



MODULE-4

Frequency Response Analysis

LESSON STRUCTURE

Nyquist Stability criterion

Nyquist criterion using Nyquist plots

Simplified forms of the Nyquist criterion

The Nyquist criterion using Bode plots

OBJECTIVES:

- ☐ To demonstrate Stability Determine Gain & Phase Margins Medium effort.
- ☐ To demonstrate applications of the frequency response to analysis of system stability (the Nyquist criterion), relating the frequency response to transient performance specifications.
- ☐ Finds number of RHP poles of $T(s)$, the closed-loop transfer function.

Nyquist Stability criterion

This graphical method, which was originally developed for the stability analysis of feedback amplifiers, is especially suitable for different control applications. With this method the closed-loop stability analysis is based on the locus of the open-loop frequency response

$G_o(j\omega)$. Since only knowledge of the frequency response $G_o(j\omega)$ is necessary, it is a versatile practical approach for the following cases:

- For many cases $G_o(j\omega)$ can be determined by series connection of elements whose parameters are known.
- Frequency responses of the loop elements determined by experiments or $G_o(j\omega)$ can be considered directly.
- Systems with dead time can be investigated.
- Using the frequency response characteristic of $G_o(j\omega)$ not only the stability analysis, but also the design of stable control systems can be easily performed.

Nyquist criterion using Nyquist plots

To derive this criterion one starts with the rational transfer function of the *open loop*

$$G_o(s) = \frac{N_o(s)}{D_o(s)}$$

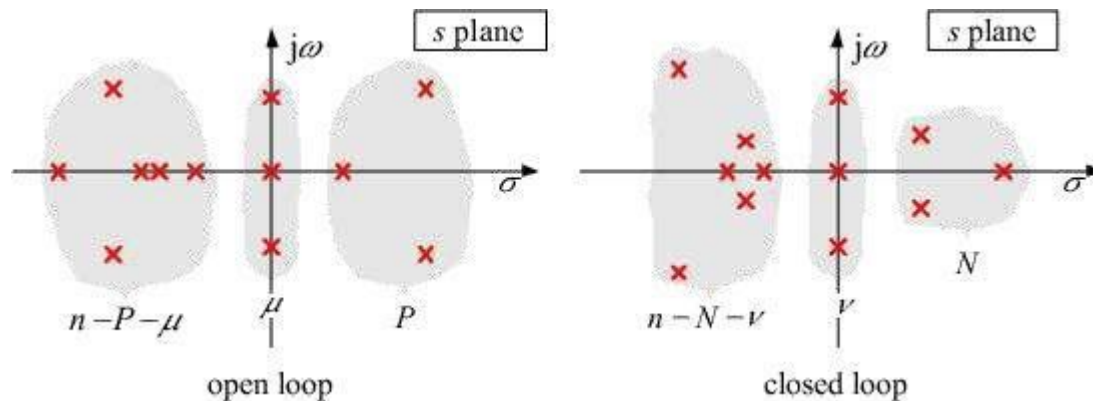


Figure: Poles of the open and closed loop in the s plane (multiple poles are counted according to their multiplicity)

To determine $\Delta\varphi_s$, the locus $G'(j\omega) = 1 + G_0(j\omega)$ can be drawn on the Nyquist diagram and the phase angle checked. Expediently one moves this curve by 1 to the left in the $G_0(j\omega)$ plane. Thus for stability analysis of the closed loop the locus $G_0(j\omega)$ of the open loop according to Figure 5.5 has to be drawn.

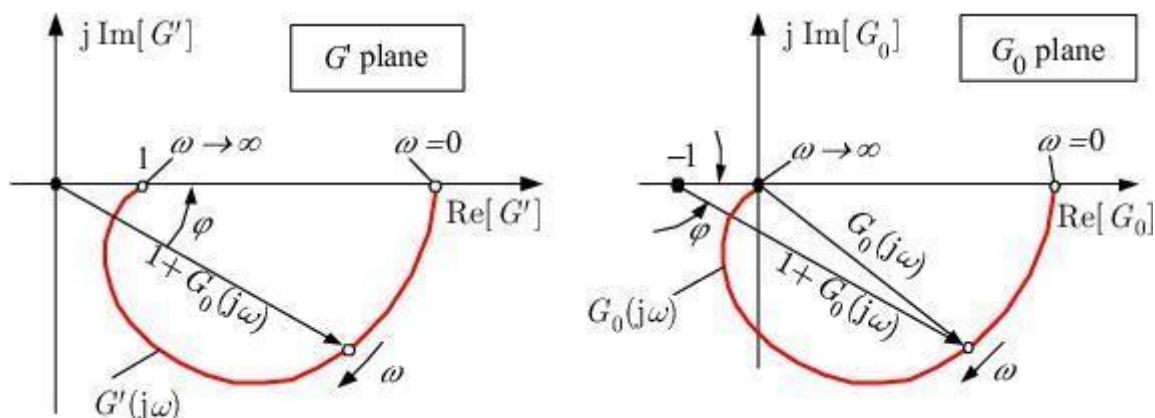


Figure : Nyquist diagrams of $G'(j\omega)$ and $G_0(j\omega)$

Here $\Delta\varphi_s$ is the continuous change in the angle of the vector from the so called *critical point* $(-1, j0)$ to the moving point on the locus of $G_0(j\omega)$ for $0 \leq \omega \leq \infty$. Points where the locus passes through the point $(-1, j0)$ or where it has points at infinity correspond to the zeros and poles of $G'(s)$ on the imaginary axis, respectively. These discontinuities are not taken into account for the derivation of $G_0(j\omega)$. Figure shows an example of $G_0(j\omega)$.

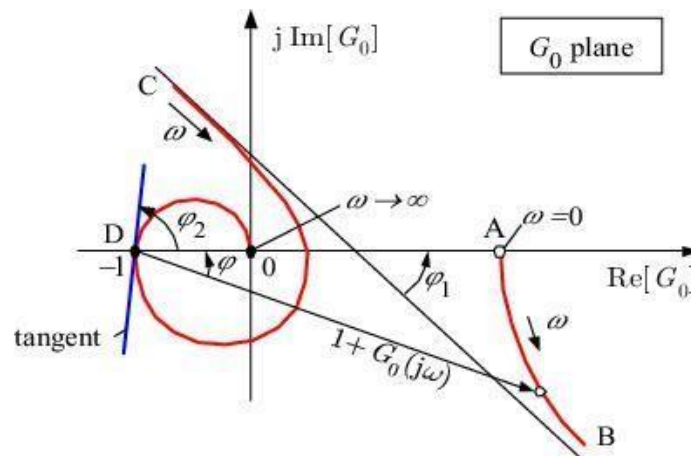


Figure: Determination of continuous changes in the angle $\Delta\varphi_s$

where two discontinuous changes of the angle occur. Thereby the continuous change of the angle consists of three parts

$$\begin{aligned}\Delta\varphi_s &= \Delta\varphi_{AB} + \Delta\varphi_{CD} + \Delta\varphi_{DO} \\ &= -\varphi_1 - (2\pi - \varphi_1 - \varphi_2) - \varphi_2 = -2\pi\end{aligned}$$

The rotation is counter clockwise positive.

As the closed loop is only asymptotically stable for $N = \nu = 0$, then from the *general case of the Nyquist criterion* follows:

The closed loop is asymptotically stable, if and only if the continuous change in the angle of the vector from the critical point $(-1, j0)$ to the moving point of the locus $G_0(j\omega)$ of the open loop is

$$\Delta\varphi_s = (P + \mu/2)\pi$$

For the case with a *negative* gain K_0 of the open loop the locus is rotated by 180° relative to the case with a *positive* K_0 . The Nyquist criterion remains valid also in the case of a *dead time* in the open loop.

Simplified forms of the Nyquist criterion

It follows from that for an open-loop stable system, that is $P = 0$ and $\mu = 0$, then $\Delta\varphi_s = 0$. Therefore the Nyquist criterion can be reformulated as follows:



If the open loop is asymptotically stable, then the closed loop is only asymptotically stable, if the frequency response locus of the open loop does neither revolve around or pass through the critical point $(-1, j0)$.

Another form of the simplified Nyquist criterion for $G_0(s)$ with poles at $s = -1$ is the so called 'left-hand rule':

The open loop has only poles in the left-half plane with the exception of a single or double pole at $(s = 0)$ or behaviour. In this case the closed loop is only stable, if the critical point $(-1, j0)$ is on the *left* hand-side of the locus $G_0(j\omega)$ in the direction of increasing values of ω .

This form of the Nyquist criterion is sufficient for most cases. The part of the locus that is significant is that closest to the critical point. For very complicated curves one should go back to the general case. The left-hand rule can be graphically derived from the generalised locus

The orthogonal (σ, ω) -net is observed and asymptotic stability of the closed loop is given, if a curve with $\sigma < 0$ passes through the critical point $(-1, j0)$. Such a curve is always on the left-hand side of $G_0(j\omega)$.

The Nyquist criterion using Bode plots

Because of the simplicity of the graphical construction of the frequency response characteristics of a given transfer function the application of the Nyquist criterion is often more simple using Bode plots. The continuous change of the angle $\Delta\varphi_s$ of the vector from the critical point $(-1, j0)$ to the locus of $G_0(j\omega)$ must be expressed by the amplitude and phase response of $G_0(j\omega)$. From figure

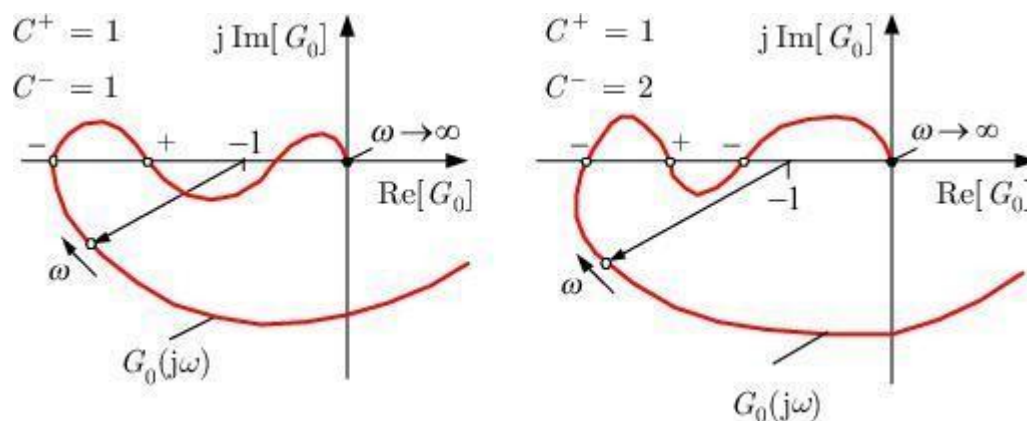


Figure : Positive (+) and negative (-) intersections of the locus $G_0(j\omega)$ with the real axis on the left-hand side of the critical point

it can be seen that this change of the angle is directly related to the count of intersections of



the locus with the real axis on the left-hand side of the critical point between $(-\infty, -1)$. The Nyquist criterion can therefore also be represented by the count of these intersections if the gain of the open loop is positive.

Regarding the intersections of the locus of $G_0(j\omega)$ with the real axis in the range $(-\infty, -1)$, the transfer from the upper to the lower half plane in the direction of increasing ω values are treated as *positive intersections* while the reverse transfer are *negative intersections* (Figure 5.7). The change of the angle is zero if the count of positive intersections S^+ is equal

to the count of negative intersections S^- . The change of the angle depends also on the number of positive and negative intersections and if the open loop does not have poles on the imaginary axis, the change of the angle is

$$\Delta\varphi_S = 2\pi(C^+ - C^-)$$

In the case of an open loop containing an integrator, i.e. a single pole in the origin of the complex plane ($\mu = 1$), the locus starts for $\omega = 0$ at $\delta - j\infty$, where an additional $+\pi/2$ is added to the change of the angle. For proportional and integral behaviour of the open loop

$$\Delta\varphi_S = 2\pi(C^+ - C^-) + \mu\pi/2 \quad \mu = 0, 1$$

is valid. In principle this relation is also valid for $\mu = 2$, but the locus starts for $\omega = 0$ at $-\infty + j\delta$ (Figure 5.8), and this intersection would be counted

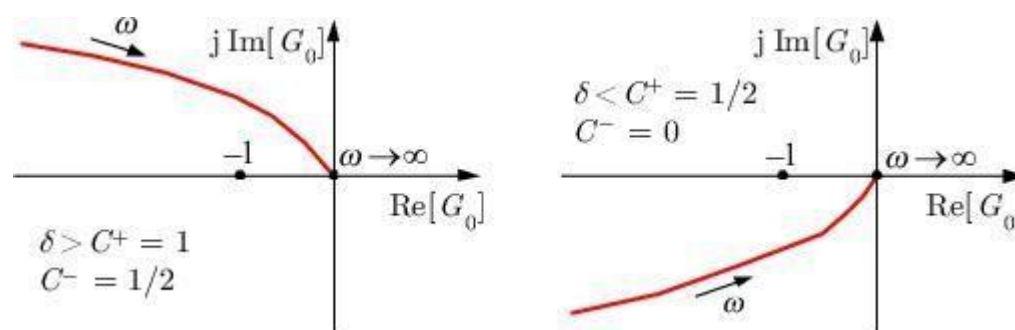


Figure : Count of the intersections on the left-hand side of the critical point for I_2 behaviour of the open loop

as a negative one if $\delta > 0$, i.e. if the locus for small ω is in the upper half plane of the real axis. But

de facto there is for $\delta > 0$ (and accordingly $\delta < 0$) no intersection. This follows from the detailed investigation of the discontinuous change of the angle, which occurs at $\omega = 0$. As only a continuous change of the angle is taken into account and because of reason of symmetry the start of the locus

at $\omega = 0$ is counted as a half intersection, positive for $\delta < 0$ and negative for $\delta > 0$, which is analogous to the definition given above For continuous changes of the angle

$$\Delta\varphi_S = 2\pi(C^+ - C^-) \quad (\mu = 2)$$



The open loop with the transfer function $G_0(s)$ has P poles in the left-half plane and possibly a single ($\mu = 1$) or double pole ($\mu = 2$) at $s = 0$. If the locus of $G_0(j\omega)$ has C^+ positive and C^- negative intersections with the real axis to the left of the critical point, then the closed loop is only asymptotically stable, if

$$D^* = C^+ - C^- = \begin{cases} \frac{P}{2} & \text{for } \mu = 0, 1 \\ \frac{P+1}{2} & \text{for } \mu = 2 \end{cases}$$

is valid. For the special case, that the open loop is stable ($P = 0$), the number of positive and negative intersections must be equal.

From this it follows that the difference of the number of positive and negative intersections in the case of $\mu = 0, 1$ is an integer and for $\mu = 2$ not an integer. From this follows immediately, that for $\mu = 0, 1$ the number P is even, for $\mu = 2$ the number $P+1$ is uneven and therefore in all cases P is an even number, such that the closed loop is asymptotically stable. This is only valid if $D^* \geq 1$.

The Nyquist criterion can now be transferred directly into the representation using frequency response characteristics. The magnitude response $A_0(\omega)_{dB}$, which corresponds to the locus $G_0(j\omega)$, is always positive at the intersections of the locus with the real axis in the range of $(-\infty, -1)$. These points of intersection correspond to the crossings of the phase response $\varphi_0(\omega)$ with lines $\pm 180^\circ$, $\pm 540^\circ$ etc., i.e. a uneven multiple of $^\circ$. In the case of a positive intersection of the locus, the phase response at the $\pm(2k+1)180^\circ$ lines crosses from below to top and reverse from top to below on a negative intersection as shown in Figure 5.9. In the following these crossings

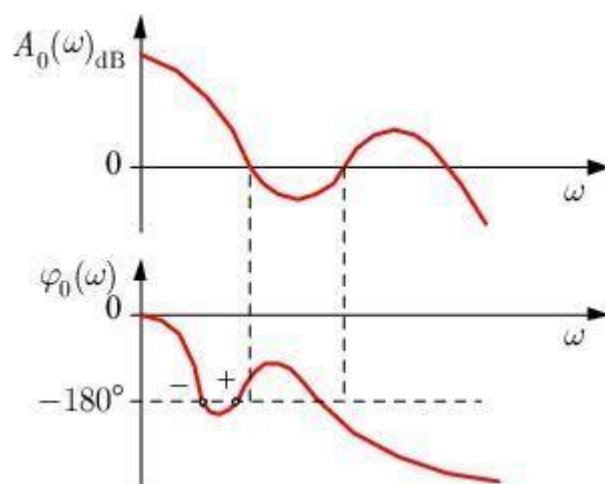


Figure : Frequency response characteristics of $G_0(j\omega) = A_0(\omega) e^{j\varphi_0(\omega)}$ and definition of positive (+) and negative (-) crossings of the phase response $\varphi_0(\omega)$ with the $^\circ$ line



will be defined as positive (+) and negative (-) crossings of the phase response $\varphi_0(\omega)$ over the particular $\pm(2k+1)180^\circ$ lines, where $k = 0, 1, 2, \dots$ may be valid. If the phase response starts at -180° this point is counted as a half crossing with the corresponding sign. Based on the discussions above the Nyquist criterion can be formulated in a form suitable for frequency response characteristics:

The open loop with the transfer function $G_0(s)$ has P poles in the right-half plane, and possibly a single or double pole at $s = 0$. C^+ are the number of positive and C^- of negative crossings of the phase response $\varphi_0(\omega)$ over the $\pm(2k+1)180^\circ$ lines in the frequency range where $A_0(\omega)_{dB} > 0$ is valid. The closed loop is only asymptotically stable, if

$$D^* = C^+ - C^- = \begin{cases} \frac{P}{2} & \text{for } \mu = 0, 1 \\ \frac{P+1}{2} & \text{for } \mu = 2 \end{cases}$$

is valid. For the special case of an open-loop stable system $P = 0$, $\mu = 0$)

$$D^* = C^+ - C^- = 0$$

must be valid.

Table 7.1: Examples of stability analysis using the Nyquist criterion with frequency response characteristics

No.	Bode Diagram	Stability Analysis
1		$\Rightarrow \left. \begin{matrix} S^+ = 1 \\ S^- = 2 \\ D^* = -1 \\ P = 2 \end{matrix} \right\} \Rightarrow D^* \neq P/2: \text{unstable}$
2		$\Rightarrow \left. \begin{matrix} S^+ = 3/2 \\ S^- = 1 \\ D^* = 1/2 \\ P = 0 \end{matrix} \right\} \Rightarrow D^* = \frac{P+1}{2}: \text{stable if 2 poles in the origin}$
3		$\Rightarrow \left. \begin{matrix} S^+ = 0 \\ S^- = 1 \\ D^* = -1 \\ P = 0 \end{matrix} \right\} \Rightarrow D^* \neq P/2: \text{unstable}$
4		$\Rightarrow \left. \begin{matrix} S^+ = 0 \\ S^- = 0 \\ D^* = 0 \\ P = 0 \end{matrix} \right\} \Rightarrow D^* = P/2: \text{stable}$



Finally the 'left-hand rule' will be given using Bode diagrams, because this version is for the most cases sufficient and simple to apply.

The open loop has only poles in the left-half s plane with the exception of possibly one single or one multiple pole at $s = 0$ (P, I or I_2 behaviour). In this case the closed loop is only asymptotically stable, if $G_0(j\omega)$ has a phase of $\varphi_0 > -180^\circ$ for the crossover frequency ω_C at $A_0(\omega_C)_{dB} = 0$.

This stability criterion offers the possibility of a practical assessment of the 'quality of stability' of a control loop. The larger the distance of the locus from the critical point the farther is the closed loop from the stability margin. As a measure of this distance the terms gain margin and phase margin are introduced according to Figure below

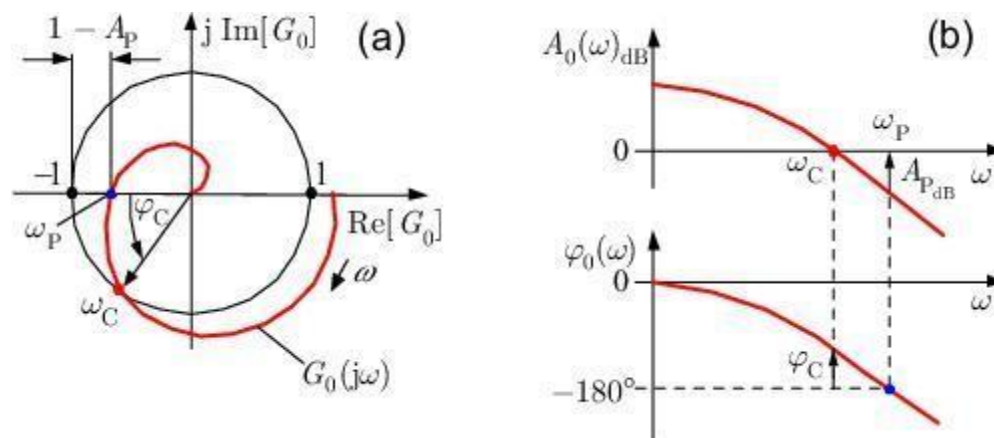


Figure : Phase and gain margin φ_C and A_P or $A_{P_{dB}}$, respectively, in the (a) Nyquist diagram and (b) Bode diagram

Example Problems:

Q1 The polar plot of the open-loop transfer of feedback control system intersects the real axis at -2 . Calculate gain margin (in dB) of the system.

Ans. Given $a = -2$

$$\begin{aligned} \text{Gain margin} &= 20 \log_{10} \left| \frac{1}{a} \right| \\ &= 20 \log_{10} |0.5| \\ \text{gain margin} &= -6.02 \text{ dB.} \end{aligned}$$



Q2. What is the gain margin of a system in decibels if its Nyquist plot cuts the negative real axis at -0.7 ?

Ans.

$$a = -0.7$$

$$\begin{aligned}\text{gain margin} &= -20 \log_{10} \left| \frac{1}{a} \right| \\ &= -20 \log_{10} \left| \frac{1}{-0.7} \right| \\ \text{gain margin} &= -3 \text{ dB.}\end{aligned}$$

Q4. Consider a feed back system with the open-loop transfer function. Given by

$$G(s) = \frac{K}{s(2s+1)}$$

Examine the stability of the closed-loop system. Using Nyquist stability theory

Ans. $G(s)H(s) = \frac{K}{s(2s+1)}$

Here poles are $s = 0, -\frac{1}{2}$. One pole is at origin and one is at $-\frac{1}{2}$. But no pole is at right half of s-plane.

$$\therefore P = 0$$

For stability,

$$N = Z - P$$

$$Z = P + N$$

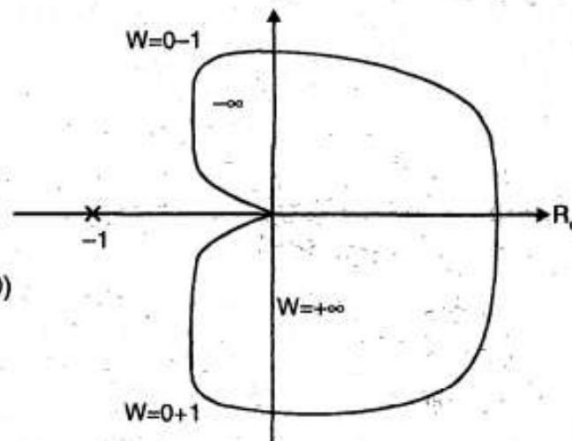
$N \Rightarrow$ Number of clockwise encirclement about $(-1 + j0)$

As there is no encirclement, so $N = 0$

$$Z = 0 + 0$$

$$= 0$$

Thus system is stable.





Q 5. Draw the Nyquist plot for the open loop transfer function given below:

$$G(s) H(s) = \frac{1}{s(1+2s)(1+s)}$$

and obtain the gain margin and phase margin

Ans. Given $G(s) H(s) = \frac{1}{s(1+2s)(1+s)}$

Put $s = j\omega$

$$G(j\omega) H(j\omega) = \frac{1}{j\omega(1+2j\omega)(1+j\omega)}$$

Rationalizing

$$G(j\omega) H(j\omega) = \frac{-3}{(1+4\omega^2)(1+\omega^2)} - j \frac{1-2\omega^2}{\omega(1+4\omega^2)(1+\omega^2)}$$

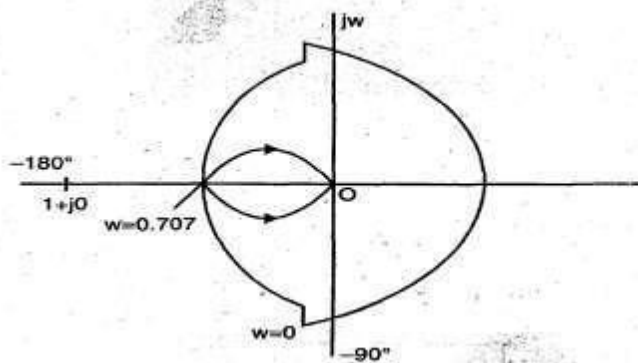
Equating imaginary parts to zero, real axis intersection is at

$$1 - 2\omega^2 = 0$$

$$\omega = 0.707$$

$$|G(j\omega) H(j\omega)|_{\omega=0.707} = 0.66$$

Nyquist plot is as shown :





Q6. Consider a feed back system with the open-loop transfer function. Given by

$$G(s) = \frac{K}{s(2s+1)}$$

Examine the stability of the closed-loop system. Using Nyquist stability theory.

Ans. $G(s)H(s) = \frac{K}{s(2s+1)}$

Here poles are $s = 0, -\frac{1}{2}$. One pole is at origin and one is at $-\frac{1}{2}$. But no pole is at right half of s-plane.

$$\therefore P = 0$$

For stability,

$$N = Z - P$$

$$Z = P + N$$

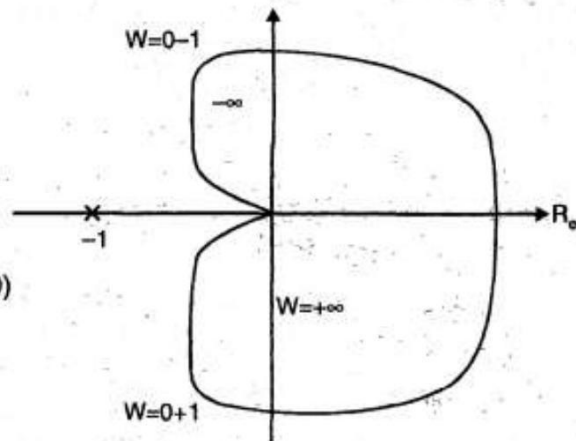
$N \Rightarrow$ Number of clockwise encirclement about $(-1 + j0)$

As there is no encirclement, so $N = 0$

$$Z = 0 + 0$$

$$= 0$$

Thus system is stable.



Q7. Sketch the Nyquist plot for the system with the open loop transfer function

$$\frac{K}{(j\omega + 1)(j\omega + 1.5)(j\omega + 2)}$$

and determine the range of K for which the system is

Ans. Given $G(s)H(s) = \frac{K}{(s+1)(s+1.5)(s+2)}$

Put $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{K}{(s+1)(s+1.5)(s+2)}$$



Rationalizing and separating real and imaginary parts

$$= \frac{(3 - 4.5 \omega^2) K}{(1 + \omega^2)(2.25 + \omega^2)(4 + \omega^2)} - \frac{JK(6.5\omega - \omega^3)}{(1 + \omega^2)(2.25 + \omega^2)(4 + \omega^2)}$$

To get point of intersection on real axis, equate imaginary part to zero.

$$\Rightarrow \frac{K(6.5\omega - \omega^3)}{(1 + \omega^2)(2.25 + \omega^2)(4 + \omega^2)} = 0$$

$$\omega = 2.55 \text{ rad/sec}$$

$$|G(j\omega)|_{\omega=2.25} = -0.24 K$$

Intersection with imaginary axis :

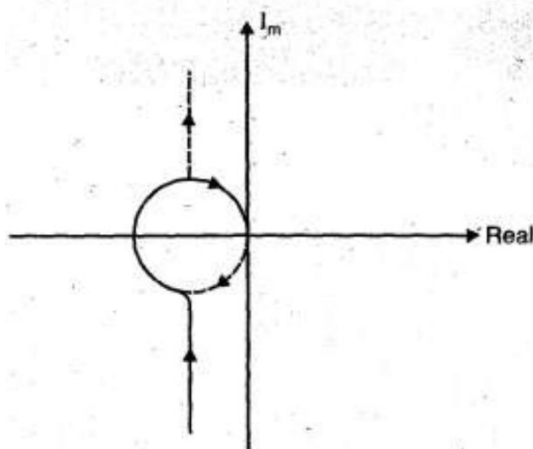
$$\omega = \sqrt{\frac{3}{4.5}} = 0.81$$

$$|G(j\omega)H(j\omega)|_{\omega=2.25} = -0.028 K$$

For stability $-0.028 K < -1$

$$K < 35.03.$$

The plot is as shown below :



Sketch the Nyquist plot for system with

$$G(s)H(s) = \frac{(1 + 0.5s)}{s^2(1 + 0.1s)(1 + 0.02s)}$$

Comment on the stability.



Ans. $G(s) H(s) = \frac{(1+0.5s)}{s^2(1+0.1s)(1+0.02s)}$

Put $s = j\omega$

$$G(j\omega) H(j\omega) = \frac{(1+0.5j\omega)}{(j\omega)^2 (1+0.1j\omega)(1+0.02j\omega)}$$

The mapping for Nyquist contour is as follow.

Along $j\omega$ axis for various values of ω , $G(j\omega) H(j\omega)$ is plotted.

ω	0	∞	0.1	1.0	2.0	4.0	10.0	20.00
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where $|G(j\omega) H(j\omega)| = \frac{\sqrt{1+0.25\omega^2}}{\omega^2 \sqrt{1+0.01\omega^2} \sqrt{1+0.0004\omega^2}}$

$$\angle G(j\omega) H(j\omega) = \tan^{-1} 0.5\omega - 180^\circ - \tan^{-1} 0.1\omega - \tan^{-1} 0.02\omega$$

Point of intersection of $G(j\omega) H(j\omega)$

$$\begin{aligned} \angle G(j\omega) H(j\omega) &= -180^\circ + \tan^{-1} 0.5\omega - 180^\circ - \tan^{-1} 0.1\omega - \tan^{-1} 0.02\omega \\ &= 180^\circ \end{aligned}$$

$$\tan^{-1} 0.5\omega = \tan^{-1} 0.1\omega + \tan^{-1} 0.02\omega$$

$$0.5\omega = \frac{(0.1)(\omega + 0.02\omega)}{1 - 0.002\omega^2}$$

$$(1 - 0.002\omega^2)(0.5\omega) = 0.1\omega + 0.02\omega$$

$$0.5\omega - 0.001\omega^3 = 0.12\omega$$

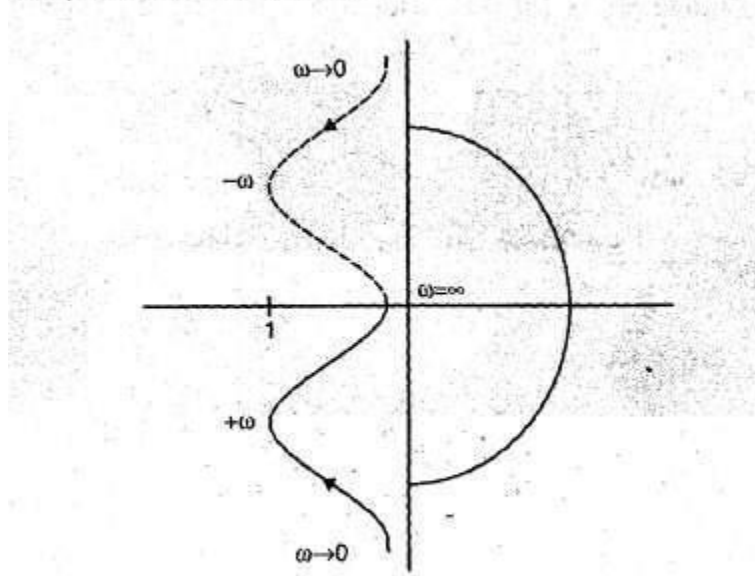
$$0.38 = 0.001\omega^2$$

$$\omega = 19.49 \text{ rad/sec}$$

$$\begin{aligned} |G(j\omega) H(j\omega)|_{\omega=19.49} &= \frac{1 + j0.25 \times 19.49}{(j19.49)^2 (1 + j0.1 \times 19.49)(1 + j0.002 \times 19.49)} \\ &= \frac{1 + 4.8725j}{(19.49j)^2 (1 + 1.949j)(1 + 0.0389j)} \end{aligned}$$



The plot is as shown below :



There is no encirclement of $1 + j0$, hence system is stable.

**Q 9. How is it possible to make assessment of relative stability using Nyquist criterion?
Construct Nyquist plot for the system whose open loop transfer function is**

$$G(s) H(s) = \frac{K(1+s)^2}{s^3}$$

Find the range of K for stability.

Ans.

- Nyquist criterion can be used to make assessment of relative
- Using the characteristic equation the Nyquist plot is drawn. A feedback system is stable if and only if, the i.e. contour in the $G(s)$ plane does not encircle the $(-1, 0)$ point when the number of poles of $G(s)$ in the right hand s plane is
- If $G(s)$ has P poles in the right hand plane, then the number of anticlockwise encirclements of the $(-1, 0)$ point must be equal to P for a stable system, $N = -P$

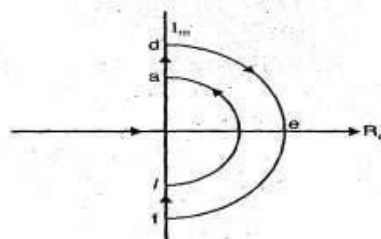
where N = No of clockwise encirclements about $(-1, 0)$ point in $C(s)$ plane
 P = No of poles $G(s)$ in RHP

Now given $G(s) H(s) = \frac{K(1+s)^2}{s^3}$

No. of poles at RHS of s -plane $P = 0$

For stability $N = 0$

Nyquist path is shown :



For path a - d, put $s = j\omega$, $0 < \omega < \infty$

$$\omega = 0, G(j\omega) H(j\omega) = \infty \angle -270^\circ$$

$$\omega = \infty, G(j\omega) H(j\omega) = 0 \angle -90^\circ$$



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Mechanical

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Notes

**Control Engineering-
15ME73**

OUTCOMES:

At the end of the unit, the students are able to:

- ☐ To Determine Gain & Phase Margins Medium effort.
- ☐ Applications of the frequency response to analysis of system stability (the Nyquist criterion), relating the frequency response to transient performance specifications.
- ☐ To find number of RHP poles of $T(s)$, the closed-loop transfer function.

SELF-TEST QUESTIONS:

1. Apply Nyquist stability criterion for the system with transfer function

$$G(S)H(S) = \frac{K}{S(S+2)(S+4)}$$

find the stability.



Module-4

FREQUENCY RESPONSE ANALYSIS USING BODE PLOTS

LESSON STRUCTURE

Bode attenuation diagrams

Stability analysis using Bode plots

OBJECTIVES:

- To demonstrate frequency response and to determine stability of control system applying using Bode plot.
- To demonstrate to plot graph of amplitude plot, usually in the log-log scale and a phase plot, which is usually a linear-log plot.

Bode attenuation diagrams

If the absolute value $A(\omega)$ and the phase $\varphi(\omega)$ of the frequency response $G(j\omega) = A(\omega) e^{j\varphi(\omega)}$

are separately plotted over the frequency ω , one obtains the

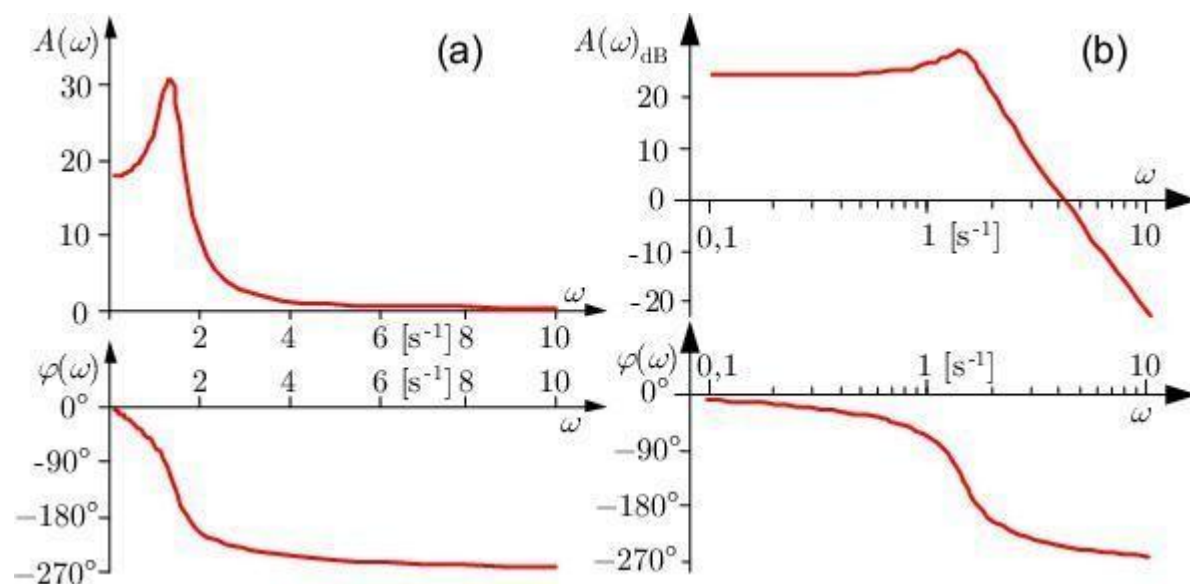


Figure 6.1: Plot of a frequency response: (a) linear, (b) logarithmic presentation (ω on a logarithmic scale) (Bode plot)

amplitude response and the phase response. Both together are the frequency response

characteristics. $A(\omega)$ and ω are normally drawn with a logarithm and $\varphi(\omega)$ with a linear scale.

This representation is called a *Bode diagram* or *Bode plot*. Usually $A(\omega)$ will be specified in decibels [dB] By definition this is

$$A(\omega)_{dB} = 20 \log_{10} A(\omega) \quad [dB]$$

The logarithmic representation of the amplitude response $A(\omega)_{dB}$ has consequently a linear scale in this diagram and is called the *magnitude*.



Stability analysis using Bode plots:

- The magnitude and phase relationship between sinusoidal input and steady state output of a system is known as frequency
- The polar plot of a sinusoidal transfer function $G(j\omega)$ is plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω varied from zero to infinity.
- The phase margin is that amount, of additional phase lag at the gain crossover frequency required to bring the system to the verge of
- The gain margin is the reciprocal of the magnitude $|G(j\omega)|$ at the frequency at which the phase angle as
- The inverse polar plot at $G(j\omega)$ is a graph of $1/G(j\omega)$ as a function of
- Bode plot is a graphical representation of the transfer function for determining the stability of control
- Bode plot is a combination of two plot - magnitude plot and phase
- The transfer function having no poles and zeros in the right -half s-plane are called minimum phase transfer
- System with minimum phase transfer function are called minimum phase systems.
- The transfer function having poles and zeros in the right half s-plane are called non-minimum phase transfer functions systems with non-minimum phase transfer function. are called non-minimum phase
- In bode plot the relative stability of the system is determined from the gain margin and phase margin. .
- If gain cross frequency is less than phase cross over frequency then gain margin and phase margin both are positive and system is
- If gain cross over frequency is greater than the phase crossover frequency then both gain margin and 'phase margin are
- If gain cross over frequency is equal to the phase cross over frequency the gain margin and phase margin are zero and system is marginally
- The maximum value of magnitude is known as resonant
- The magnitude of resonant peak gives the information about the relative stability of the
- The frequency at which magnitude has maximum value is known as resonant frequency.
- Bandwidth is defined as the range of frequencies in which the magnitude of closed loop does not drop —3 db.

**Example Problems:**

Q1. Sketch the Bode Plot for the transfer function given by,

$$G(s) H(s) = \frac{2(s+0.25)}{s^2(s+1)(s+0.5)}$$

and from Plot find (a) Phase and Gain cross over frequencies (b) Gain Margin and Phase Margin. Is this System Stable?

Ans. Given $G(s) H(s) = \frac{2(s+0.25)}{s^2(s+1)(s+0.5)}$

$$= \frac{\frac{2 \times 0.25}{0.5} \left[\frac{s}{0.25} + 1 \right]}{s^2(s+1) \left[\frac{s}{0.5} + 1 \right]}$$

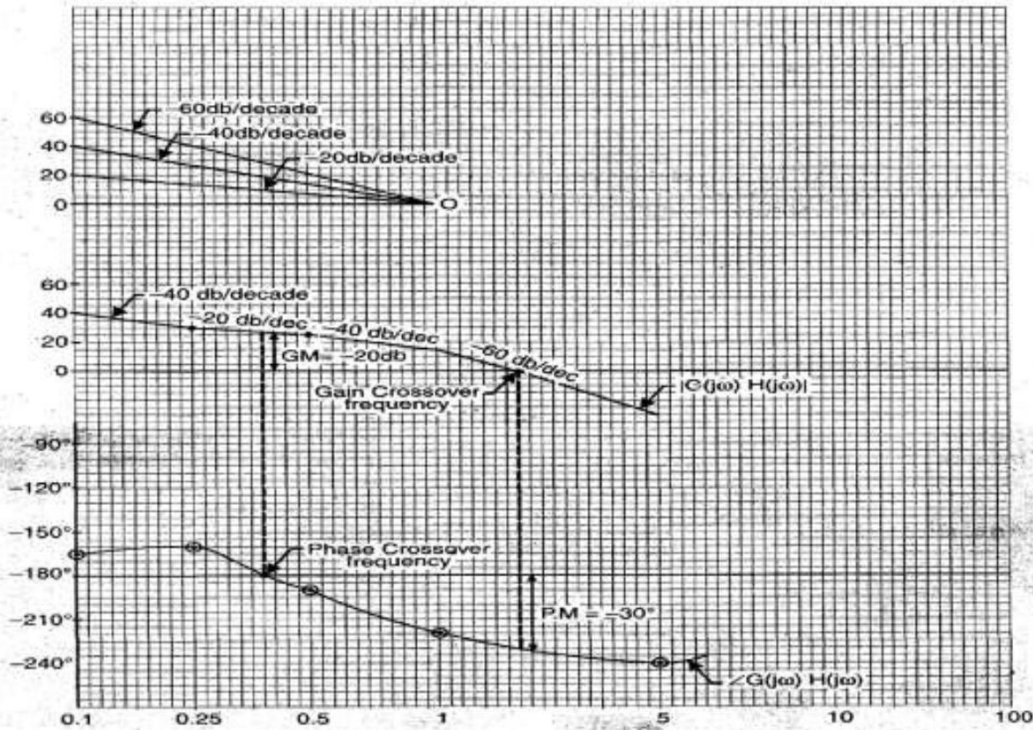
$$= \frac{1(4s+1)}{s^2(s+1)(2s+1)}$$

Put $s = j\omega$

$$G(j\omega) H(j\omega) = \frac{(j4\omega+1)}{(j\omega^2)(j\omega+1)(2j\omega+1)}$$



This is type 2 system, hence initial slope of bode plot = -40 dB/decade and the plot intersects 0 dB axis at $\omega = \sqrt{K} = \sqrt{1} = 1$ rad/sec. The corner frequencies are :



$$\omega = \frac{1}{4} = 0.25 \text{ rad/sec}$$

$$\omega = \frac{1}{2} = 0.5 \text{ rad/sec}$$

$$\omega = 1 \text{ rad/sec.}$$

Frequency range is considered from $\omega = 0.1$ rad/sec to $\omega = 10$ rad/sec.

The plot is as shown.

As initial slope of plot is -40 dB/dec and corner frequency is 0.25 rad/sec. The plot after $\omega = 0.25$ has slope = -20 dB/dec.

After $\omega = 0.5$, slope is -40 dB/decade

After $\omega = 1$, slope is -60 dB/dec.

Phase Angle :

$$\angle G(j\omega) H(j\omega) = \tan^{-1}(4\omega) - 180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega.$$

The phase angle for frequency range considered are calculated as :

ω	0.1	0.25	0.5	1	5
$\angle G(j\omega) H(j\omega)$	-175.2	-175.2	-188	-212.4	-225



The gain crosses 0db axis at $\omega_c = 1.24$ rad/sec, the gain crossover frequency is $\omega_c =$ rad/sec.

The phase crosses -180° line at $\omega_c = 0.4$ rad/sec, therefore phase crossover frequency is $\omega_c = 0.4$ rad/sec.

At phase cross over the gain is 20 dB, therefore gain margin is -20 dB.

At gain crossover the phase angle is -215° , the phase margin is $180^\circ + (-215^\circ) = -35^\circ$.

As both gain and phase margins are negative, the system is unstable.

Q3. Sketch the bode plot for the transfer function given by

$$G(s) = \frac{23.7 (1+j\omega) (1+0.2j\omega)}{(j\omega) (1+3j\omega) (1+0.5j\omega) (1+0.1j\omega)}$$

and from plot find gain margin and phase margin.

Ans.

On 0)-axis mark the point at 23.7 rad/sec. since in denominator (jw) term is having power one, from 23.7 draw a line of slope -20 db/decade to meet y-axis. This will be the starting point.

Step 1.

From the starting point to I corner frequency (0.33) the slope of the line is -20 db/decade.

From I corner frequency (0.33) to second corner frequency (1) the slope of the line will be $-20 \div (-20) = -40$ db/decade.

From II corner frequency to IV corner frequency (2) the slope of the line be $-40 + (\div 20) = -20$ db/decade.

From III corner frequency to IV corner frequency, the slope of line will be $-20 + (-20) = -40$ db/decade.

From IV corner frequency (5) to V corner frequency the slope will be $-40 \div (+20) = -20$ db/decade.

After V corner frequency, the slope will be $(-20) \div (-20) = -40$ db/decade.

Step 2.

Draw the phase plot.

Step 3.

From graph

Phase margin = $+34^\circ$

Gain margin = infinity



$$G(j\omega) H(j\omega) = \frac{k}{j\omega (j0.1\omega + 1) (j0.05\omega + 1)}$$

Ans. Advantages of Bode Plot :

Please refer to Q. No. 1 (i) of May 2009.

$$\text{As } G(j\omega) H(j\omega) = \frac{k}{j\omega (j0.1\omega + 1) (j0.05\omega + 1)}$$

Corner frequencies are

$$\omega_1 = \frac{1}{0.1} = 10 \text{ rad/sec}$$

$$\omega_2 = \frac{1}{0.05} = 20 \text{ rad/sec}$$

Draw magnitude plot without K.

For phase plot

ω	Arg $j\omega$ ϕ_1	Arg $(1 + j0.1\omega)$ ϕ_2	Arg $(1 + j0.05\omega)$ ϕ_3	Resultant $\phi_1 + \phi_2 + \phi_3$
4	-90°	-21.8°	-11.3°	-123.1°
6	-90°	-30.96°	-16.69°	-137.65°
8	-90°	-38.56°	-21.8°	-150.36°
10	-90°	-45°	-26.56°	-161.56°
12	-90°	-50.19°	-30.96°	-171.46°
14	-90°	-54.46°	-35°	-179.48°
16	-90°	-60.9°	-42°	-192.9°
20	-90°	-63.43°	-45°	-198.43°

ω	$-\tan^{-1} j\omega$	$-\tan^{-1} 3\omega$	$-\tan^{-1} 0.5\omega$	$-\tan^{-1} 0.1\omega$	$\tan^{-1} \omega$	$\tan^{-1} 2\omega$	Resultant
0.1	-90°	-16.7°	-2.86°	-0.57°	+5.71°	1.14°	-103°
0.2	-90°	-31°	-5.71°	-1.14°	+11.3°	2.3°	-114.25°
0.5	-90°	-56.3°	-14.03°	-2.86°	+26.56°	5.71°	-130.92°
0.8	-90°	-67.4°	-21.8°	-4.57°	+38.65°	9.09°	-136.03°
1.0	-90°	-71.56°	-26.56°	-5.71°	+45°	11.3°	-137.5°
2.0	-90°	-80.54°	-45°	-11.3°	+63.43°	21.8°	-141.61°
5.0	-90°	-86.18°	-68.19°	-26.56°	+78.7°	45°	-147.23°
8.0	-90°	-87.61°	-76°	-38.65°	+82.87°	58°	-151.39°
10.0	-90°	-88°	-78.7°	-45°	+84.3°	63.4°	-154.0°
20.0	-90°	-89°	-84.3°	-63.43°	+87.13°	76°	-163.6°

OUTCOMES:

At the end of the unit, the students are able to:



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Mechanical

Academic

Notes

**Control Engineering-
15ME73**

- ☐ Determine stability of control system applying Nyquist stability criterion and using Bode plot.
- ☐ Plot a graph of amplitude plot, usually in the log-log scale and a phase plot, which is usually a linear-log plot.

SELF-TEST QUESTIONS:

1. The open loop transfer function of a system is given by $G(S)H(S) = \frac{10(S+10)}{S(S+2)(S+5)}$.

Draw Bode diagram, Find Gain cross over frequency (GCF), Phase cross over frequency (PCF), Gain margin (GM), Phase margin (PM). Find stability of the system.

2. The open loop transfer function of a system is given by

$$G(S)H(S) = \frac{50K}{S(S+10)(S+6)(S+1)}$$

Draw Bode diagram, Find Gain cross over frequency (GCF), Phase cross over frequency (PCF), Gain margin (GM), Phase margin (PM). Find the value of **K** to have GM=10 decibels.



ROOT-LOCUS TECHNIQUES

LESSON STRUCTURE

Definition of root loci

Analysis using root locus plots

General rules for constructing root loci

OBJECTIVES:

- To teach the relationship of the poles (and zeros) of a system's transfer function to its response, including classical time-domain transient performance specifications such as overshoot, rise-time, and settling-time.
- To demonstrate the construction of root loci from open loop transfer functions of control systems and Analyze the behaviour of roots with system gain.

Definition of root loci

The root locus of a feedback system is the graphical representation in the complex s -plane of the possible locations of its closed-loop poles for varying values of a certain system parameter. The points that are part of the root locus satisfy the angle condition. The value of the parameter for a certain point of the root locus can be obtained using the magnitude condition.

In **root locus technique in control system** we will evaluate the position of the roots, their locus of movement and associated information. These information will be used to comment upon the system performance.

Analysis using root locus plots.

A designer can determine whether his design for a control system meets the specifications if he knows the desired time response of the controlled variable. By deriving the differential equations for the control system and solving them, an accurate solution of the system's performance can be obtained, but this approach is not feasible for other than simple systems. It is not easy to determine from this solution just what parameters in the system should be changed to improve the response. A designer wishes to be able to predict the performance by an analysis that does not require the actual solution of the differential equations.

The first thing that a designer wants to know about a given system is whether or not it is stable. This can be determined by examining the roots obtained from the characteristic equation

$$1 + G_0(s) = 0 \quad (5.1)$$



of the closed loop. The work involved in determining the roots of this equation can be avoided by applying the Hurwitz or Routh criterion. Determining in this way whether the system is stable or unstable does not satisfy the designer, because it does not indicate the degree of stability of the system, i.e., the amount of overshoot and the settling time of the controlled variable for a step input. Not only must the system be stable, but the overshoot must be maintained within prescribed bounds and transients must die out in a sufficiently short time.

The root-locus method described in this section not only indicates whether a system is stable or unstable but, for a stable system, also shows the degree of stability. The root locus is a plot of the roots of the characteristic equation of the closed loop as a function of the gain. This graphical approach yields a clear indication of the effect of gain adjustment with relatively small effort.

With this method one determines the closed-loop poles in the s plane - these are the roots of Eq.(5.1) - by using the known distribution of the poles and zeros of the open-loop transfer

function $G_o(s)$. If for instance a parameter is varied, the roots of the characteristic equation will move on certain curves in the s plane as shown by the example in Figure 5.1. On these curves lie all

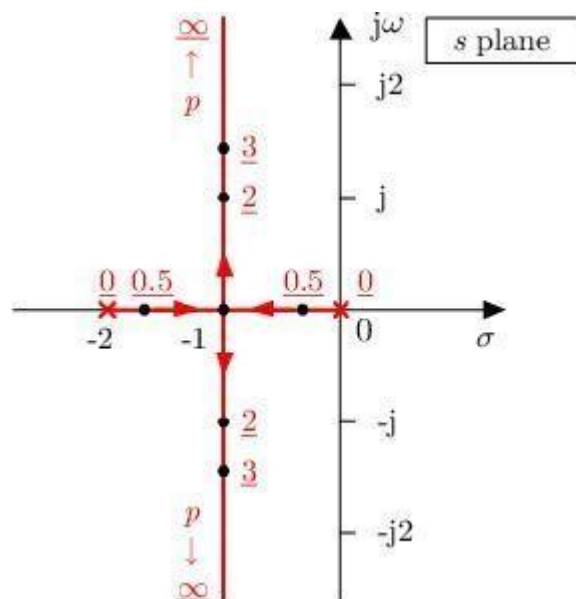


Figure 5.1: Plot of all roots of the characteristic equation $s^2 + 2s + p = 0$ for $0 \leq p < \infty$.

Values of p are red and underlined.

possible roots of the characteristic equation for all values of the varied parameter from zero to infinity. These curves are defined as the *root-locus plot* of the closed loop. Once this plot is obtained, the roots that best fit the system performance specifications can be selected. Corresponding to the selected roots there is a required value of the parameter which can be determined from the plot. When the roots have been selected, the time response can be obtained. Since the process of finding the root locus by calculating the roots for various values of a parameter becomes tedious, a simpler method of obtaining the root locus is desired. The graphical method for determining the root-locus plot is shown in the following.



An open-loop transfer function with k poles at the origin of the s -plane is often described by

$$G_0(s) = \frac{K_0}{s^k} \frac{1 + \beta_1 s + \dots + \beta_m s^m}{1 + \alpha_1 s + \dots + \alpha_{n-k} s^{n-k}} \quad m \leq n, \quad (5.2)$$

where K_0 is the gain of the open loop. In order to represent this transfer function in terms of the open-loop poles and zeros it is rewritten as

$$G_0(s) = k_0 \frac{\prod_{\mu=1}^m (s - s_{Z_\mu})}{\prod_{\nu=1}^n (s - s_{P_\nu})} = k_0 G(s) \quad (5.3)$$

or

$$G_0(s) = k_0 \frac{\prod_{\mu=1}^m (-s_{Z_\mu})}{\prod_{\nu=1}^{n-k} (-s_{P_\nu})} \frac{1}{s^k} \frac{\prod_{\mu=1}^m \left(1 + \frac{s}{-s_{Z_\mu}}\right)}{\prod_{\nu=1}^{n-k} \left(1 + \frac{s}{-s_{P_\nu}}\right)} \quad (5.4)$$

with $k_0 > 0$ and $s_{Z_\mu} \neq s_{P_\nu}$. The relationship between the factor k_0 and the open-loop gain K_0 is

$$K_0 = k_0 \frac{\prod_{\mu=1}^m (-s_{Z_\mu})}{\prod_{\nu=1}^{n-k} (-s_{P_\nu})} \frac{1}{s^k} \quad (5.5)$$

The characteristic equation of the closed loop using Eq. (5.3) is

$$1 + k_0 G(s) = 0 \quad (5.6)$$

or

$$G(s) = -\frac{1}{k_0} \quad (5.7)$$

All complex numbers $s_i = s_i(k_0)$, which fulfil this condition for $0 \leq k_0 \leq \infty$, represent the root locus.



From the above it can be concluded that the magnitude of $k_0 G(s)$ must always be unity and its phase angle must be an odd multiple of π . Consequently, the following two conditions are formalised for the root locus for all positive values of k_0 from zero to infinity:

a)

Magnitude condition:

$$|G(s)| = \frac{1}{k_0} \quad (5.8)$$

b)

Angle condition

$$\varphi(s) = \arg G(s) = \pm 180^\circ (2k + 1) \quad \text{for} \quad \begin{matrix} k = 0, 1, 2, \dots \\ k_0 \geq 0 \end{matrix} \quad (5.9)$$



In a similar manner, the conditions for negative values of k_0 ($-\infty \leq k_0 < 0$) can be determined. The magnitude condition is the same, but the angle must satisfy the

c)

Angle condition

$$\angle G(s) = \arg G(s) = \pm k 360^\circ \quad \text{for } \begin{matrix} k = 0, 1, 2, \dots \\ k_0 < 0 \end{matrix} \quad (5.10)$$

Apparently the angle condition is independent of k_0 . All points of the s -plane that fulfil the angle condition are the loci of the poles of the closed loop by varying k_0 . The calibration of the curves by the values of k_0 is obtained by the magnitude condition according to Eq. (5.8). Based upon this interpretation of the conditions the root locus can be constructed in a graphical/numerical way.

Once the open-loop transfer function $G_0(s)$ has been determined and put into the proper form, the poles and zeros of this function are plotted in the s -plane.

- The plot of the locus of the closed loop poles as a function of the open loop gain K , when K is varied from 0 to $+\infty$
- When system gain K is varied from 0 to $+\infty$, the locus is called direct root
- When system gain K is varied from $-\infty$ to 0, the locus is called as inverse root locus.
- The root locus is always symmetrical about the real axis i.e. x -axis.
- The number of separate branches of the root locus equals either the number of open loop poles or number of open-loop zeros whichever is less
- A section of root locus lies on the real axis if the total number of open-loop poles and zeros to the right of the section is odd
- If the root locus intersects the imaginary axis then the point of intersection are conjugate. From the open loop complex pole the root locus departs making an angle with the horizontal line.

-
- The root locus starts from open-loop
 - The root locus terminates either on open loop zero or
 - infinity. The number of branches of root locus

N if $P > Z$
and M if $P < Z$

where $N \rightarrow$ No. of poles $_P$

$M \rightarrow$ No. of zeros $_Z$

- Centroid is the centre of asymptotes. It is given by

$$\sigma_c = \frac{\sum P - \sum Z}{N - M}$$



Angle of asymptotes is denoted by

$$\phi = \frac{(2K+1)}{N-M} \times 180^\circ$$

☞ Breakaway/saddle point is the point at which the root locus comes out of the real axis.
To find breakaway point

$$\text{Put } \frac{dK}{ds} = 0$$

- Angle of departure is tangent to root locus at complex

$$\phi_d = 180^\circ - (\phi_p - \phi_z)$$

Angle of arrival is tangent to the root locus at the complex zero.

$$\phi_a = 180^\circ - (\phi_z - \phi_p)$$

Where ϕ_z = sum of all angles subtended by remaining zeros,

ϕ_p = sum of all angles subtended by remaining poles.

Based on the pole and zero distributions of an open-loop system the stability of the closed-loop system can be discussed as a function of one scalar parameter. The root-locus method shown in this module is a technique that can be used as a tool to design control systems. The basic ideas and its relevancy to control system design are introduced and illustrated. Ten general rules for constructing root loci for positive and negative gain are shortly presented such that they can be easily applied. This is demonstrated by some discussed examples, by a table with sixteen examples and by a comprehensive design of a closed-loop system of higher order.



Example Problems:

Consider the example

$$G_0(s) = \frac{K_0}{s(s+2)} = \frac{k_0}{(s-s_{P_1})(s-s_{P_2})}$$

with $s_{P_1} = 0$, $s_{P_2} = -2$ and $k_0 = K_0$. The poles of the closed-loop transfer function

$$G_W(s) = \frac{K_0}{s^2 + 2s + K_0}$$

are the roots s_1 and s_2 of the characteristic equation

$$P(s) = s^2 + 2s + K_0 = 0$$

and are given by

$$s_{1,2} = -1 \pm \sqrt{1 - K_0}$$

As $s_1 = s_{P_1} = 0$ and $s_2 = s_{P_2} = -2$ it can be seen that for $K_0 = 0$ the poles of the closed loop transfer function are identical with those of the open-loop transfer function $G_0(s)$. For other values K_0 the following two cases are considered:

a)

$K_0 \leq 1$: Both roots s_1 and s_2 are real and lie on the real axis in the range of $-2 \leq \sigma \leq -1$ and $-1 \leq \sigma \leq 0$;

b)

$K_0 > 1$: The roots s_1 and s_2 are conjugate complex with the real part $\text{Re } s_{1,2} = -1$, which does not depend on K_0 , and the imaginary part $s_{1,2} = \pm \sqrt{K_0 - 1}$.

The curve has two branches as shown in Figure 6.2.

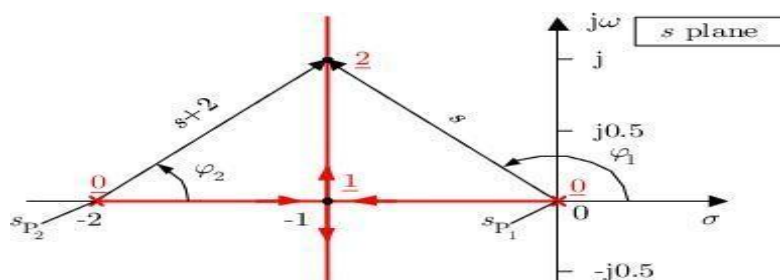




Figure 5.2: Root locus of a simple second-order system

At $(s_{P_1} + s_{P_2})/2 = -1$ is the *breakaway point* of the two branches. Checking the angle condition the condition

$$\varphi(s) = \arg\{G(s)\} = \arg\left\{\frac{1}{s(s+2)}\right\} = -\arg s - \arg(s+2) \stackrel{!}{=} \pm 180^\circ(2k+1)$$

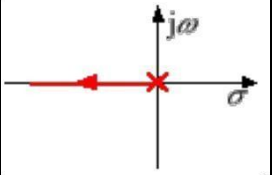
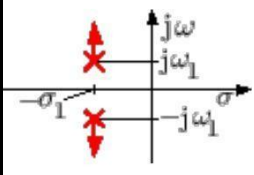
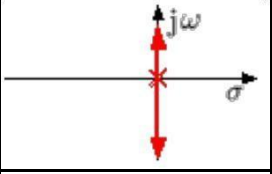
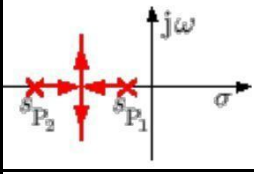
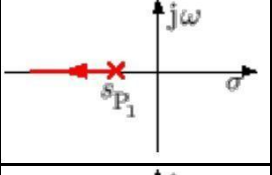
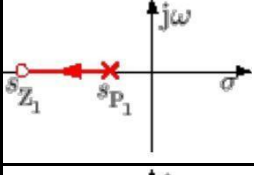
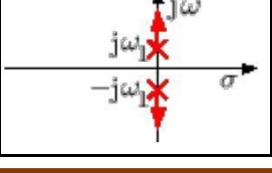
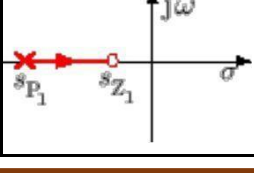
must be valid. The complex numbers s and $(s+2)$ have the angles φ_1 and φ_2 and the magnitudes $|s|$ and $|s+2|$. The triangle $(-2, 0, -1+j)$ in Figure 6.2 yields the angle condition. Evaluating the magnitude condition according to Eq. (6.8)

$$|G(s)| = \left|\frac{1}{s(s+2)}\right| = \frac{1}{K_0}$$

one obtains the value K_0 on the root locus. E.g. for $s = -1+j$ the gain of the open loop is $K_0 = |s(s+2)|_{s=-1+j} = 2$.

The value of K_0 at the breakaway point $s_B = -1$ is $K_0 = |-1(-1+2)| = 1$.

Table 5.1 shows further examples of some 1st- and 2nd-order systems.

Table 5.1: Root loci of 1st- and 2nd-order systems			
$G_0(s)$	root locus	$G_0(s)$	root locus
$\frac{k_0}{s}$		$\frac{k_0}{(s+\sigma_1)^2 + \omega_1^2}$	
$\frac{k_0}{s^2}$		$\frac{k_0}{(s-s_{P_1})(s-s_{P_2})}$	
$\frac{k_0}{s-s_{P_1}}$		$\frac{k_0(s-s_{Z_1})}{(s-s_{P_1})} \quad s_{Z_1} > s_{P_1} $	
$\frac{k_0}{s^2 + \omega_1^2}$		$\frac{k_0(s-s_{Z_1})}{(s-s_{P_1})} \quad s_{Z_1} < s_{P_1} $	



5.3. General rules for constructing root loci

To facilitate the application of the root-locus method for systems of higher order than 2nd, rules can be established. These rules are based upon the interpretation of the angle condition and the analysis of the characteristic equation. The rules presented aid in obtaining the root locus by expediting the manual plotting of the locus. But for automatic plotting using a computer these rules provide checkpoints to ensure that the solution is correct.

Though the angle and magnitude conditions can also be applied to systems having dead time, in the following we restrict to the case of the open-loop rational transfer functions according to Eq. (5.3)

$$G_0(s) = k_0 \frac{(s - s_{Z_1})(s - s_{Z_2}) \dots (s - s_{Z_m})}{(s - s_{P_1})(s - s_{P_2}) \dots (s - s_{P_n})}, \quad k_0 \geq 0 \quad (5.11)$$

or

$$G_0(s) = k_0 \frac{b_0 + b_1 s + \dots + b_{m-1} s^{m-1} + s^m}{a_0 + a_1 s + \dots + a_{n-1} s^{n-1} + s^n} = k_0 \frac{N_0(s)}{D_0(s)} \quad (5.12)$$

As this transfer function can be written in terms of poles and zeros s_{P_ν} and s_{Z_μ} ($\nu = 1, 2, \dots, n$; $\mu = 1, 2, \dots, m$) $G_0(s)$ can be represented by their magnitudes and angles

$$G_0(s) = k_0 \frac{|s - s_{Z_1}| e^{j\varphi_{Z_1}} |s - s_{Z_2}| e^{j\varphi_{Z_2}} \dots |s - s_{Z_m}| e^{j\varphi_{Z_m}}}{|s - s_{P_1}| e^{j\varphi_{P_1}} |s - s_{P_2}| e^{j\varphi_{P_2}} \dots |s - s_{P_n}| e^{j\varphi_{P_n}}}$$

or

$$G_0(s) = k_0 \frac{\prod_{\mu=1}^m |s - s_{Z_\mu}|}{\prod_{\nu=1}^n |s - s_{P_\nu}|} e^{j \left(\sum_{\mu=1}^m \varphi_{Z_\mu} - \sum_{\nu=1}^n \varphi_{P_\nu} \right)} \quad (5.13)$$

From Eq. (5.8) the *magnitude condition*

$$\frac{\prod_{\mu=1}^m |s - s_{Z_\mu}|}{\prod_{\nu=1}^n |s - s_{P_\nu}|} = \frac{1}{k_0} \quad (5.14)$$

and from Eq. (5.9) the *angle condition*

$$\varphi(s) = \sum_{\mu=1}^m \varphi_{Z_\mu} - \sum_{\nu=1}^n \varphi_{P_\nu} = \pm 180^\circ (2k + 1) \quad k = 0, 1, 2, \dots \quad (5.15) \text{ for}$$



follows. Here φ_{Z_μ} and φ_{P_ν} denote the angles of the complex values $(s - s_{Z_\mu})$ and $(s - s_{P_\nu})$, respectively. All angles are considered positive, measured in the counterclockwise sense. If for each point the sum of these angles in the plane is calculated, just those particular points that fulfil the condition in Eq. (5.15) are points on the root locus. This principle of constructing a root-locus curve - as shown in Figure 5.3 - is mostly used for automatic root-locus plotting.

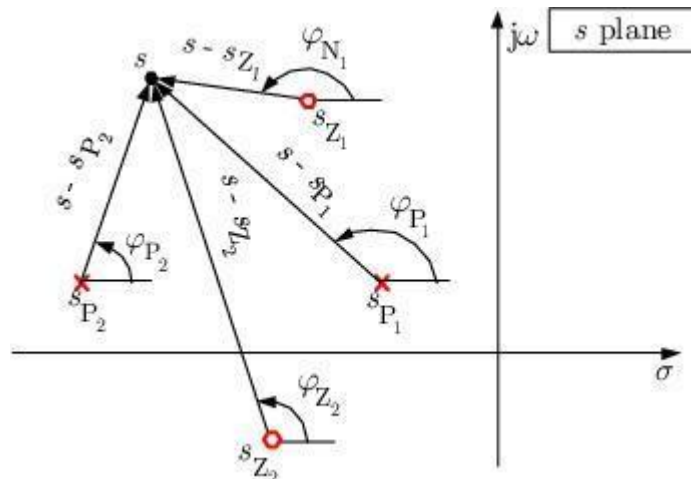


Figure 5.3: Pole-zero diagram for construction of the root locus

In the following the most important rules for the construction of root loci for $k_0 > 0$ are listed:

Rule 1 Symmetry

As all roots are either real or complex conjugate pairs so that the root locus is symmetrical to the real axis.

Rule 2 Number of branches

The number of branches of the root locus is equal to the number of poles n of the open-loop transfer function.

Rule 3 Locus start and end points

The locus starting points ($k_0 = 0$) are at the open-loop poles and the locus ending points ($k_0 = \infty$) are at the open-loop zeros. $(n - m)$ branches end at infinity. The number of starting branches from a pole and ending branches at a zero is equal to the multiplicity of the poles and zeros, respectively. A point at infinity is considered as an equivalent zero of multiplicity equal to $n - m$.

Rule 4 Real axis locus

If the total number of poles and zeros to the right of a point on the real axis is odd, this point lies on the locus.

Rule 5 Asymptotes

There are $n - m$ asymptotes of the root locus with a slope of

$$\alpha_k = \arg s = \frac{\pm 180^\circ (2k + 1)}{n - m} \quad (5.16)$$



For $(n - m) = 1, 2, 3$ and 4 one obtains the asymptote configurations as shown in Figure 5.4.

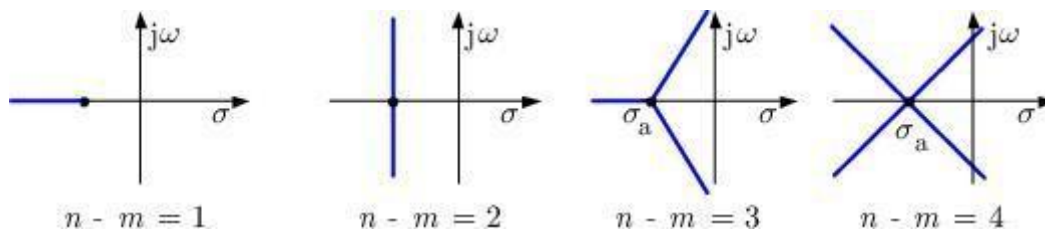


Figure 5.4: Asymptote configurations of the root locus

Rule 6 Real axis intercept of the asymptotes

The real axis crossing $(\sigma_a, j0)$ of the asymptotes is at

$$\sigma_a = \frac{1}{n - m} \left\{ \sum_{\nu=1}^n \text{Re } s_{P_\nu} - \sum_{\mu=1}^m \text{Re } s_{Z_\mu} \right\} \quad (5.17)$$

Rule 7 Breakaway and break-in points on the real axis

At least one breakaway or break-in point $(\sigma_B, j0)$ exists if a branch of the root locus is on the real axis between two poles or zeros, respectively. Conditions to find such real points are based on the fact that they represent multiple real roots. In addition to the characteristic equation (6.1) for multiple roots the condition

$$\frac{d}{ds} [1 + G_0(s)] = \frac{d}{ds} G_0(s) = 0 \quad (5.18)$$

must be fulfilled, which is equivalent to

$$\sum_{\nu=1}^n \frac{1}{s - s_{P_\nu}} = \sum_{\mu=1}^m \frac{1}{s - s_{Z_\mu}} \quad (5.19)$$

for $s = \sigma_B$. If there are no poles or zeros, the corresponding sum is zero.

Rule 8 Complex pole/zero angle of departure/entry

The angle of departure of pairs of poles with multiplicity r_{P_ϱ} is

$$\varphi_{P_\varrho, D} = \frac{1}{r_{P_\varrho}} \left\{ - \sum_{\substack{\nu=1 \\ \nu \neq \varrho}}^n \varphi_{P_\nu} + \sum_{\mu=1}^m \varphi_{Z_\mu} \pm 180^\circ (2k + 1) \right\} \quad (6.20)$$



$$\varphi_{Z_p, E} = \frac{1}{r_{Z_p}} \left\{ - \sum_{\substack{\mu=1 \\ \mu \neq p}}^m \varphi_{Z_\mu} + \sum_{\nu=1}^n \varphi_{P_\nu} \pm 180^\circ (2k+1) \right\} \quad (5.21)$$

Rule 9 Root-locus calibration

The labels of the values of k_0 can be determined by using

$$k_0 = \frac{\prod_{\nu=1}^n |s - s_{P_\nu}|}{\prod_{\mu=1}^m |s - s_{Z_\mu}|} \quad (5.22)$$

For $m = 0$ the denominator is equal to one.

Rule 10 Asymptotic stability

The closed loop system is asymptotically stable for all values of k_0 for which the locus lies in the left-half s plane. From the imaginary-axis crossing points the critical values k_{0crit} can be determined.

The rules shown above are for positive values of k_0 . According to the angle condition of Eq. (5.10) for negative values of k_0 some rules have to be modified. In the following these rules are numbered as above but labelled by a *.

Rule 3* Locus start and end points

The locus starting points ($k_0 = 0$) are at the open-loop poles and the locus ending points ($k_0 = -\infty$) are at the open-loop zeros. $(n-m)$ branches end at infinity. The number of starting branches from a pole and ending branches at a zero is equal to the multiplicity of the poles and zeros, respectively. A point at infinity is considered as an equivalent zero of multiplicity equal to $n-m$.

Rule 4* Real axis locus

If the total number of poles and zeros to the right of a point on the real axis is even including zero, this point lies on the locus.

Rule 5* Asymptotes

There are $n-m$ asymptotes of the root locus with a slope of

$$\alpha_k = \arg s = \frac{\pm 360^\circ k}{n-m} \quad (5.23)$$

**Rule 8* Complex pole/zero angle of departure/entry**

The angle of departure of pairs of poles with multiplicity $r_{p\varrho}$ is

$$\varphi_{p\varrho,D} = \frac{1}{r_{p\varrho}} \left\{ - \sum_{\substack{\nu=1 \\ \nu \neq \varrho}}^n \varphi_{p\nu} + \sum_{\mu=1}^m \varphi_{z\mu} \pm 360^\circ k \right\} \quad (5.24)$$

and the angle of entry of the pairs of zeros with multiplicity $r_{z\varrho}$

$$\varphi_{z\varrho,E} = \frac{1}{r_{z\varrho}} \left\{ - \sum_{\substack{\mu=1 \\ \mu \neq \varrho}}^m \varphi_{z\mu} + \sum_{\nu=1}^n \varphi_{p\nu} \pm 360^\circ k \right\} \quad (5.25)$$

The root-locus method can also be applied for other cases than varying k_0 . This is possible as long as $G_0(s)$ can be rewritten such that the angle condition according to Eq. (5.15) and the rules given above can be applied. This will be demonstrated in the following two examples.

Given the closed-loop characteristic equation

$$a_0 + a_1 s + \dots + a_{n-1} s^{n-1} + s^n = 0,$$

the root locus for varying the parameter a_1 is required. The characteristic equation is therefore rewritten as

$$1 + a_1 \frac{s}{a_0 + a_2 s^2 + \dots + s^n} = 0.$$

This form then corresponds to the standard form

$$1 + G_0(s) = 1 + a_1 \frac{N_0(s)}{D_0(s)} = 0$$

to which the rules can be applied. ■

Given the closed-loop characteristic equation

$$s^3 + (3 + \alpha) s^2 + 2s + 4 = 0,$$



it is required to find the effect of the parameter on the position of the closed-loop poles. The equation is rewritten into the desired form

$$1 + K \frac{s^2}{s^3 + 3s^2 + 2s + 4} = 0$$

Using the rules 1 to 10 one can easily predict the geometrical form of the root locus based on the distribution of the open-loop poles and zeros. Table 6.2 shows some typical distributions of open-loop poles and zeros and their root loci.

Table 6.2: Typical distributions of open-loop poles and zeros and the root loci

No.	root locus	No.	root locus
1		9	
2		10	
3		11	
4		12	
5		13	
6		14	
7		15	
8		16	



For the qualitative assessment of the root locus one can use a physical analogy. If all open-loop poles are substituted by a negative electrical charge and all zeros by a commensurate positive one and if a massless negative charged particle is put onto a point of the root locus, a movement is observed. The path that the particle takes because of the interplay between the repulsion of the poles and the attraction of the zeros lies just on the root locus. Comparing the root locus examples 3 and 9 of Table 5.2 the 'repulsive' effect of the additional pole can be clearly seen.

The systematic application of the rules from section 5.2 for the construction of a root locus is shown in the following non-trivial example for the open-loop transfer function

$$G_0(s) = \frac{k_0(s+1)}{s(s+2)(s^2+12s+40)} \quad (5.26)$$

The degree of the numerator polynomial is $m = 1$. This means that the transfer function has one zero ($s_{Z_1} = -1$). The degree of the denominator polynomial is $n = 4$ and we have the four poles ($s_{P_1} = 0$, $s_{P_2} = -2$, $s_{P_3} = -6 + j2$, $s_{P_4} = -6 - j2$). First the poles (x) and the zeros (o) of the open loop are drawn on the s -plane as shown in Figure 5.5. According to rule 3 these poles are just

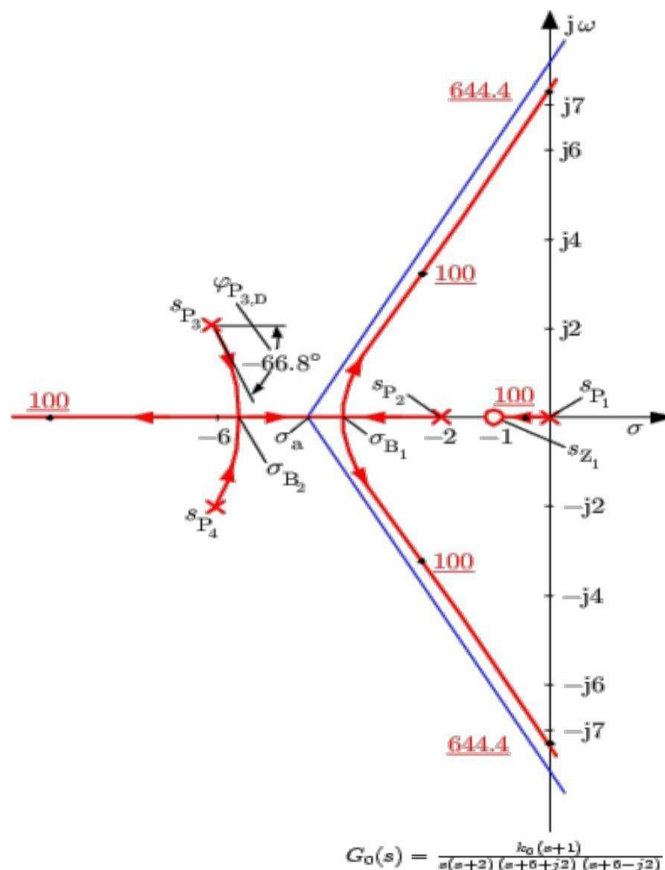




Figure 5.5: Root locus of $G(s)H(s)$. Values of k_0 are in red and underlined.

those points of the root locus where $k_0 = 0$ and the zeros where $k_0 \rightarrow \infty$. We have $(n - m) = 3$ branches that go to infinity and the asymptotes of these three branches are lines which intercept the real axis according to rule 6. From Eq. (5.17) the crossing is at

$$\sigma_a = \frac{(0 - 2 - 6 - 6) - (-1)}{3} = -\frac{13}{3} = -4.33 \quad (5.27)$$

and the slopes of the asymptotes are according to Eq. (5.16)

$$\alpha_k = \frac{\pm 180^\circ(2k+1)}{3} = \pm 60^\circ(2k+1) \quad k = 0, 1, 2, \dots \quad (5.28)$$

i.e. $\alpha_0 = 60^\circ, \quad \alpha_1 = +180^\circ, \quad \alpha_2 = -60^\circ$

The asymptotes are shown in Figure 5.5 as blue lines. Using Rule 4 it can be checked which points on the real axis are points on the root locus. The points σ with $-1 < \sigma < 0$ and $\sigma < -2$ belong to the root locus, because to the right of them the number of poles and zeros is odd. According to rule 7 breakaway and break-in points can only occur pairwise on the real axis to the left of -2. These points are real solutions of the Eq. (5.19). Here we have

$$\frac{1}{s} + \frac{1}{s+2} + \frac{1}{s+6-j2} + \frac{1}{s+6+j2} = \frac{1}{s+1} \quad (5.29)$$

or

$$3s^4 + 32s^3 + 106s^2 + 128s + 80 = 0$$

This equation has the solutions $s_{B1} = -3.68$, $s_{B2} = -5.47$ and $s_{B3,4} = -0.76 \pm j0.866$. The real roots $s_{B1} = -3.68$ and $s_{B2} = -5.47$ are the positions of the breakaway and the break-in point.

The angle of departure $\varphi_{P3,D}$ of the root locus from the complex pole at $s_{P3} = -6 + j2$ can be determined from Figure 5.6 according to Eq. (5.20):

$$\begin{aligned} \varphi_{P3,D} &= -90^\circ - 153.4^\circ - 161.6^\circ + 158.2^\circ \pm 180^\circ(2k+1) \\ &= -246.8^\circ + 180^\circ = -66.8^\circ \end{aligned} \quad (5.30)$$

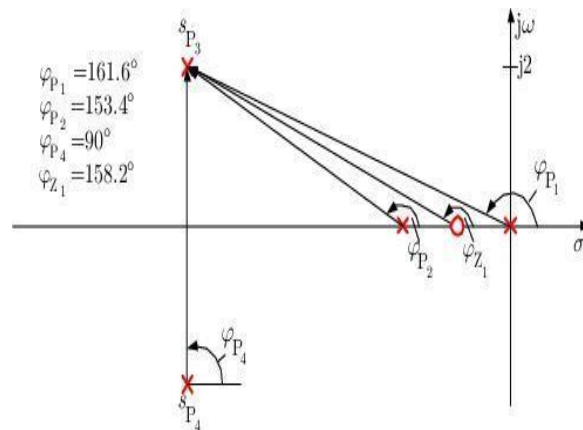


Figure 5.6: Calculating the angle of departure $\varphi_{P_3,D}$ of the complex pole $s_{P_3} = -6 + j2$

With this specifications the root locus can be sketched. Using rule 9 the value of k_O can be determined for some selected points. The value at the intersection with the imaginary axis is

$$k_{O,crit} = \frac{7.2 \cdot 7.4 \cdot 7.9 \cdot 11.1}{7.25} = 644.4$$

OUTCOMES:

At the end of the unit, the students are able to:

- Construct root loci from open loop transfer functions of control systems and Analyze the behaviour of roots with system gain.
- Assess the stability of closed loop systems by means of the root location in s-plane and their effects on system performance.

SELF-TEST QUESTIONS:

1. Sketch the root locus for $G(S)H(S) = \frac{K}{S(S+2)(S+4)}$ show all details on it.
2. Sketch the root locus for $G(S)H(S) = \frac{10K}{S(S+2)(S+6)}$ show all details on it.
3. Sketch the root locus for $G(S)H(S) = \frac{K(S+1)}{S(S+2)(S+4)}$ show all details on it.