from where the rays are emanating. Now the number of mesh covered by the plan is counted. In case of partly covered mesh an intelligent judgement of the fraction of mesh covered is required. Let the total number of mesh be equal to „n" . Then the vertical stress at the desired depth is given by:

$$\sigma_z = I \times n \times q$$

Where I = Influence value = $1/(c \times s)$

               n = Number of meshes under the loaded area

               q = uniformly distributed load

               c = No. of concentric areas

               s = No. of radial lines

Approximate method

The method discussed in the preceding sections are relatively more accurate, but are time consuming. Sometimes, the engineer is interested to estimate the vertical stresses approximately. For preliminary designs, thus saving time and labour without sacrificing accuracy to any significant degree.

They are also used to determine the stress distribution in soil under the influence of complex loading and/ or shapes of loaded areas.

 Two commonly used approximate methods are:

Equivalent point- load method

The vertical stress at a point under a loaded area of any of any shape can be determined by dividing the loaded area into small area and replacing the distributed load on each on small area by an equivalent point load acting at the centroid of the small area. The principle of superposition is then applied and the required stress at a specified point is obtained by summing up the contributions of the individual. Point loads from each of the units by applying the approximate point load formula, such as that of Boussinesq"s or Westergaard"s.
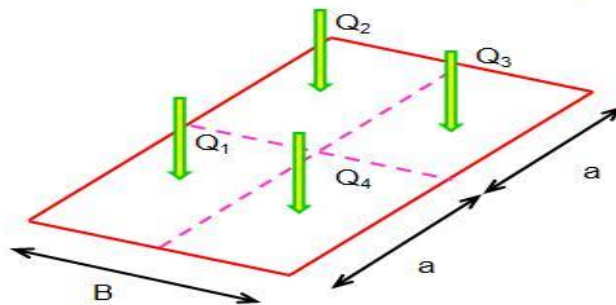

Fig.2.17 Equivalent point- load method

As shown in the Fig.2.17, if a square area of size B is acted on by a uniform load q, the same area can be divided into four small area. And the load on each area can be converted into an equivalent point load assumed to act at its centroid. Then the vertical stress at any point below or outside the loaded area is equal to the sum of the vertical stresses due to these equivalent point loads. Then

$$\sigma_Z = \frac{[Q_1(I_B)_1 + Q_2(I_B)_2 + Q_3(I_B)_3 + \ldots\ldots + Q_n(I_B)_n]}{Z^2}$$

$$\sigma_Z = \frac{1}{Z^2} \sum_{i=1}^{n} Q_i(I_B)_i$$

2.12 Contact pressure

The upward pressure due to soil on the underside of the footing or foundation is termed contact pressure.

In the derivations of vertical stress below the loaded areas using Boussinesq"s theory or Westergaard"s theory, it has been assumed that the footing is flexible and the contact pressure distribution is uniform and equal to „q". Actual footings are not flexible as assumed. The actual distribution of the contact pressure depends on a number of factors.

2.12.1 Factors affecting contact pressure distribution
The factors are:
1. Flexural rigidity of base of footing
2. Type of soil
3. Confinement

Flexural rigidity of base of footing

Uniform loading on a flexible base induces uniform contact pressure on any type of soil, while a rigid base induces non-uniform pressure. Foundation bases are usually thick massive concrete structures, which cannot be treated as ideally flexible.

Type of soil

The contact pressure distribution also depends on the elastic properties of the soil. The elastic properties of soil depends on the type of soil.
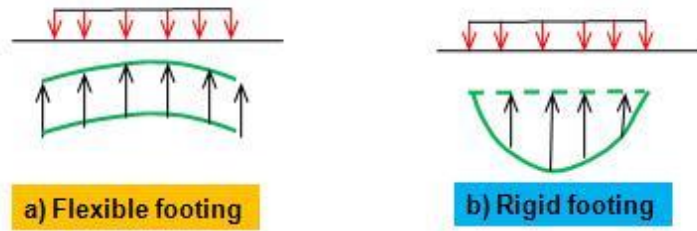
a. Sandy soil

Fig. 2.19 Contact pressure diagram on sand

Fig 2.19 a & b shows the qualitative contact pressure distribution under flexible and rigid footing resting on a sandy soil and subjected to a uniformly distributed load q. when the footing is flexible, the edges undergo a large settlement than at centre. The soil at centre is confined and therefore has a high modulus of elasticity and deflects less for the same contact pressure. The contact pressure is uniform.

When the footing is rigid the settlement is uniform. The contact pressure is parabolic with zero intensity at the edge sand maximum at the centre.
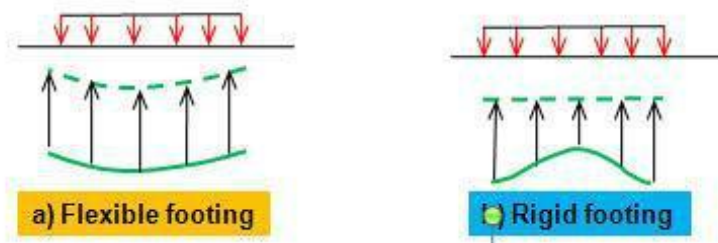
b. Clayey soils

Fig. 2.20 Contact pressure diagram on saturated clay

Fig. 2.20 shows the qualitative contact pressure distribution under flexible and rigid footings resting on saturated clay and subjected to a uniformly distributed load q.

When the footing is flexible, it deforms into the shape of a bowel, with the maximum deflection at the centre. The contact pressure distribution is uniform

If the footing is rigid, the settlement is uniform. The contact pressure distribution is minimum at the centre and the maximum at the edges(infinite theoretically). The stresses at the edges in real soil cannot be infinite as theoretically determined for an elastic mass. In real soils, beyond a certain limiting values of stress, the plastic flow occurs and the pressure becomes infinite as shown in Fig.2.20

c. C-Ø soil

For a c – Ø soil, the contact pressure for a flexible footing will be uniform as shown in Fig. 3.21(a). For a rigid footing, the pressure distribution will be as shown in the Fig. 3.21 (b), it is more at the edge and less at the centre.
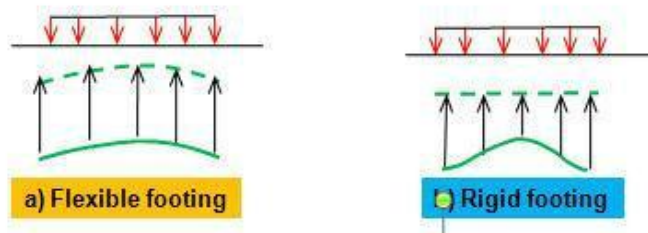

Fig. 2.21 Contact pressure diagram on c – Ø soil

Confinement

For surface loading in sand contact pressure is zero and for clayey soils, it is very high. When the footings are confined then the edge stresses and the contact pressure distribution changes. In sand, if the foundation is embedded or confined, then there would be some finite contact pressure at the edges. In clayey soil the contact pressure at the edges slightly reduces as confinement increase at the edges to surface loading

The more the foundation is below the surface of the sand, the more the shear resistance developed at the edges due to increase in the overburden pressure and as a consequence, the contact pressure distribution tends to be more uniform as compared to being parabolic to surface loading.

2.13 Assignment Questions

1. List the assumption of Boussinesq analysis for the pressure distribution in a soil layer.

2. What do you understand by "pressure bulb"? Illustrate with sketches.

3. Using Boussinesq"s equation, construct isobar of intensity 0.1 Q, where Q is point load acting on surface.

4. Derive an expression to find vertical pressure under a uniformly loaded circular area

2.14 OUTCOMES
- Various methods to determine stresses in soils
- Understanding of contact pressure
- Knowledge of estimation of settlement in soils

2.15 Further reading

- http://nptel.ac.in/courses/105103097/20
- http://environment.uwe.ac.uk/geocal/SoilMech/stresses/stresses.htm