Operation Research

Module 1

1.1 Origin of Operations Research

The term Operations Research (OR) was first coined by MC Closky and Trefthen in 1940 in a small town, Bowdsey of UK. The main origin of OR was during the second world war – The military commands of UK and USA engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations.

Their mission was to formulate specific proposals and to arrive at the decision on optimal utilization of scarce military resources and also to implement the decisions effectively. In simple words, it was to uncover the methods that can yield greatest results with little efforts. Thus it had gained popularity and was called “An art of winning the war without actually fighting it”

The name Operations Research (OR) was invented because the team was dealing with research on military operations. The encouraging results obtained by British OR teams motivated US military management to start with similar activities. The work of OR team was given various names in US: Operational Analysis, Operations Evaluation, Operations Research, System Analysis, System Research, Systems Evaluation and so on.

The first method in this direction was simplex method of linear programming developed in 1947 by G.B Dantzig, USA. Since then, new techniques and applications have been developed to yield high profit from least costs.

Now OR activities has become universally applicable to any area such as transportation, hospital management, agriculture, libraries, city planning, financial institutions, construction management and so forth. In India many of the industries like Delhi cloth mills, Indian Airlines, Indian Railway, etc are making use of OR activity.

1.2 Concept and Definition of OR

Operations research signifies research on operations. It is the organized application of modern science, mathematics and computer techniques to complex military, government, business or industrial problems arising in the direction and management of large systems of men, material, money and machines. The purpose is to provide the management with explicit quantitative understanding and assessment of complex situations to have sound basics for arriving at best decisions.
Operations research seeks the optimum state in all conditions and thus provides optimum solution to organizational problems.

Definition: OR is a scientific methodology – analytical, experimental and quantitative – which by assessing the overall implications of various alternative courses of action in a management system provides an improved basis for management decisions.

1.3 Characteristics of OR (Features)

The essential characteristics of OR are

1. **Inter-disciplinary team approach** – The optimum solution is found by a team of scientists selected from various disciplines.
2. **Wholistic approach to the system** – OR takes into account all significant factors and finds the best optimum solution to the total organization.
3. **Imperfectness of solutions** – Improves the quality of solution.
4. **Use of scientific research** – Uses scientific research to reach optimum solution.
5. **To optimize the total output** – It tries to optimize by maximizing the profit and minimizing the loss.

1.4 Applications of OR

Some areas of applications are

- Finance, Budgeting and Investment
  - Cash flow analysis , investment portfolios
  - Credit polices, account procedures
- Purchasing, Procurement and Exploration
  - Rules for buying, supplies
  - Quantities and timing of purchase
  - Replacement policies
- Production management
  - Physical distribution
  - Facilities planning
  - Manufacturing
  - Maintenance and project scheduling
- Marketing
  - Product selection, timing
  - Number of salesman, advertising
- Personnel management
  - Selection of suitable personnel on minimum salary
  - Mixes of age and skills
- Research and development
  - Project selection
  - Determination of area of research and development
  - Reliability and alternative design

1.5 Phases of OR

OR study generally involves the following major phases

1. Defining the problem and gathering data
2. Formulating a mathematical model
3. Deriving solutions from the model
4. Testing the model and its solutions

5. Preparing to apply the model
6. Implementation

Defining the problem and gathering data

- The first task is to study the relevant system and develop a well-defined statement of the problem. This includes determining appropriate objectives, constraints, interrelationships and alternative course of action.
- The OR team normally works in an **advisory capacity**. The team performs a detailed technical analysis of the problem and then presents recommendations to the management.
- Ascertaining the appropriate **objectives** is very important aspect of problem definition.
  Some of the objectives include maintaining stable price, profits, increasing the share in market, improving work morale etc.
- OR team typically spends huge amount of time in gathering relevant data.
  - To gain accurate understanding of problem
  - To provide input for next phase.
- OR teams uses Data mining methods to search large databases for interesting patterns that may lead to useful decisions.

Formulating a mathematical model

This phase is to reformulate the problem in terms of mathematical symbols and expressions. The mathematical model of a business problem is described as the system of equations and related mathematical expressions. Thus

1. **Decision variables** (x_1, x_2 … x_n) – ‘n’ related quantifiable decisions to be made.
2. **Objective function** – measure of performance (profit) expressed as mathematical function of decision variables. For example $P=3x_1 +5x_2 + \ldots + 4x_n$
3. **Constraints** – any restriction on values that can be assigned to decision variables in terms of inequalities or equations. For example $x_1 +2x_2 \geq 20$
4. **Parameters** – the constant in the constraints (right hand side values)

The advantages of using mathematical models are

- Describe the problem more concisely
- Makes overall structure of problem comprehensible
- Helps to reveal important cause-and-effect relationships
- Indicates clearly what additional data are relevant for analysis
- Forms a bridge to use mathematical technique in computers to analyze

Deriving solutions from the model

This phase is to develop a procedure for deriving solutions to the problem. A common theme is to search for an optimal or best solution. The main goal of OR team is to obtain an optimal solution which minimizes the cost and time and maximizes the profit.

Herbert Simon says that "Satisficing is more prevalent than optimizing in actual practice". Where satisficing = satisfactory + optimizing

Samuel Eilon says that "Optimizing is the science of the ultimate; Satisficing is the art of the feasible".

To obtain the solution, the OR team uses

- **Heuristic procedure** (designed procedure that does not guarantee an optimal solution) is used to find a good suboptimal solution.
- **Metaheuristics** provides both general structure and strategy guidelines for designing a specific heuristic procedure to fit a particular kind of problem.
- **Post-Optimality analysis** is the analysis done after finding an optimal solution. It is also referred as **what-if analysis**. It involves conducting **sensitivity analysis** to determine which parameters of the model are most critical in determining the solution.

Testing the model

After deriving the solution, it is tested as a whole for errors if any. The process of testing and improving a model to increase its validity is commonly referred as **Model validation**. The OR group doing this review should preferably include at least one individual who did not participate in the formulation of model to reveal mistakes.

A systematic approach to test the model is to use **Retrospective test**. This test uses historical data to reconstruct the past and then determine the model and the resulting solution. Comparing the effectiveness of this hypothetical performance with what actually happened, indicates whether the model tends to yield a significant improvement over current practice.

Preparing to apply the model

After the completion of testing phase, the next step is to install a well-documented system for applying the model. This system will include the model, solution procedure and operating procedures for implementation.

The system usually is computer-based. **Databases** and **Management Information System** may provide up-to-date input for the model. An interactive computer based system called **Decision Support System** is installed to help the manager to use data and models to support their decision making as needed. A **managerial report** interprets output of the model and its implications for applications.

Implementation

The last phase of an OR study is to implement the system as prescribed by the management. The success of this phase depends on the support of both top management and operating management.

The implementation phase involves several steps

1. OR team provides a detailed explanation to the operating management
2. If the solution is satisfied, then operating management will provide the explanation to the personnel, the new course of action.
3. The OR team monitors the functioning of the new system
4. Feedback is obtained
5. Documentation

2.1 Introduction to Linear Programming

A linear form is meant a mathematical expression of the type $a_1x_1 + a_2x_2 + \ldots. + a_nx_n$, where a_1, a_2, …, a_n are constants and x_1, x_2 … x_n are variables. The term Programming refers to the process of determining a particular program or plan of action. So Linear Programming (LP) is one of the most important optimization (maximization / minimization) techniques developed in the field of Operations Research (OR).

The methods applied for solving a linear programming problem are basically simple problems; a solution can be obtained by a set of simultaneous equations. However a unique solution for a set of simultaneous equations in n-variables (x_1, x_2 … x_n), at least one of them is non-zero, can be obtained if there are exactly *n* relations. When the number of relations is greater than or less than *n*, a unique solution does not exist but a number of trial solutions can be found.

In various practical situations, the problems are seen in which the number of relations is not equal to the number of the number of variables and many of the relations are in the form of inequalities (≤ or ≥) to maximize or minimize a linear function of the variables subject to such conditions. Such problems are known as Linear Programming Problem (LPP).

Definition – The general LPP calls for optimizing (maximizing / minimizing) a linear function of variables called the '**Objective function**' subject to a set of linear equations and / or inequalities called the '**Constraints**' or '**Restrictions'**.

2.2 General form of LPP

We formulate a mathematical model for general problem of allocating resources to activities. In particular, this model is to select the values for x_1, x_2 … x_n so as to maximize or minimize

$$Z = c_1x_1 + c_2x_2 + \ldots\ldots\ldots\ldots + c_nx_n$$

subject to restrictions

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_1nx_n \ (\leq \text{ or } \geq)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_2nx_n \ (\leq \text{ or } \geq)\ b_2$$
$$.$$
$$.$$
$$.$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n \ (\leq \text{ or } \geq)\ b_m$$

and

$$x_1 \geq 0,\ x_2 \geq 0, \ldots, x_n \geq$$

Where
Z = value of overall measure of performance x_j =
level of activity (for j = 1, 2, ..., n)
c_j = increase in Z that would result from each unit increase in level of activity j
b_i = amount of resource i that is available for allocation to activities (for i = 1,2, …, m)
a_{ij} = amount of resource i consumed by each unit of activity j

Resource	Resource usage per unit of activity	Amount of resource available
	Activity	
	1 2 …………………….. n	
1	a_{11} a_{12} …………………….a_{1n}	b_1
2	a_{21} a_{22} …………………….a_{2n}	b_2
.	.	.
.	.	.
.	.	.
m	a_{m1} a_{m2} …………………….a_{mn}	b_m
Contribution to Z per unit of activity	c_1 c_2 ………………………..c_n	

Data needed for LP model

- The level of activities x_1, x_2………x_n are called **decision variables**.
- The values of the c_j, b_i, a_{ij} (for i=1, 2 … m and j=1, 2 … n) are the **input constants** for the model. They are called as **parameters** of the model.
- The function being maximized or minimized Z = $c_1x_1 + c_2x_2 +\ldots +c_nx_n$ is called **objective function**.
- The restrictions are normally called as **constraints**. The constraint $a_{i1}x_1 + a_{i2}x_2 \ldots a_{in}x_n$ are sometimes called as **functional constraint** (L.H.S constraint). $x_j \geq 0$ restrictions are called **non-negativity constraint**.

2.3 Assumptions in LPP

a) Proportionality
b) Additivity
c) Multiplicativity
d) Divisibility
e) Deterministic

2.4 Applications of Linear Programming

1. Personnel Assignment Problem
2. Transportation Problem
3. Efficiency on Operation of system of Dams
4. Optimum Estimation of Executive Compensation
5. Agriculture Applications
6. Military Applications

7. Production Management
8. Marketing Management
9. Manpower Management10. Physical distribution

2.5 Advantages of Linear Programming Techniques

1. It helps us in making the optimum utilization of productive resources.
2. The quality of decisions may also be improved by linear programming techniques.
3. Provides practically solutions.
4. In production processes, high lighting of bottlenecks is the most significant advantage of this technique.

2.6 Formulation of LP Problems

Example 1
A firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires 1 minute of processing time on G and 2 minutes on H; type B requires 1 minute on G and 1 minute on H. The machine G is available for not more than 6 hours 40 minutes while machine H is available for 10 hours during any working day. Formulate the problem as a linear programming problem.

Solution
Let
x_1 be the number of products of type A
x_2 be the number of products of type B

After understanding the problem, the given information can be systematically arranged in the form of the following table.

	Type of products (minutes)		
Machine	Type A (x_1 units)	Type B (x_2 units)	Available time (mins)
G	1	1	400
H	2	1	600
Profit per unit	Rs. 2	Rs. 3	

Since the profit on type A is Rs. 2 per product, 2 x_1 will be the profit on selling x_1 units of type A. similarly, $3x_2$ will be the profit on selling x_2 units of type B. Therefore, total profit on selling x_1 units of A and x_2 units of type B is given by

Maximize $Z = 2x_1 + 3x_2$ (**objective function**)

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required on machine G is given by $x_1 + x_2$.

Similarly, the total number of minutes required on machine H is given by $2x_1 + 3x_2$.

But, machine G is not available for more than 6 hours 40 minutes (400 minutes). Therefore,

$x_1 + x_2 \leq 400$ (**first constraint**)

Also, the machine H is available for 10 hours (600 minutes) only, therefore,

$2x_1 + 3x_2 \leq 600$ (**second constraint**)

Since it is not possible to produce negative quantities

$x_1 \geq 0$ and $x_2 \geq 0$ (**non-negative restrictions**)

Hence

Maximize $Z = 2x_1 + 3x_2$

Subject to restrictions

$x_1 + x_2 \leq 400$

$2x_1 + 3x_2 \leq 600$

and non-negativity constraints

$x_1 \geq 0$, $x_2 \geq 0$

Example 2

A company produces two products A and B which possess raw materials 400 quintals and 450 labour hours. It is known that 1 unit of product A requires 5 quintals of raw materials and 10 man hours and yields a profit of Rs 45. Product B requires 20 quintals of raw materials, 15 man hours and yields a profit of Rs 80. Formulate the LPP.

Solution

Let

x_1 be the number of units of product A

x_2 be the number of units of product B

	Product A	Product B	Availability
Raw materials	5	20	400
Man hours	10	15	450
Profit	Rs 45	Rs 80	

Hence

Maximize $Z = 45x_1 + 80x_2$

Subject to

$5x_1 + 20x_2 \leq 400$

$10x_1 + 15x_2 \leq 450$

$x_1 \geq 0$, $x_2 \geq 0$

Example 3

A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has 2 machines and below is given the required processing time in minutes for each machine on each product.

	Products		
Machine	A	B	C

X	4	3	5
Y	2	2	4

Machine X and Y have 2000 and 2500 machine minutes. The firm must manufacture 100 A's, 200 B's and 50 C's type, but not more than 150 A's.

Solution

Let

x_1 be the number of units of product A
x_2 be the number of units of product B
x_3 be the number of units of product C

	Products			
Machine	A	B	C	Availability
X	4	3	5	2000
Y	2	2	4	2500
Profit	3	2	4	

Max $Z = 3x_1 + 2x_2 + 4x_3$

Subject to

$4x_1 + 3x_2 + 5x_3 \leq 2000$
$2x_1 + 2x_2 + 4x_3 \leq 2500$
$100 \leq x_1 \leq 150$
$x_2 \geq 200$
$x_3 \geq 50$

Example 4

A company owns 2 oil mills A and B which have different production capacities for low, high and medium grade oil. The company enters into a contract to supply oil to a firm every week with 12, 8, 24 barrels of each grade respectively. It costs the company Rs 1000 and Rs 800 per day to run the mills A and B. On a day A produces 6, 2, 4 barrels of each grade and B produces 2, 2, 12 barrels of each grade. Formulate an LPP to determine number of days per week each mill will be operated in order to meet the contract economically.

Solution

Let

x_1 be the no. of days a week the mill A has to work
x_2 be the no. of days per week the mill B has to work

Grade	A	B	Minimum requirement
Low	6	2	12
High	2	2	8
Medium	4	12	24
Cost per day	Rs 1000	Rs 800	

Minimize $Z = 1000x_1 + 800\ x_2$

Subject to

$6x_1 + 2x_2 \geq 12$

$2x_1 + 2x_2 \geq 8$

$4x_1 + 12x_2 \geq 24$

$x_1 \geq 0\ ,\ x_2 \geq 0$

Example 5

A company has 3 operational departments weaving, processing and packing with the capacity to produce 3 different types of clothes that are suiting, shirting and woolen yielding with the profit of Rs. 2, Rs. 4 and Rs. 3 per meters respectively. 1m suiting requires 3mins in weaving 2 mins in processing and 1 min in packing. Similarly 1m of shirting requires 4 mins in weaving 1 min in processing and 3 mins in packing while 1m of woolen requires 3 mins in each department. In a week total run time of each department is 60, 40 and 80 hours for weaving, processing and packing department respectively. Formulate a LPP to find the product to maximize the profit.

Solution

Let

x_1 be the number of units of suiting

x_2 be the number of units of shirting

x_3 be the number of units of woolen

	Suiting	Shirting	Woolen	Available time
Weaving	3	4	3	60
Processing	2	1	3	40
Packing	1	3	3	80
Profit	2	4	3	

Maximize $Z = 2x_1 + 4x_2 + 3x_3$

Subject to

$3x_1 + 4x_2 + 3x_3 \leq 60$

$2x_1 + 1x_2 + 3x_3 \leq 40$

$x_1 + 3x_2 + 3x_3 \leq 80$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

Example 6

ABC Company produces both interior and exterior paints from 2 raw materials m1 and m2. The following table produces basic data of problem.

	Exterior paint	Interior paint	Availability
M1	6	4	24
M2	1	2	6
Profit per ton	5	4	

A market survey indicates that daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also maximum daily demand for interior paint is 2 tons. Formulate

LPP to determine the best product mix of interior and exterior paints that maximizes the daily total profit.

Solution

Let

x_1 be the number of units of exterior paint

x_2 be the number of units of interior paint

Maximize $Z = 5x_1 + 4x_2$

Subject to

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$x_2 - x_1 \leq 1$

$x_2 \leq 2$

$x_1 \geq 0,\ x_2 \geq 0$

b) The maximum daily demand for exterior paint is atmost 2.5 tons

$x_1 \leq 2.5$

c) Daily demand for interior paint is atleast 2 tons

$x_2 \geq 2$

d) Daily demand for interior paint is exactly 1 ton higher than that for exterior paint.

$x_2 > x_1 + 1$

Example 7

A company produces 2 types of hats. Each hat of the I type requires twice as much as labour time as the II type. The company can produce a total of 500 hats a day. The market limits daily sales of I and II types to 150 and 250 hats. Assuming that the profit per hat are Rs.8 for type A and Rs. 5 for type B. Formulate a LPP models in order to determine the number of hats to be produced of each type so as to maximize the profit.

Solution

Let x_1 be the number of hats produced by type A

Let x_2 be the number of hats produced by type B

Maximize $Z = 8x_1 + 5x_2$

Subject to

$2x_1 + x_2 \leq 500$ (labour time)

$x_1 \leq 150$

$x_2 \leq 250$

$x_1 \geq 0,\ x_2 \geq 0$

Example 8

A manufacturer produces 3 models (I, II and III) of a certain product. He uses 2 raw materials A and B of which 4000 and 6000 units respectively are available. The raw materials per unit of 3 models are given below.

Raw materials	I	II	III
A	2	3	5

B	4	2	7

The labour time for each unit of model I is twice that of model II and thrice that of model III. The entire labour force of factory can produce an equivalent of 2500 units of model I. A model survey indicates that the minimum demand of 3 models is 500, 500 and 375 units respectively. However the ratio of number of units produced must be equal to 3:2:5. Assume that profits per unit of model are 60, 40 and 100 respectively. Formulate a LPP.

Solution

Let

x_1 be the number of units of model I
x_2 be the number of units of model II
x_3 be the number of units of model III

Raw materials	I	II	III	Availability
A	2	3	5	4000
B	4	2	7	6000
Profit	60	40	100	

$x_1 + 1/2x_2 + 1/3x_3 \leq 2500$ [Labour time]

$x_1 \geq 500$, $x_2 \geq 500$, $x_3 \geq 375$ [Minimum demand]

The given ratio is $x_1: x_2: x_3 = 3: 2: 5$
$x_1 / 3 = x_2 / 2 = x_3 / 5 = k$
$x_1 = 3k; x_2 = 2k; x_3 = 5k$
$x_2 = 2k \rightarrow k = x_2 / 2$
Therefore $x_1 = 3\, x_2 / 2 \rightarrow 2\, x_1 = 3\, x_2$
Similarly $2\, x_3 = 5\, x_2$

Maximize $Z= 60x_1 + 40x_2 + 100x_3$
Subject to $2x_1 + 3x_2 + 5x_3 \leq 4000$
$4x_1 + 2x_2 + 7x_3 \leq 6000$
$x_1 + 1/2x_2 + 1/3x_3 \leq 2500$
$2\, x_1 = 3\, x_2$
$2\, x_3 = 5\, x_2$
and $x_1 \geq 500$, $x_2 \geq 500$, $x_3 \geq 375$

3.1 Graphical Solution Procedure

The graphical solution procedure

1. Consider each inequality constraint as equation.
2. Plot each equation on the graph as each one will geometrically represent a straight line.
3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality constraint corresponding to that line is '≤' then the region below the line lying in the first quadrant is shaded. Similarly for '≥' the region above the line is shaded. The points lying in the common region will satisfy the constraints. This common region is called **feasible region.**

4. Choose the convenient value of Z and plot the objective function line.
5. Pull the objective function line until the extreme points of feasible region.
    a. In the maximization case this line will stop far from the origin and passing through at least one corner of the feasible region.
    b. In the minimization case, this line will stop near to the origin and passing through at least one corner of the feasible region.
6. Read the co-ordinates of the extreme points selected in step 5 and find the maximum or minimum value of Z.

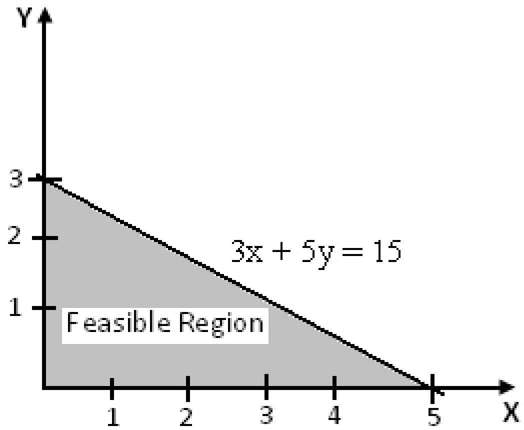
3.2 Definitions

1. **Solution** – Any specification of the values for decision variable among (x_1, x_2… x_n) is called a solution.
2. **Feasible solution** is a solution for which all constraints are satisfied.
3. **Infeasible solution** is a solution for which atleast one constraint is not satisfied.
4. **Feasible region** is a collection of all feasible solutions.
5. **Optimal solution** is a feasible solution that has the most favorable value of the objective function.
6. **Most favorable value** is the largest value if the objective function is to be maximized, whereas it is the smallest value if the objective function is to be minimized.
7. **Multiple optimal solution** – More than one solution with the same optimal value of the objective function.
8. **Unbounded solution** – If the value of the objective function can be increased or decreased indefinitely such solutions are called unbounded solution.
9. **Feasible region** – The region containing all the solutions of an inequality
10. **Corner point feasible solution** is a solution that lies at the corner of the feasible region.

3.3 Example problems

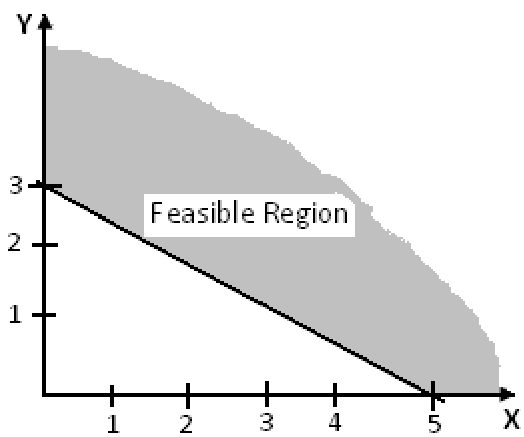
Example 1
Solve $3x + 5y < 15$ graphically

Solution
Write the given constraint in the form of equation i.e. $3x + 5y = 15$
Put x=0 then the value y=3
Put y=0 then the value x=5
Therefore the coordinates are (0, 3) and (5, 0). Thus these points are joined to form a straight line as shown in the graph.
Put x=0, y=0 in the given constraint then
0<15, the condition is true. (0, 0) is solution nearer to origin. So shade the region below the line, which is the feasible region.



Example 2
Solve 3x + 5y >15

Solution
Write the given constraint in the form of equation i.e. 3x + 5y = 15
Put x=0, then y=3
Put y=0, then x=5
So the coordinates are (0, 3) and (5, 0)
Put x =0, y =0 in the given constraint, the condition turns out to be false i.e. 0 > 15 is false.
So the region does not contain (0, 0) as solution. The feasible region lies on the outer part of the line as shown in the graph.



Example 3

Max Z = $80x_1 + 55x_2$
Subject to
$4x_1 + 2x_2 \leq 40$
$2x_1 + 4x_2 \leq 32$
$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $4x_1 + 2x_2 \leq 40$, written in a form of equation
$4x_1 + 2x_2 = 40$

Put $x_1 = 0$, then $x_2 = 20$
Put $x_2 = 0$, then $x_1 = 10$

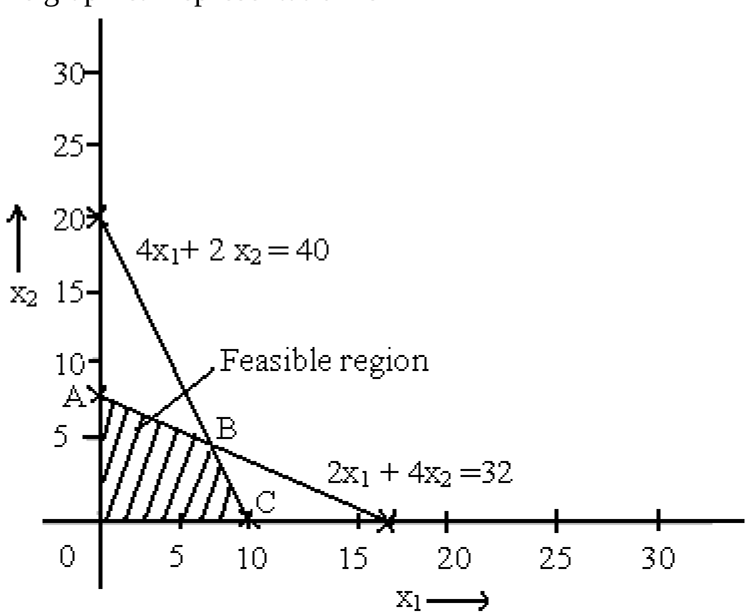
The coordinates are (0, 20) and (10, 0)

The second constraint $2x_1 + 4x_2 \leq 32$, written in a form of equation
$2x_1 + 4x_2 = 32$

Put $x_1 = 0$, then $x_2 = 8$
Put $x_2 = 0$, then $x_1 = 16$

The coordinates are (0, 8) and (16, 0)

The graphical representation is



The corner points of feasible region are A, B and C. So the coordinates for the corner points are
A (0, 8)
B (8, 4) (Solve the two equations $4x_1 + 2x_2 = 40$ and $2x_1 + 4x_2 = 32$ to get the coordinates)
C (10, 0)

We know that Max $Z = 80x_1 + 55x_2$

At A (0, 8)
$Z = 80(0) + 55(8) = 440$

At B (8, 4)
$Z = 80(8) + 55(4) = 860$

At C (10, 0)
$Z = 80(10) + 55(0) = 800$

The maximum value is obtained at the point B. Therefore Max Z = 860 and $x_1 = 8$, $x_2 = 4$

Example 4

Minimize $Z = 10x_1 + 4x_2$
Subject to

$3x_1 + 2x_2 \geq 60$
$7x_1 + 2x_2 \geq 84$
$3x_1 + 6x_2 \geq 72$
$x_1 \geq 0$, $x_2 \geq 0$

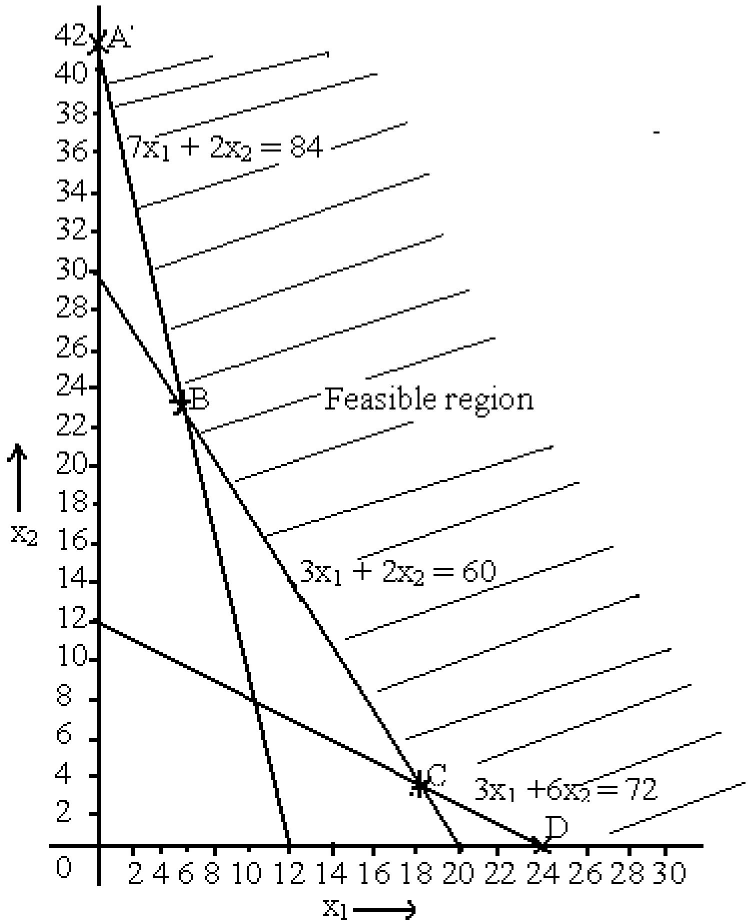
Solution

The first constraint $3x_1 + 2x_2 \geq 60$, written in a form of equation
$3x_1 + 2x_2 = 60$
Put $x_1 = 0$, then $x_2 = 30$
Put $x_2 = 0$, then $x_1 = 20$
The coordinates are (0, 30) and (20, 0)

The second constraint $7x_1 + 2x_2 \geq 84$, written in a form of equation
$7x_1 + 2x_2 = 84$
Put $x_1 = 0$, then $x_2 = 42$
Put $x_2 = 0$, then $x_1 = 12$
The coordinates are (0, 42) and (12, 0)

The third constraint $3x_1 + 6x_2 \geq 72$, written in a form of equation
$3x_1 + 6x_2 = 72$
Put $x_1 = 0$, then $x_2 = 12$
Put $x_2 = 0$, then $x_1 = 24$
The coordinates are (0, 12) and (24, 0)

The graphical representation is



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points are

A (0, 42)

B (6, 21) (Solve the two equations $7x_1 + 2x_2 = 84$ and $3x_1 + 2x_2 = 60$ to get the coordinates)

C (18, 3) Solve the two equations $3x_1 + 6x_2 = 72$ and $3x_1 + 2x_2 = 60$ to get the coordinates)

D (24, 0)

We know that Min $Z = 10x_1 + 4x_2$

At A (0, 42)

Z = 10(0) + 4(42) = 168

At B (6, 21)

Z = 10(6) + 4(21) = 144

At C (18, 3)

Z = 10(18) + 4(3) = 192

At D (24, 0)
Z = 10(24) + 4(0) = 240

The minimum value is obtained at the point B. Therefore Min Z = 144 and $x_1 = 6$, $x_2 = 21$

Example 5

A manufacturer of furniture makes two products – chairs and tables. Processing of this product is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and a table is Rs 2 and Rs 10 respectively. What should be the daily production of each of two products?

Solution

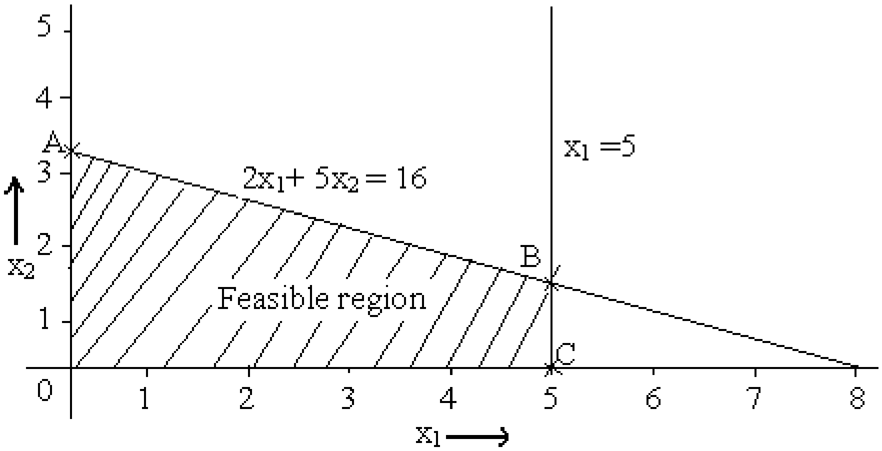
Let x_1 denotes the number of chairs
Let x_2 denotes the number of tables

	Chairs	Tables	Availability
Machine A	2	5	16
Machine B	6	0	30
Profit	Rs 2	Rs 10	

LPP
Max $Z = 2x_1 + 10x_2$
Subject to
$2x_1 + 5x_2 \leq 16$
$6x_1 + 0x_2 \leq 30$
$x_1 \geq 0$, $x_2 \geq 0$

Solving graphically
The first constraint $2x_1 + 5x_2 \leq 16$, written in a form of equation
$2x_1 + 5x_2 = 16$
Put $x_1 = 0$, then $x_2 = 16/5 = 3.2$
Put $x_2 = 0$, then $x_1 = 8$
The coordinates are (0, 3.2) and (8, 0)

The second constraint $6x_1 + 0x_2 \leq 30$, written in a form of equation
$6x_1 = 30 \rightarrow x_1 = 5$



The corner points of feasible region are A, B and C. So the coordinates for the corner points are
A (0, 3.2)
B (5, 1.2) (Solve the two equations $2x_1 + 5x_2 = 16$ and $x_1 = 5$ to get the coordinates)
C (5, 0)

We know that Max $Z = 2x_1 + 10x_2$
At A (0, 3.2)
$Z = 2(0) + 10(3.2) = 32$

At B (5, 1.2)
$Z = 2(5) + 10(1.2) = 22$

At C (5, 0)
$Z = 2(5) + 10(0) = 10$

Max $Z = 32$ and $x_1 = 0$, $x_2 = 3.2$
The manufacturer should produce approximately 3 tables and no chairs to get the max profit.

3.4 Special Cases in Graphical Method

3.4.1 Multiple Optimal Solution

Example 1
Solve by using graphical method

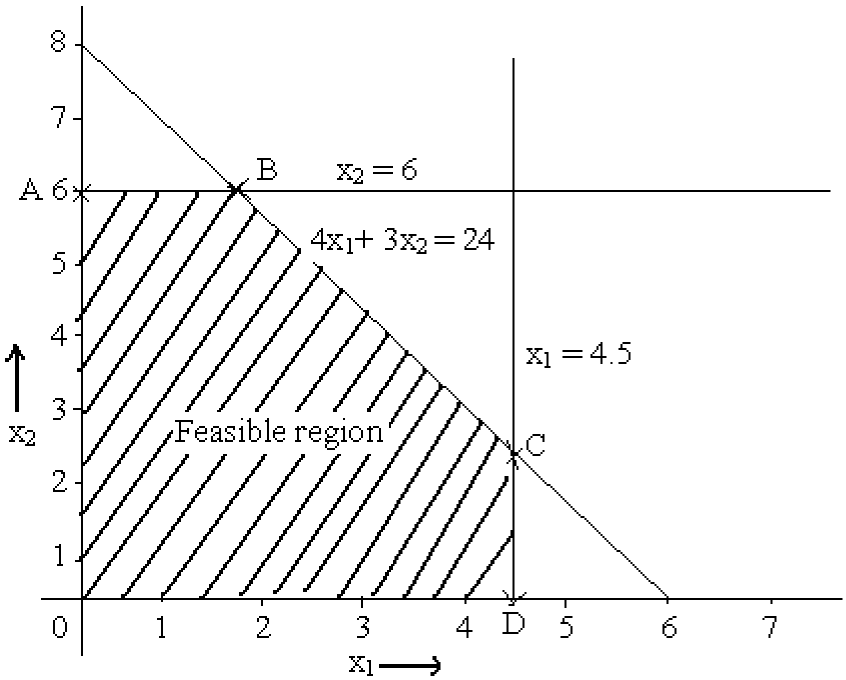
Max $Z = 4x_1 + 3x_2$
Subject to
$4x_1 + 3x_2 \leq 24$
$x_1 \leq 4.5$
$x_2 \leq 6$
$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $4x_1 + 3x_2 \le 24$, written in a form of equation
$4x_1 + 3x_2 = 24$
Put $x_1 = 0$, then $x_2 = 8$
Put $x_2 = 0$, then $x_1 = 6$
The coordinates are (0, 8) and (6, 0)

The second constraint $x_1 \le 4.5$, written in a form of equation
$x_1 = 4.5$

The third constraint $x_2 \le 6$, written in a form of equation
$x_2 = 6$



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points are
A (0, 6)
B (1.5, 6) (Solve the two equations $4x_1 + 3x_2 = 24$ and $x_2 = 6$ to get the coordinates)
C (4.5, 2) (Solve the two equations $4x_1 + 3x_2 = 24$ and $x_1 = 4.5$ to get the coordinates)
D (4.5, 0)

We know that Max $Z = 4x_1 + 3x_2$
At A (0, 6)
Z = 4(0) + 3(6) = 18

At B (1.5, 6)
Z = 4(1.5) + 3(6) = 24

At C (4.5, 2)
Z = 4(4.5) + 3(2) = 24

At D (4.5, 0)
Z = 4(4.5) + 3(0) = 18

Max Z = 24, which is achieved at both B and C corner points. It can be achieved not only at B and C but every point between B and C. Hence the given problem has multiple optimal solutions.

3.4.2 No Optimal Solution

Example 1

Solve graphically

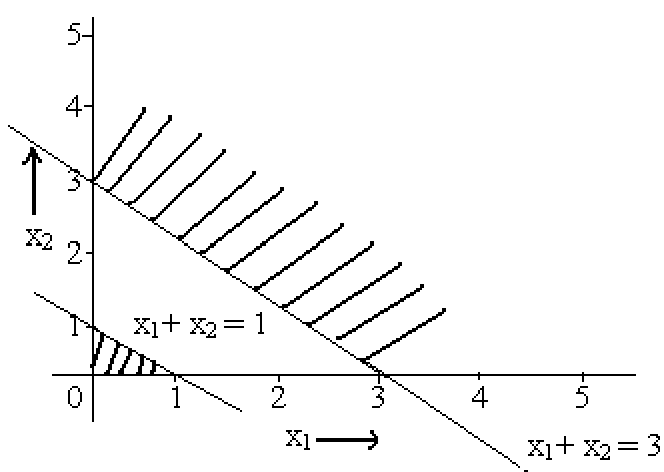
Max Z = $3x_1 + 2x_2$
Subject to

$x_1 + x_2 \leq 1$
$x_1 + x_2 \geq 3$
$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $x_1 + x_2 \leq 1$, written in a form of equation
$x_1 + x_2 = 1$
Put $x_1 = 0$, then $x_2 = 1$
Put $x_2 = 0$, then $x_1 = 1$
The coordinates are (0, 1) and (1, 0)

The first constraint $x_1 + x_2 \geq 3$, written in a form of equation
$x_1 + x_2 = 3$
Put $x_1 = 0$, then $x_2 = 3$
Put $x_2 = 0$, then $x_1 = 3$
The coordinates are (0, 3) and (3, 0)



There is no common feasible region generated by two constraints together i.e. we cannot identify even a single point satisfying the constraints. Hence there is no optimal solution.

3.4.3 Unbounded Solution

Example
Solve by graphical method

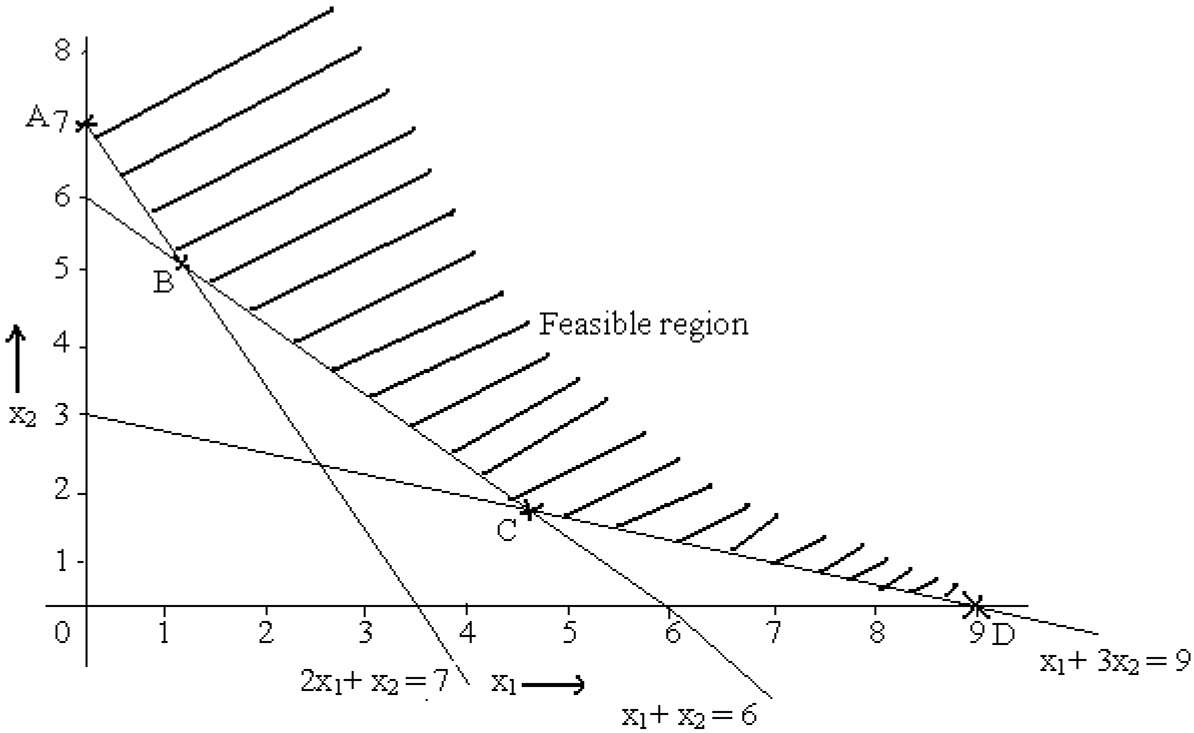
Max $Z = 3x_1 + 5x_2$
Subject to
$2x_1 + x_2 \geq 7$
$x_1 + x_2 \geq 6$
$x_1 + 3x_2 \geq 9$
$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $2x_1 + x_2 \geq 7$, written in a form of equation
$2x_1 + x_2 = 7$
Put $x_1 = 0$, then $x_2 = 7$
Put $x_2 = 0$, then $x_1 = 3.5$
The coordinates are (0, 7) and (3.5, 0)

The second constraint $x_1 + x_2 \geq 6$, written in a form of equation
$x_1 + x_2 = 6$
Put $x_1 = 0$, then $x_2 = 6$
Put $x_2 = 0$, then $x_1 = 6$
The coordinates are (0, 6) and (6, 0)

The third constraint $x_1 + 3x_2 \geq 9$, written in a form of equation
$x_1 + 3x_2 = 9$
Put $x_1 = 0$, then $x_2 = 3$
Put $x_2 = 0$, then $x_1 = 9$
The coordinates are (0, 3) and (9, 0)



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points ar
e
A (0,
7)
B (1, 5) (Solve the two equations $2x_1 + x_2 = 7$ and $x_1 + x_2 = 6$ to get the coordinates)
C (4.5, 1.5) (Solve the two equations $x_1 + x_2 = 6$ and $x_1 + 3x_2 = 9$ to get the coordinates)
D (9, 0)

We know that Max $Z = 3x_1 + 5x_2$
At A (0, 7)
Z = 3(0) + 5(7) = 35

At B (1, 5)
Z = 3(1) + 5(5) = 28

At C (4.5, 1.5)
Z = 3(4.5) + 5(1.5) = 21

At D (9, 0)
Z = 3(9) + 5(0) = 27
The values of objective function at corner points are 35, 28, 21 and 27. But there exists infinite number of points in the feasible region which is unbounded. The value of objective function will be more than the value of these four corner points i.e. the maximum value of the objective function occurs at a point at ∞. Hence the given problem has unbounded solution.