15 CV 33 FLUID MECHANICS NOTES

MODULE-2

- **Module-2A : Hydrostatic forces on Surfaces**
- **Module-2B :Fundamentals of fluid flow (Kinematics)**

by

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Module-2A: Hydrostatic forces on Surfaces
Definition, Total pressure, centre of pressure, total pressure on horizontal, vertical and inclined plane surface, total pressure on curved surfaces, water pressure on gravity dams, Lock gates. Numerical Problems.

2.0 Definitions

Pressure or Pressure intensity (p): It is the Fluid pressure force per unit area of application. Mathematically, $P = \frac{p}{A}$. Units are Pascal or N/m^2.

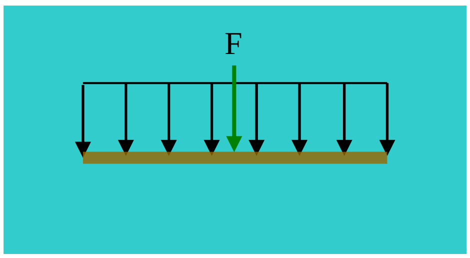
Total Pressure (P): This is that force exerted by the fluid on the contact surface (of the submerged surfaces), when the fluid comes in contact with the surface always acting normal to the contact surface. Units are N.

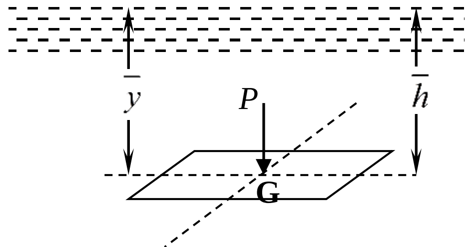
Centre of Pressure: It is defined as the point of application of the total pressure on the contact surface.

The submerged surface may be either plane or curved. In case of plane surface, it may be vertical, horizontal or inclined. Hence, the above four cases may be studied for obtaining the total pressure and centre of pressure.

2.1 Hydrostatic Forces on Plane Horizontal Surfaces:

- If a plane surface immersed in a fluid is horizontal, then
    - Hydrostatic pressure is uniform over the entire surface.
    - The resultant force acts at the centroid of the plane.





Consider a horizontal surface immersed in a static fluid as shown in Fig. As all the points on the plane are at equal depth from the free surface of the liquid, the pressure intensity will be equal on the entire surface and given by $\boldsymbol{p = \rho g \bar{y}}$, where $\bar{y}$ is the depth of the fluid surface

Let $\boldsymbol{A}$ = Area of the immersed surface

The total pressure force acting on the immersed surface is $\boldsymbol{P}$

$\boldsymbol{P = p}$ x Area of the surface = $\boldsymbol{\rho g \bar{y} A}$

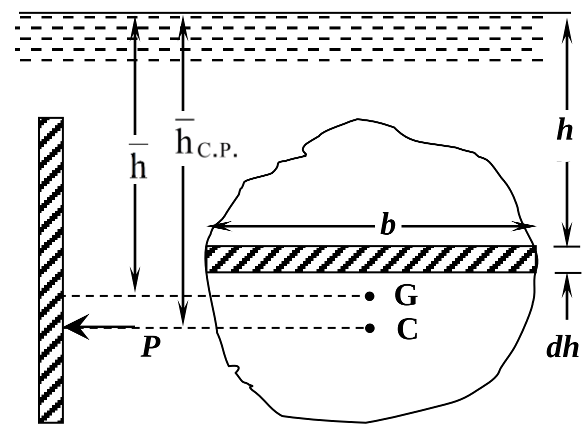
$\boldsymbol{P = \rho g A \bar{y}}$

Where $\bar{y}$ is the centroidal distance immersed surface from the free surface of the liquid and $\bar{h}$ is the centre of pressure.

2.2 Hydrostatic Forces on Vertical Plane Surface:

Vertical Plane surface submerged in liquid

Consider a vertical plane surface of some arbitrary shape immersed in a liquid of mass density ρ as shown in Figure below:



End View Elevation

Let, A = Total area of the surface

$\bar{h}$ = Depth of Centroid of the surface from the free surface

G = Centroid of the immersed surface

C = Centre of pressure

$\bar{h}_{C.P.}$ = Depth of centre of pressure

Consider a rectangular strip of breadth b and depth dy at a depth y from the free surface.

Total Pressure:

The pressure intensity at a depth y acting on the strip is $p = \rho gh$

Total pressure force on the strip = $dP = (\rho gh)dA$

$\therefore$ The Total pressure force on the entire area is given by integrating the above expression over the entire area $P = \int dP = \int (\rho gh)dA = \rho g\int h\, dA$ Eq.(1)

But $\int y\, dA$ is the Moment of the entire area about the free surface of the liquid given by

$$\int h dA = A\bar{h}$$

Substituting in Eq.(1), we get $P = \rho gA\bar{h} = \gamma A\bar{h}$ Eq.(2)

Where γ is the specific weight of Water

For water, ρ=1000 kg/m^3 and g = 9.81 m/s^2. The force will be expressed in Newtons (N)

2.3 Hydrostatic Force on a Inclined submerged surface:

The other important utility of the hydrostatic equation is in the determination of force acting upon submerged bodies. Among the innumerable applications of this is the force calculation in storage tanks, ships, dams etc.

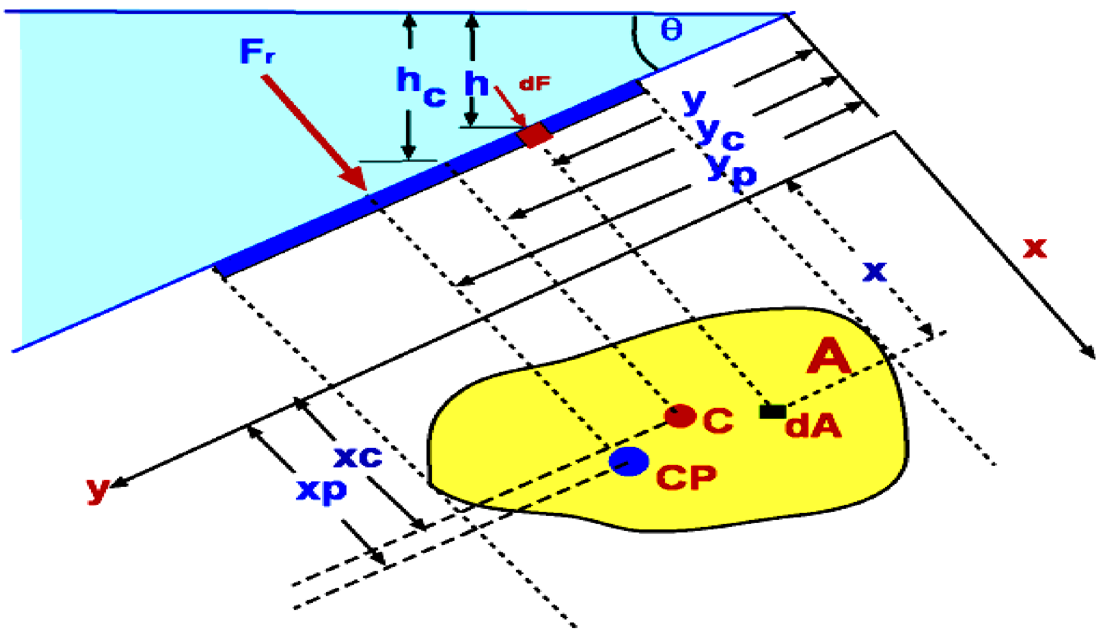


Figure 3.4 : Force upon a submerged object

First consider a planar arbitrary shape submerged in a liquid as shown in the figure. The plane makes an angle θ with the liquid surface, which is a free surface. The depth of water over the plane varies linearly. This configuration is efficiently handled by prescribing a coordinate frame such that the y-axis is aligned with the submerged plane. Consider an infinitesimally small area at a (x,y). Let this small area be located at a depth h from the free surface.dA = dx.dy

Differential Force acting on the differential area dA of plane,

$$dF = (\text{Pressure}) \cdot (\text{Area}) = (\gamma h) \cdot (dA) \qquad \text{(Perpendicular to plane)}$$

Then, Magnitude of total resultant force F_R

$$F_R = \int_A \gamma h dA = \int_A \gamma (y \sin\theta) dA$$ Where $h = y \sin\theta$

$$= \gamma \sin\theta \int_A \boxed{y dA}$$

1^st^ moment of the area

- Related with the center of area

※ $\int_A y dA = y_c A$ whereyc: y coordinate of the center of area (Centroid)

c.f.Centeror 1st moment

$\int_M x dm = MX_C$ & $\int_M y dm = MY_C$ (XC & YC: Center of Mass)

$\int_A x dA$ = xc & $\int_A y dA$ = yc (xc&yc: Center of Area)

Moment of inertia or 2nd moment

$\int_M r^2 dm = I$ (2nd moment of Mass)

$\int_A y^2 dA = I_x$ & $\int_A x^2 dA = I_y$ (2nd moment of Area)

Then,

$$F_R = \gamma A y_c \sin\theta = (\gamma h_c) A$$

Where γh_c : Pressure at the centroid = (Pressure at the centroid) × Area

- Magnitude of a force on an INCLINED plane
- Dependent on γ □, Area, and Depth of centroid
- Perpendicular to the surface (Direction)

i) Position of FR on y-axis ‘yR’ : y coordinate of the point of action of FR

Moment about x axis:

$$F_R y_R = (\cancel{\gamma} A y_c \cancel{\sin}\theta) \cancel{y_R} = \int_A y dF = \int_A \gamma \sin\theta y^2 dA = \cancel{\gamma} \cancel{\sin}\theta \int_A y^2 dA$$

$$\therefore h_R = \frac{\int_A h^2 dA}{h_c A} = \frac{I_x}{h_c A}$$ where $I_x = \int_A y^2 dA$:2nd moment of area

or, by using the parallel-axis theorem, $I_x = I_{xc} + Ay_c^{\,2}$

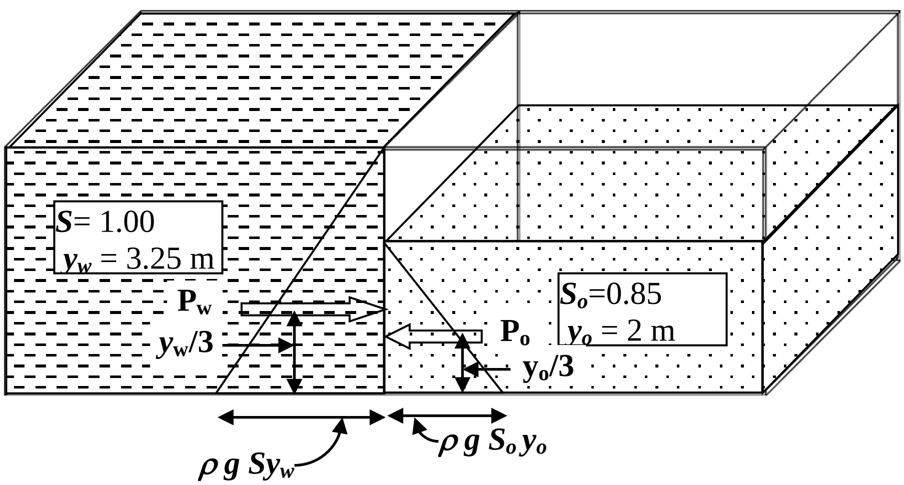
$$\therefore h_{C.P.} = \bar{h} + \frac{I_G \sin^2\theta^\circ}{A\bar{h}}$$

(The centre of pressure below the centroid)

Solved Examples:

Q1. A rectangular tank 10 m x 5 m and 3.25 m deep is divided by a partition wall parallel to the shorter wall of the tank. One of the compartments contains water to a depth of 3.25 m and the other oil of specific gravity 0.85 to a depth of 2 m. Find the resultant pressure on the partition.

Solution:



The problem can be solved by considering hydrostatic pressure distribution diagram for both water and oil as shown in Fig.

From hydrostatic law, the pressure intensity ***p*** at any depth y_w is given by

$p = S_o \rho\, g\, y_w$

where ρ is the mass density of the liquid

Pressure force ***P = p x Area***

$P_w = 1000 \times 10 \times 3.25 \times 5 \times 3.25 = 528.125\ kN\ (\rightarrow)$

Acting at ***3.25/3 m*** from the base

$P_o = 0.85 \times 1000 \times 10 \times 2.0 \times 5 \times 2.0 = 200$ kN ($\leftarrow$)

Acting at ***2/3 m*** from the base.

Net Force $P = P_w - P_o = 528.125 - 200.0 = 328.125$ kN ($\rightarrow$)

Location:

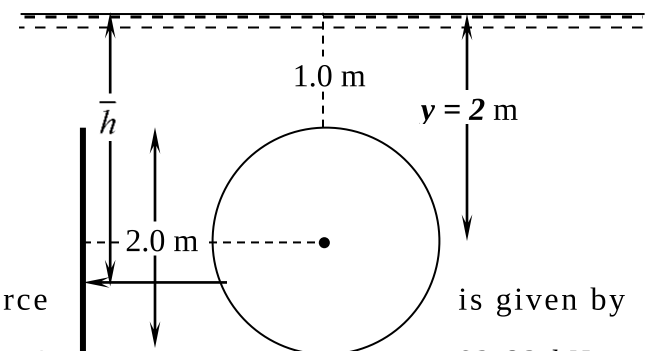
Let ***P*** act at a distance ***y*** from the base. Taking moments of P_w, P_o and ***P*** about the base, we get

$P \times y = P_w \times y_w / 3 - P_o \times y_o / 3$

$328.125\, y = 528.125 \times (3.25/3) - 200 \times (2/3)$ or $y = 1.337$ m.

Q2. Determine the total force and location of centre of pressure for a circular plate of 2 m dia immersed vertically in water with its top edge 1.0 m below the water surface

Solution:



$$A = \frac{\pi \times D^2}{4} = \frac{\pi \times 2^2}{4} = 3.142 \, \text{m}$$

Assume

$\rho = 1000$ kg/m^3 and g = 10 m/s^2

We know that the total pressure force is given by

$P = S_o \rho g A \bar{y} = 1000 \times 10 \times 3.142 \times 2 = 62.83$ kN

Centre of Pressure

The Centre of pressure is given by

$$\bar{h} = \bar{y} + \frac{I_g}{A \bar{y}}$$

$$I_g = \frac{\pi R^4}{4} = \frac{\pi \times 1^4}{4} = 0.785 \text{ m}^4$$

$$\bar{h} = 2 + \frac{0.785}{3.142 \times 2} = 2.125 \, \text{m}$$

Q.3 *A large tank of sea water has a door in the side 1 m square. The top of the door is 5 m below the free surface. The door is hinged on the bottom edge. Calculate the force required at the top to keep it closed. The density of the sea water is 1033 kg/m3.*

Solution: The total hydrostatic force $F = \gamma_{sea\ \ water} A\ h_c$

$$\gamma_{sea\ \ water} = 1033\ x 9.81 = 10133.73 N / m^3$$

Given A = 1m X 1m = 1m^2

$$h_c = 5 + \frac{1}{2} = 5.5m$$

$$F = 10133.73X1X5.5 = 55735.5N$$

Acting at centre of pressure $(y_{c.p})$:

From the above $h_c = 5.5m,\ A = 1m^2$

$$(I_c)_{xx} = \frac{BD^3}{12} = \frac{1X1^3}{12} = 0.08333m^4$$

$$h_{C.P.} = h_c + \frac{(I_c)_{xx}}{Ah_c} = 5.5 + \frac{0.08333}{1X5.5} = 5.515m$$

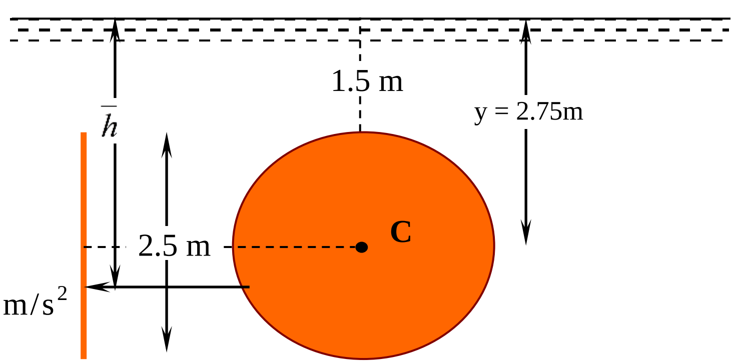
Distance of Hydrostatic force (F) from the bottom of the hinge = 6-5.515 = 0.48485m

The force 'P' required at the top of gate (1m from the hinge)

PX1 = FX0.48485 = 55735.5X0.48485

P = 27023.4 N = 27.023 kN

Q.4 Calculate the total hydrostatic force and location of centre of pressure for a circular plate of 2.5 m diameter immersed vertically in water with its top edge 1.5 m below the oil surface (Sp. Gr.=0.9)



Solution:

$$A = \frac{\pi \times D^2}{4} = \frac{\pi \times 2^2}{4} = 4.91\ \mathbf{m^2}$$

Assume

ρ = 0.9X1000=900 kg/m^3,g=9.8 m/s^2

$$\gamma_{oil} = 900\ X\ 9.81 = 8829\ N / m^3$$

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$h_c = 2.75m$

We know that the total pressure force is given by 'F'

$F = \gamma_{oil}\ A\ h_c$ = 8829 x 4.91 x 2.75 = 238184 N = 238.184 kN

Centre of Pressure:

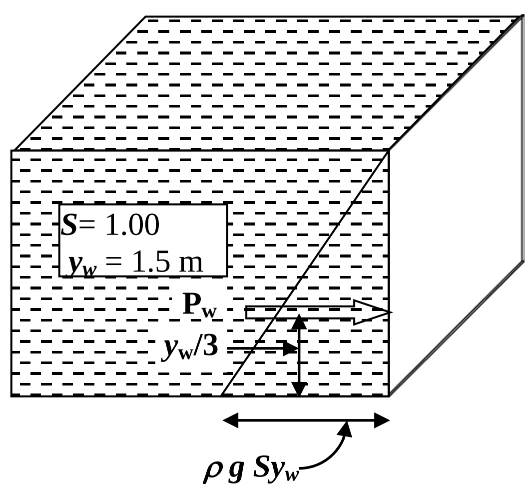
The Centre of pressure is given by

$$h_{C.P.} = h_c + \frac{(I_c)_{x\ x}}{Ah_c}$$

$$I_g = \frac{\pi R^4}{4} = \frac{\pi \times 1.25^4}{4} = 1.9175\ \mathbf{m^4}$$

$$h_{C.P.} = 2.75 + \frac{1.9175}{4.91 \times 2.75} = 2.892\mathbf{m}$$

Q.5 A square tank with 2 m sides and 1.5 m high contains water to a depth of 1 m and a liquid of specific gravity 0.8 on the water to a depth of 0.5 m. Find the magnitude and location of hydrostatic pressure on one face of tank.



Solution:

The problem can be solved by considering hydrostatic pressure distribution diagram for water as shown in Fig. From hydrostatic law, the pressure intensity ***p*** at any depth y_w is given by $\boldsymbol{p = S_o \rho\ g\ y_w}$

where ρ is the mass density of the liquid

Pressure force ***P = p x Area***

$$P_w = 1000 \times 10 \times 2.0 \times 1.5 \times 1.5 = 45\ kN\ (\rightarrow)$$

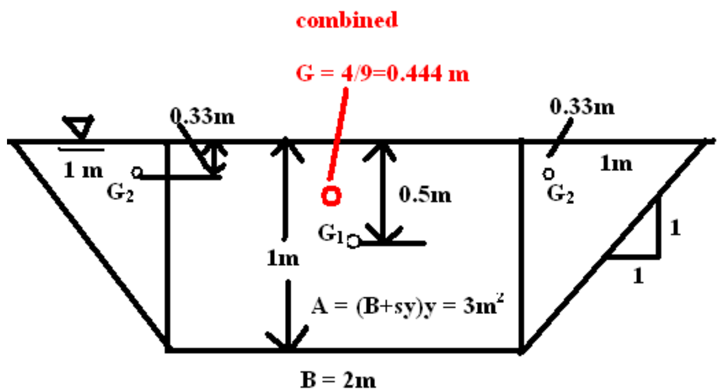
cting at ***1.5/3=0.5 m*** from the base

Q.6 A trapezoidal channel 2m wide at the bottom and 1m deep has side slopes 1:1. Determine: i) Total pressure ii) Centre of pressure, when it is full of water

Ans: Given B = 2m Area of flow A = (B+sy)y = $3m^2$
The combined centroid will be located based on two triangular areas and one rectangle (shown as G_1, G_2, G_2)

$$\bar{y} = \frac{A_1 \times h_1 + A_2 \times h_2 + A_3 \times h_3}{A_1 + A_2 + A_3}$$

The total area A = $3m^2$
Area of rectangle = 2 ×1 = $2m^2$
Area of Triangle = $\frac{1}{2} \times 1 \times 1 = 0.5m^2$



$$\bar{y} = \frac{\left((2\times1)\times0.5 + \left[\frac{1}{2}\times(1\times1)\right]\times0.333 + \left[\frac{1}{2}\times(1\times1)\right]\times 0.\right.}{3}$$

i) The total pressure P = $\gamma_w \times A \times \bar{y}$ = 9810×3×0.444 = 13080N
ii) Centre of pressure

The centroidal moment of Inertia of Rectangle and Triangle is

$$I_{G1} = \frac{2\times1^3}{12} = 0.1667m^4 \quad at \quad 0.5m \ \ from\ water - surface$$

$$I_{G1} = \frac{1\times1^3}{36} = 0.028m^4 \quad at \quad 0.333m \ \ from\ water - surface$$

$$\bar{h} = \bar{y} + \frac{I_g}{A\,\bar{y}} \cdots Eq.1$$

The moment of Inertia about combined centroid can be obtained by using parallel axis theorem

$$I_G = \left((I_{G1}) + A_1 d_1^2\right) + \left((I_{G2}) + A_2 d_2^2\right) + \left((I_{G2}) + A_2 d_2^2\right)$$ (as both the triangles are similar)

$$I_G = (0.1667 + 0.00618) + 2\left((I_{G2}) + A_2 d_2^2\right)$$

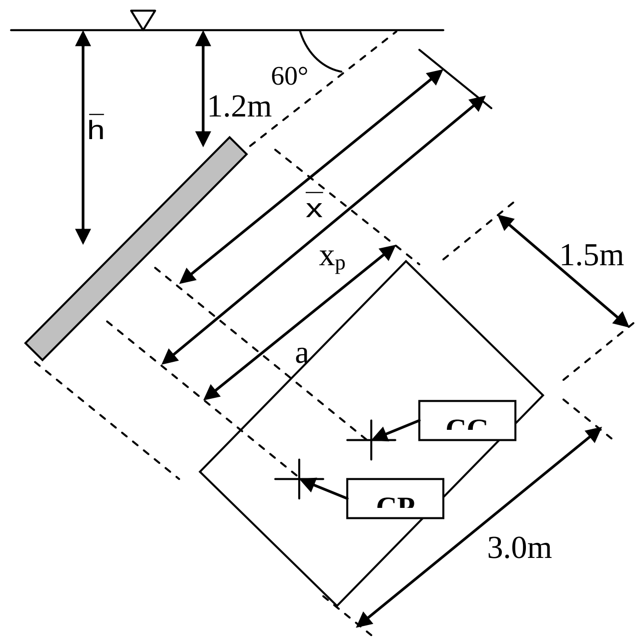
$$I_G = (0.1667 + 0.00618) + 2(0.028 + 0.0062) = 0.2408m^4$$

Substituting in Eq.1, The centre of pressure from free surface of water

$$\bar{h} = \bar{y} + \frac{I_g}{A\,\bar{y}} \cdots Eq.1$$

$$\bar{h} = 0.444 + \frac{0.2408}{3\times0.444} = 0.6252m$$

Q.7. A rectangular plate 1.5m x 3.0m is submerged in water and makes an angle of 60° with the horizontal, the 1.5m sides being horizontal. Calculate the magnitude of the force on the plate and the location of the point of application of the force, with reference to the top edge of the plate, when the top edge of the plate is 1.2m below the water surface.



Solution:

$$\bar{h} = \frac{1.2}{\sin 60^\circ} + 1.5 = 1.386 + 1.5 = 2.886\text{m}$$

$$A = 3\text{m X } 1.5\text{m} = 4.5\text{m}^2$$

$$\bar{h} = y \sin 60^\circ = 2.886 \sin 60^\circ = 2.499\text{m}$$

$$F = \rho g \bar{h} A = 1000 \times 9.81 \times 2.499 \times 3 \times 1.5$$

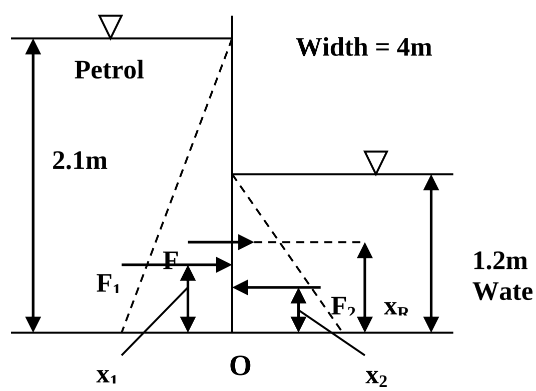
$$\therefore \quad F = 109.92 \times 10^3 \text{ N} = 109.92 \text{ kN}$$

$$h_{C.P.} = \bar{h} + \frac{I_G \sin^2 60^\circ}{A\bar{h}}$$

$$\therefore \; h_{C.P.} = 2.886 + \frac{3^2}{12 \times 2.886} = 2.886 + 0.260 = 3.146m$$

From the top edge of the plate, $a = 3.146 - 1.386 = 1.760\text{m}$

Q.8 A vertical bulkhead 4m wide divides a storage tank. On one side of the bulkhead petrol (S.G. = 0.78) is stored to a depth of 2.1m and on the other side water is stored to a depth of 1.2m. Determine the resultant force on the bulkhead and the position where it acts.



Solution:

$$F = \rho g \bar{h} A = \rho g \frac{h}{2} \cdot bh = \tfrac{1}{2} \rho g h^2 \cdot b$$

$$F_1 = \tfrac{1}{2} \times 780 \times 9.81 \times 2.1^2 \times 4 \text{ N} = 67.5 \text{ kN}$$

$$F_2 = \tfrac{1}{2} \times 1000 \times 9.81 \times 1.2^2 \times 4 \text{ N} = 28.25 \text{ kN}$$

Hence the resultant force

$$F_R = F_1 - F_2 = 67.5 - 28.25 = 39.25 \text{ kN}$$

$$h_{C.P.} = \bar{h} + \frac{I_G}{A\bar{h}} = \frac{h}{2} + \frac{bh^3}{12} \quad \frac{1}{bh(h/2)} = \frac{h}{2} + \frac{h}{6} = \tfrac{2}{3}h$$

From the diagram, $y = h - {}^{2}/_{3}h = {}^{1}/_{3}h$

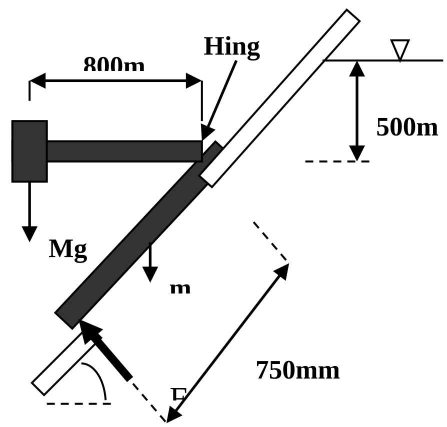
Hence, $y_1 = 2.1 / 3 = 0.7\text{m}$ and $y_2 = 1.2 / 3 = 0.4\text{m}$

Taking moments about 'O', $\boxed{F_R \cdot y_R = F_1 \cdot y_1 - F_2 \cdot y_2}$

i.e. $39.25 \times y_R = 67.5 \times 0.7 - 28.25 \times 0.4$ and hence $y_R = 0.916\text{m}$

Q.9 *A* hinged, circular gate 750mm in diameter is used to close the opening in a sloping side of a tank, as shown in the diagram in **Error! Reference source not found.**. The gate is kept closed against water pressure partly by its own weight and partly by a weight on the lever arm. Find the mass M required to allow the gate to begin to open when the water level is 500mm above the top of the gate. The mass of the gate is 60 kg. (Neglect the weight of the lever arm.)

Solution:



$$a = \frac{500}{\sin 45°} = 707\text{mm}$$

$$\bar{x} = a + 375 = 1082\text{mm}$$

$$\bar{h} = \bar{x}\sin 45° = 765\text{mm}$$

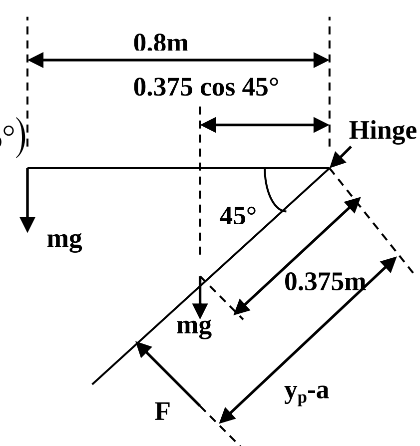
$$F = \rho g \bar{h} A = 1000 \times 9.81 \times 0.765 \times \left(\pi \times 0.75^2/4\right)$$

$$\therefore \quad F = 3.315 \times 10^3 \text{ N} = 3.315 \text{ kN}$$

$$x_P = \bar{x} + \frac{I_G}{A\bar{x}} = \bar{x} + \frac{\pi d^4}{64} \cdot \frac{4}{\pi d^2 \bar{x}} = \bar{x} + \frac{d^2}{16 \cdot \bar{x}}$$

$$x_P = 1.082 + \frac{0.75^2}{16 \times 1.082} = 1.082 + 0.032 = 1.114\text{m}$$

Taking moments about the hinge



$$F(x_P - a) = Mg \times 0.8 + mg \times 0.375\cos 45°$$

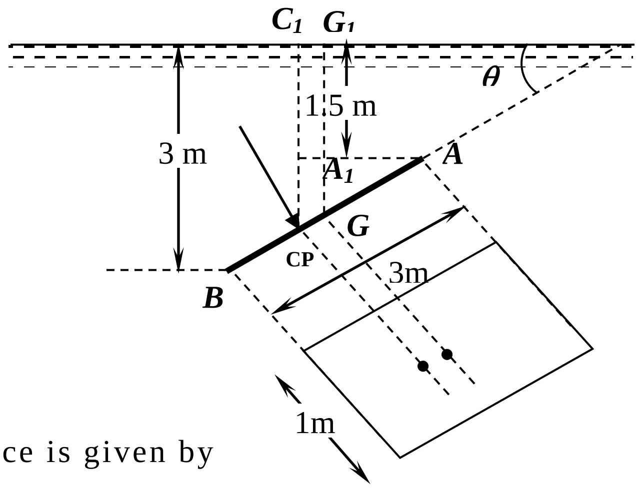
$$3315(1.114 - 0.707) = 9.81(M \times 0.8 + 60 \times 0.375\cos 45°)$$

$$M \times 0.8 = \frac{3315(1.114 - 0.707)}{9.81} - 60 \times 0.375\cos 45°$$

$$M \times 0.8 = 137.5 - 16 = 121.5$$

$$\therefore M = \frac{121.5}{0.8} = 152 \text{ kg}$$

Q.10. A rectangular plate 1 m x 3 m is immersed in water such that its upper and lower edge is at depths 1.5 m and 3 m respectively. Determine the total pressure acting on the plate and locate it.



Solution:

$A = 1 \times 3 = 3\ m^2$

$\gamma_w = 9810\ N/m^3$

$$h_c = \frac{3m + 1.5m}{2} = 2.25m$$

We know that the total pressure force is given by

$\boldsymbol{F = \gamma_{water} A h_c}$ $= 9810 \times 3.0 \times 2.25 = 66217.5$ N

Sin θ = 1.5 / 3 = 0.5

θ = 30°

Centre of Pressure; The Centre of pressure is given by

$$\left(I_C\right)_{x\ x} = \frac{bd^3}{12} = \frac{1 \times 3^3}{12} = 2.25 m^4$$

$h_c = 2.25m$

$$CC_1 = h_{C.P.} = h_c + \frac{\left(I_c\right)_{x\ x} Sin^2\theta}{Ah_c}$$

$$CC_1 = h_{C.P.} = 2.25 + \frac{2.25\ Sin^2 30^o}{3\ X\ 2.25}$$

$$CC_1 = h_{C.P.} = 2.33333\ m$$

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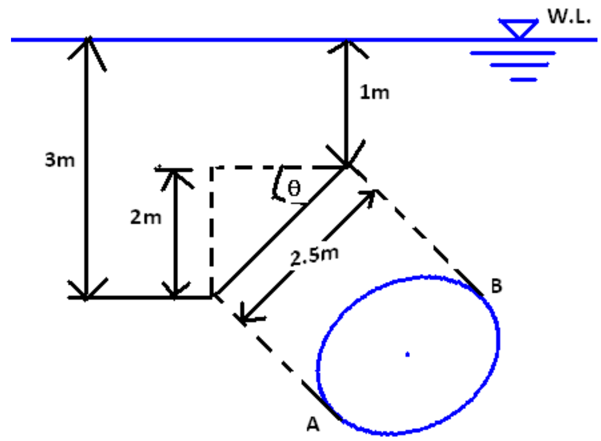
Q 11. A circular plate 2.5m diameter is immersed in water, its greatest and least depth below the free surface being 3m and 1m respectively. Find
(i) The total pressure on one face of the plate and (ii) Position of centre of pressure



Ans: Given d = 2.5m,

$$\theta = Sin^{-1}\left(\frac{2}{2.5}\right)$$

$$\theta = 53.13^o$$

$$\bar{h} = 1 + 1 = 2m$$

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 2.5^2 = 4.909m^2$$

$$I_G = \frac{\pi}{64}d^4 = \frac{\pi}{64}\times(2.5)^4 = 1.917m^4$$

F = $\gamma_w A \bar{h}$ = 9.81×4.909×2 = 96.31 kN

$$\boldsymbol{h_{c.p.}} = \boldsymbol{\bar{h}} + \frac{\boldsymbol{I_G \times Sin^2\theta}}{\boldsymbol{A \times \bar{h}}} = 2 + \frac{1.917\times \boldsymbol{Sin}^2 53.13^o}{4.909\times 2}$$

$$\boldsymbol{h_{c.p.}} = 2.125\boldsymbol{m}$$

Q.12. A 2m wide and 3m deep rectangular plane surface lies in water in such a way the top of and bottom edges are at a distance of 1.5m and 3m respectively from the surface. Determine the hydrostatic force and centre of pressure

Ans: Given A = 3m × 2m = 6m^2,

$$I_G = \frac{2\times 3^3}{12} = 4.5m^4$$

Hydrostatic force

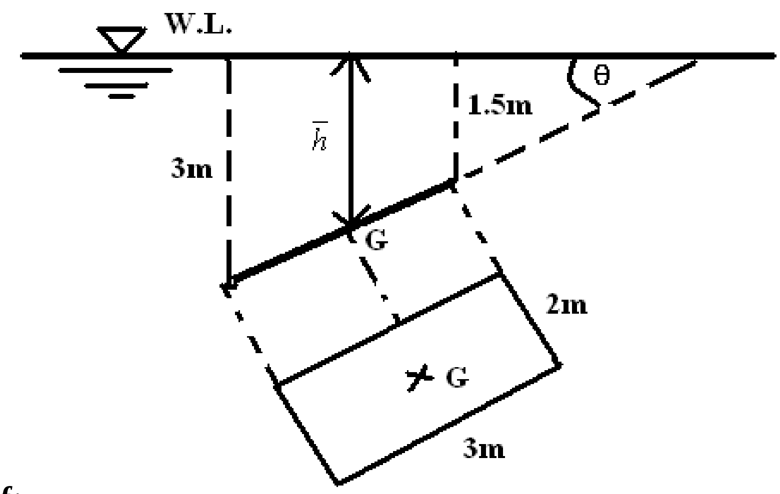
$$P = \gamma_w \times A \times \bar{h}$$

$$P = 9.81\times 6\times\left(\frac{3+1.5}{2}\right)$$

P = 132.435 kN

$$\sin\theta = \frac{(3.0-1.5)}{3} = 0.5$$

$$\theta = 30°$$



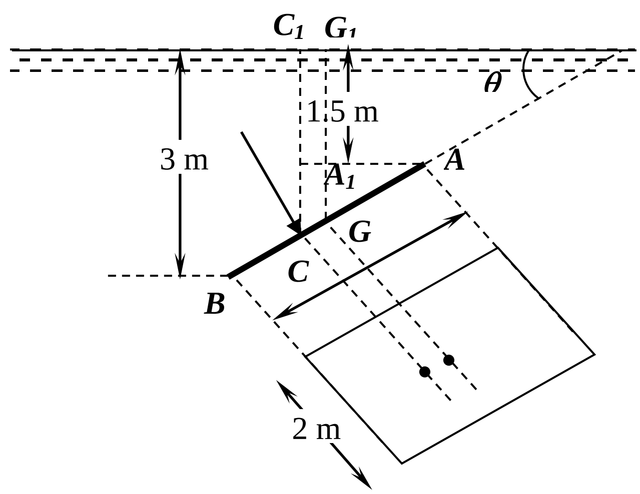
The centre of pressure

$$h_{C.P} = \bar{h} + \frac{I_G \times Sin^2\theta}{A\,\bar{h}}$$

$$h_{C.P} = 2.25 + \frac{4.5 \times \left(1/4\right)}{6 \times 2.25} = 2.33m$$

Q.13 A rectangular plate 2 m x 3 m is immersed in oil of specific gravity 0.85 such that its ends are at depths 1.5 m and 3 m respectively. Determine the total pressure acting on the plate and locate it.

Solution:



A = 2 x 3 = 6 m^2

S_o = 0.85

Assume

ρ = 1000 kg/m^3

g = 10 m/s^2

$\bar{y} = GG_1$

$\bar{h} = CC_1$

Sin θ = 1.5 / 3 = 0.5

θ = 30^o

$GG_1 = G_1A_1 + A_1G = G_1A_1 + AG\ Sin\ \theta$

GG_1 = 1.5 + (3/2) Sin 30 = 2.25 m

We know that the total pressure force is given by

$P = S_o\rho g A\bar{y}$ *= 0.85 x 1000 x 10 x 6 x 2.25 = 114.75 kN*

Centre of Pressure

The Centre of pressure is given by

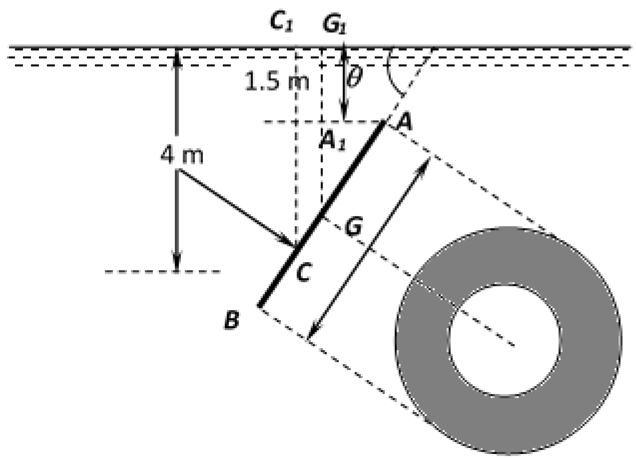
$$\bar{h} = \bar{y} + \frac{I_g}{A\,\bar{y}}\sin^2\theta$$

$$I_g = \frac{b\,d^3}{12} = \frac{2\times 3^3}{12} = 4.5\,m^4$$

$$\bar{h} = 2.25 + \frac{4.5}{6\times 2.25}\sin^2 30 = 2.33\,m$$

Q.14. A Circular plate with a concentric hole is immersed in water in such a way that its greatest and least depth below water surface are 4 m and 1.5 m respectively. Determine the total pressure on the plate and locate it if the diameter of the plate and hole are 3 m and 1.5 m respectively.

Solution:



Assume

ρ = 1000 kg/m^3 and g = 10 m/s^2

$$A = \frac{\pi}{4}\left(D^2 - d^2\right) = \frac{\pi}{4}\left(3^2 - 1.5^2\right) = 5.3014 \text{ m}^2$$

$\bar{y}$ = GG_1

$\bar{h}$ = CC_1

Sin θ = 2.5 / 3 = 0.833 and ***θ = 30°***

$GG_1 = G_1A_1 + A_1G = G_1A_1 + AG \sin\theta$

$GG_1 = 1.5 + (3/2)\, 0.833 = 2.75$ m

We know that the total pressure force is given by

$P = S_o \rho g A \bar{y}$ $= 1000 \times 10 \times 5.3014 \times 2.75 = 144.7885$ *kN*

Centre of Pressure

The Centre of pressure is given by

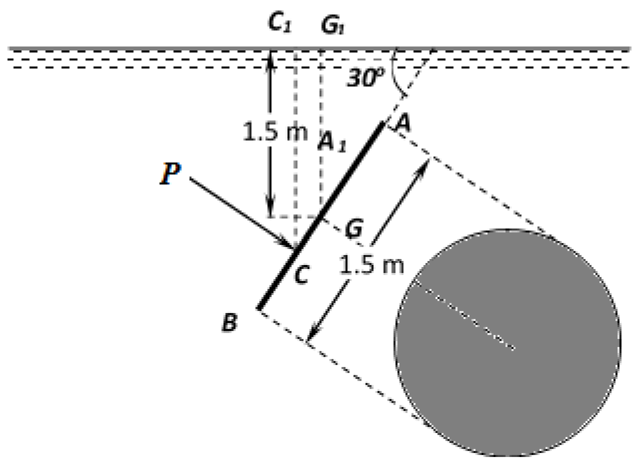
$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}}\sin^2\theta$$

$$I_g = \frac{\pi}{4}\left(R^4 - r^4\right) = \frac{\pi}{4}\left(1.5^4 - 0.75^4\right) = 3.728\,\text{m}^4$$

$$\bar{h} = 2.75 + \frac{3.728}{5.3014\times 2.75}\sin^2 30 = 2.814 \text{ m}$$

Q.15. A circular plate of dia 1.5 m is immersed in a liquid of relative density of 0.8 with its plane making an angle of 30^o with the horizontal. The centre of the plate is at a depth of 1.5 m below the free surface. Calculate the total force on one side of the plate and location of centre of pressure.

Solution:



Assume

ρ = 1000 kg/m^3 and g = 10 m/s^2

S_o = 0.80

$$A = \frac{\pi D^2}{4} = \frac{\pi \times 1.5^2}{4} = 1.767 \text{ m}^2$$

$\bar{y} = GG_1$

$\bar{h} = CC_1$

$\boldsymbol{\theta = 30^o}$

$\boldsymbol{GG_1 = G_1A_1 + A_1G = G_1A_1 + AG \sin\theta}$

$\boldsymbol{GG_1 = 1.5 + (3/2)\, 0.833 = 2.75 \text{ m}}$

We know that the total pressure force is given by

$\boldsymbol{P = S_o \rho g A \bar{y}} = 0.8 \times 1000 \times 10 \times 1.767 \times 2.75 = 38.874 \text{ kN}$

Centre of Pressure

The Centre of pressure is given by

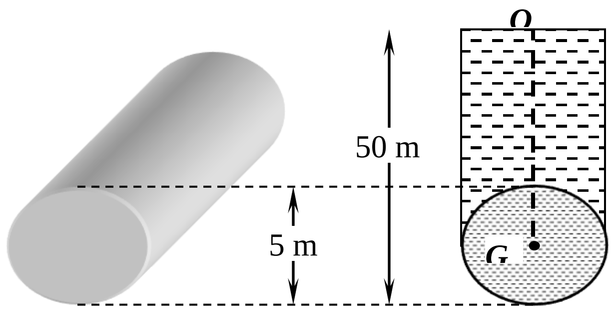
$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}} \sin^2\theta$$

$$I_g = \frac{\pi R^4}{4} = \frac{\pi \times 0.75^4}{4} = 0.2485 \text{ m}^4$$

$$\bar{h} = 2.75 + \frac{0.2485}{1.767 \times 2.75} \sin^2 30 = 2.763 \text{ m}$$

Q.16 A vertical gate closes a circular tunnel of 5 m diameter running full of water, the pressure at the bottom of the gate is 0.5 MPa. Determine the hydrostatic force and the position of centre of pressure.

Solution: Assume ρ = 1000 kg/m^3 and g = 10 m/s^2



Pressure intensity at the bottom of the gate is = $\boldsymbol{p = S_o \rho g y}$

Where $\boldsymbol{y}$ is the depth of point from the free surface.

$\boldsymbol{0.5 \times 10^6 = 1000 \times 10 \times y}$

$\boldsymbol{y = 50\ m}$

Hence the free surface of water is at 50 m from the bottom of the gate

$$A = \frac{\pi D^2}{4} = \frac{\pi \times 5^2}{4} = 19.635\ \text{m}^2$$

$\bar{y} = OG = \boldsymbol{50 - 2.5 = 47.5\ m}$

We know that the total pressure force is given by

$\boldsymbol{P = S_o \rho g A}\bar{y} = 1000 \times 10 \times 19.635 \times 47.5 = 9326.625\ kN$

Centre of Pressure

The Centre of pressure is given by

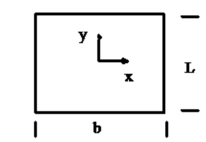
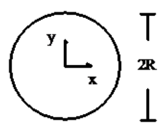
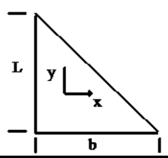
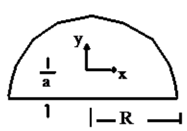
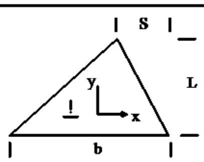
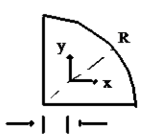
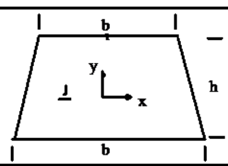
$$\bar{h} = \bar{y} + \frac{I_g}{A\bar{y}}$$

$$I_g = \frac{\pi R^4}{4} = \frac{\pi \times 2.5^4}{4} = 30.68\,\text{m}^4$$

$$\bar{h} = 47.5 + \frac{30.68}{19.635 \times 47.5} = 47.533\ \text{m}$$

i.e. 50.0 – 47.533 = 2.677 m from the bottom of the gate or tunnel.

PROPERTIES OF PLANE SECTIONS

Geometry	Centroid	Moment of Inertia I_{xx}	Product of Inertia I_{xy}	Area
	$b/2,\ L/2$	$\dfrac{bL^3}{12}$	0	$b \cdot L$
	$0, 0$	$\dfrac{\pi R^4}{4}$	0	πR^2
	$b/3,\ L/3$	$\dfrac{bL^3}{36}$	$-\dfrac{b^2L^2}{72}$	$\dfrac{b \cdot L}{2}$
	$0, a = \dfrac{4R}{3\pi}$	$R^4\left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right)$	0	$\dfrac{\pi R^2}{2}$
	$a = \dfrac{L}{3}$	$\dfrac{bL^3}{36}$	$\dfrac{b(b-2s)L^2}{72}$	$\dfrac{1}{2} b \cdot L$
	$a = \dfrac{4R}{3\pi}$	$\left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)R^4$	$\left(\dfrac{1}{8} - \dfrac{4}{9\pi}\right)R^4$	$\dfrac{\pi R^2}{4}$
	$a = \dfrac{h(b+2b_1)}{3(b+b_1)}$	$\dfrac{h^3(b^2+4bb_1+b_1^2)}{36(b+b_1)}$	0	$(b+b_1)\dfrac{h}{2}$

Fluid Specific Weight

	lbf/ft^3	N/m^3		lbf/ft^3	N/m^3
Air	.0752	11.8	Seawater	64.0	10,050
Oil	57.3	8,996	Glycerin	78.7	12,360
Water	62.4	9,790	Mercury	846.	133,100
Ethyl	49.2	7,733	Carbon	99.1	15,570

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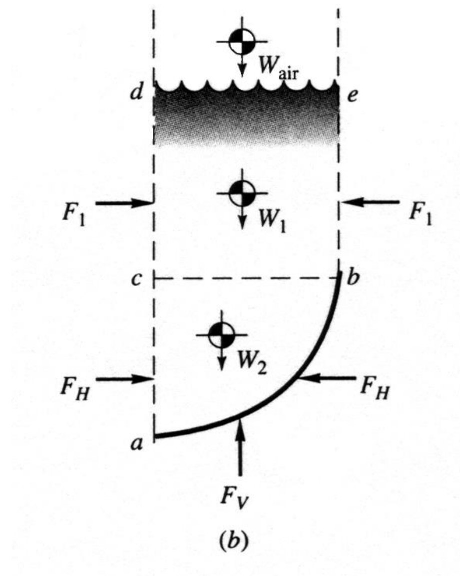
2.4 Hydrostatic Forces on Curved Surfaces

Since this class of surface is curved, the direction of the force is different at each location on the surface. Therefore, we will evaluate the x and y components of net hydrostatic force separately.

Consider curved surface, a-b. Force balances in x & y directions yield

$F_h = F_H$

$Fv = W_{air} + W_1 + W_2$



(b)

From this force balance, the basic rules for determining the horizontal and vertical component of forces on a curved surface in a static fluid can be summarized as follows:

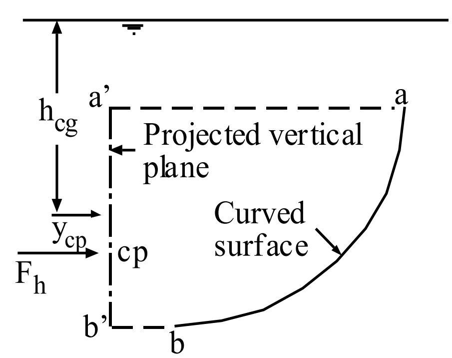
Horizontal Component, F_h

The horizontal component of force on a curved surface equals the force on the plane area formed by the projection of the curved surface onto a vertical plane normal to the component.

The horizontal force will act through the c.p. (not the centroid) of the projected area.

from the Diagram:

All elements of the analysis are performed with the vertical plane. The original curved surface is important only as it is used to define the projected vertical plane.



Therefore, to determine the horizontal component of force on a curved surface in a hydrostatic fluid:

Vertical Component - Fv

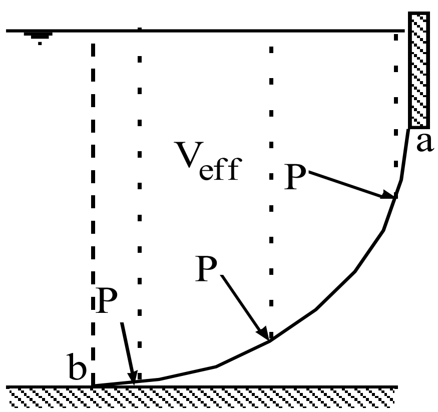
The vertical component of force on a curved surface equals the weight of the effective column of fluid necessary to cause **the pressure on the surface**.

The use of the words **effective column of fluid** is important in that there may not always actually be fluid directly above the surface. (See graphics below)

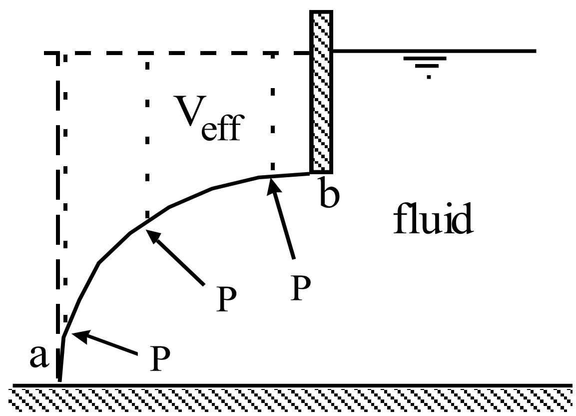
This effective column of fluid is specified by identifying the column of fluid that would be required to cause the pressure at each location on the surface.

Thus, to identify the Effective Volume - V_{eff}:

$$\mathbf{F_V = V_{eff}}$$



Fluid above the surface

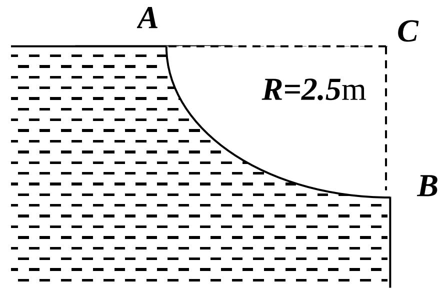


No fluid actually above surface

$$R = \sqrt{\left(\sum F_x^2\right) + \left(\sum F_y^2\right)} \qquad \theta = \tan^{-1}\left(\frac{\sum F_y}{\sum F_x}\right)$$

Q.17 Find the horizontal and vertical component of force and its point of application due to water per meter length of the gate AB having a quadrant shape of radius 2.5 m shown in Fig. Find also the resultant force in magnitude and direction.

Solution:



Assume

ρ = 1000 kg/m^3 and g = 9.81 m/s^2

R = 2.5 m, Width of gate = 1 m

Horizontal force F_x

F_h = Force on the projected area of the curved surface on the vertical plane

= Force on ***BC***

$A = 2.5 \times 1 = 2.5\,\text{m}^2$

$$\bar{y} = \frac{2.5}{2} = 1.25\,\text{m}$$

$F = \gamma_{water} A h_c = 9810 \times 2.5 \times 1.25 = 30656$ N = 30.656kN

This will act at a distance $\bar{h} = \frac{2}{3} \times 2.5 = \frac{5}{3}\,\text{m}$ from the free surface of liquid AC

Vertical Force F_y

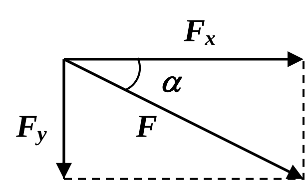
F_y= Weight of water (imaginary) supported by ***AB***

= γ_{water} x Area of ***ACB*** x Length of gate

$$= 9810 \times \frac{\pi \times 2.5^2}{4} \times 1 = 48154\,N = 48.154\,kN$$

This will act at a distance $\bar{x} = \frac{4 \times 2.5}{3\pi} = 1.061\,\text{m}$ from ***CB***

The Resultant force

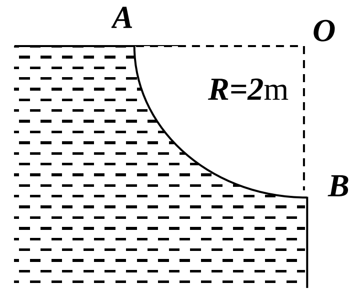


$F = \sqrt{F_x^2 + F_y^2} = \sqrt{30.656^2 + 48.154^2} = 57.084\,kN$ and its inclination is given by

$$\alpha = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{48.154}{30.656} = 57.51^{o}$$

Q.18 Find the horizontal and vertical component of force and its point of application due to water per meter length of the gate AB having a quadrant shape of radius 2m shown in Fig. Find also the resultant force in magnitude and direction.

Solution:



Assume

ρ = 1000 kg/m^3 and g = 10 m/s^2

R = 2 m, Width of gate = 1 m

Horizontal force F_x

F_x = Force on the projected area of the curved surface on the vertical plane

= Force on ***BO = P = $S_o \rho g A \bar{y}$***

A = 2 x 1 = 2 m^2

$\bar{y} = \frac{2}{2} = 1\,\text{m}$

F_x*= 1000 x 10 x 2 x 1 = 20 kN*

This will act at a distance $\bar{h} = \frac{2}{3} \times 2 = \frac{4}{3}\,\text{m}$ from the free surface of liquid

Vertical Force F_y

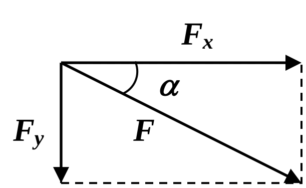
F_y= Weight of water (imaginary) supported by ***AB***

= ***$S_o \rho g$*** *x* Area of ***AOB****x* Length of gate

= ***1000 x 10 x*** $\frac{\pi \times 2^2}{4}$ ***x1= 31.416 kN***

This will act at a distance $\bar{x} = \frac{4 \times 2}{3\pi} = 0.848\ \text{m}$ from ***OB***

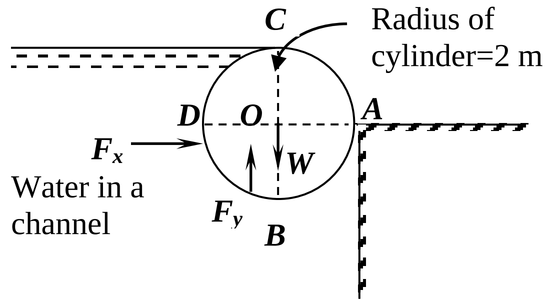
Resultant force $F = \sqrt{F_x^2 + F_y^2} = \sqrt{20^2 + 31.426^2} = 37.25$ ***kN*** and its inclination is given by



$$\alpha = \tan^{-1}\left[\frac{F_y}{F_x}\right] = \tan^{-1}\left[\frac{31.426}{20}\right] = 57.527^o$$

Q.19. A cylinder holds water in a channel as shown in Fig. Determine the weight of 1 m length of the cylinder.

Solution:



Radius of Cylinder = R = 2m

Length of cylinder = 1 m

Weight of Cylinder = W

Horizontal force exerted by water= F_x

F_x = Force on vertical area ***BOC***

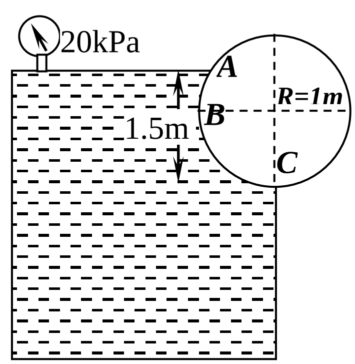
$= S_o \rho g A \bar{y} = 1000 \times 10 \times (4 \times 1) \times (2/2) = 40\ kN\ (\rightarrow)$

The vertical force exerted by water=F_y =Weight of water enclosed in ***BDCOB***

$$F_y = S_o \rho g \left(\frac{\pi \times 2^2}{4}\right) x L = 1000 x 10 x 3.142 = 31.416 kN\ (\uparrow)$$

For equilibrium of the cylinder the weight of the cylinder must be equal to the force exerted by the water on the cylinder. Hence, the weight of the cylinder is 31***.416 kN*** per meter length**.**

Q.20. Fig. shows the cross section of a tank full of water under pressure. The length of the tank is 2 m. An empty cylinder lies along the length of the tank on one of its corner as shown. Find the resultant force acting on the curved surface of the cylinder.



Solution:

R=1 m

L = 2 m

$p = \rho g h = 1000 \times 10 \times h = 20 \times 10^3$

$h = 2$ m

For this pressure, the free surface should be 2 m above A

Horizontal component of force F_x

$\boldsymbol{F_x = S_o \rho g A \bar{y}}$

$A = 1.5 \times 2.0 = 3 \text{ m}^2$

$$\bar{y} = 2 + \frac{1.5}{2} = 2.75 \text{ m}$$

$F_x = 1000 \times 10 \times 3.0 \times 2.75 = 82.5 \text{ kN } (\rightarrow)$

The vertical force exerted by water = $\boldsymbol{F_y}$

$\boldsymbol{F_y}$ = Weight of water enclosed in ***ABC***

= Weight of water enclosed in ***CODEABC***

= Weight of water enclosed in (***CODFBC – AEFB***)

But Weight of water enclosed in ***CODFBC***

= Weight of water enclosed in (***COB+ODFBO)***

$$= \rho g \left[\frac{\pi R^2}{4} + BO \times OD \right] \times 2 = 1000 \times 10 \left[\frac{\pi \times 1^2}{4} + 1 \times 2.5 \right] \times 2 = 65.708 \text{ kN}$$

Weight of water in***AEFB = $S_o \rho g$[***Area of ***AEFB]x 2.0***

= ***$S_o \rho g$[***Area of (***AEFG+AGBH-AHB]x 2.0***

sin θ = AH/AO = 0.5/1.0 = 0.5. ∴ θ = 30°

BH = BO – HO = 1.0 – AO cos θ = 1.0 – 1 ***x*** cos 30° = 0.134

Area ABH = Area ABO – Area AHO

$$= \pi R^2 \times \frac{30}{360} - \frac{AH \times HO}{2.0} = \pi \times 1^2 \times \frac{1}{12} - \frac{0.5 \times 0.866}{2.0} = 0.0453$$

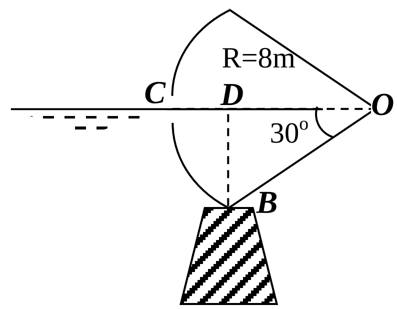
∴ Weight of water in AEFB = 1000***x***10[AE***x***AG+AG***x***AH–0.0453] ***x*** 0.2

= 1000***x***10[2.0***x***0.134+0.134***x***0.5–.0453] ***x*** 0.2

= 5794 N

F_y =65708 – 5794 = 59914 N (Ans)

Q.21. Calculate the resultant water pressure on the Tainter gate of radius 8 m and width unity as shown in Fig.



Solution:

Horizontal component of force F_x

$F_x = S_o \rho g A \bar{y}$

$DB = OB \sin 30 = 8 \text{ x } 0.5 = 4.0$ m

$A = 4 \text{x } 1.0 = 4 \text{ m}^2$

$$\bar{y} = \frac{4}{2} = 2\,\text{m}$$

The Horizontal force exerted by water = F_x

$F_x = 1000 \text{ x } 10 \text{ x } 4.0 \text{x } 2.0 = 80.0 \text{ kN } (\rightarrow)$

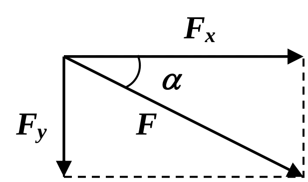
The vertical force exerted by water = F_y

$F_y =$ Weight of water enclosed in ***CDBC***

= Weight of water enclosed in (***CDOBC – DOB***)

$$= S_o \rho g \left[\pi R^2 \times \frac{30}{360} - \frac{BD \times DO}{2.0} \right] = 1000 \times 10 \left[\pi \times 8^2 \times \frac{1}{12} - \frac{4.0 \times 8.8 \cos 30}{2.0} \right] = 15.13\,\text{kN}$$

Resultant force $F = \sqrt{F_x^2 + F_y^2} = \sqrt{80^2 + 15.13^2} = 81.418\,\text{kN}$



kN and its inclination is given by

$$\alpha = \tan^{-1}\left[\frac{F_y}{F_x}\right] = \tan^{-1}\left[\frac{15.13}{80}\right] = 10.71^o$$

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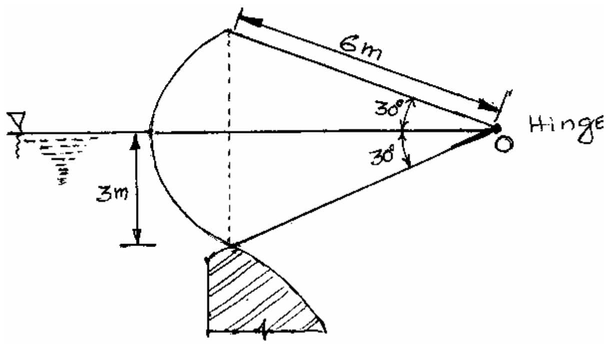
Q.22 Length of a Tainter gate perpendicular to paper is 0.50m. Find:

i) Total horizontal thrust of water on gate.

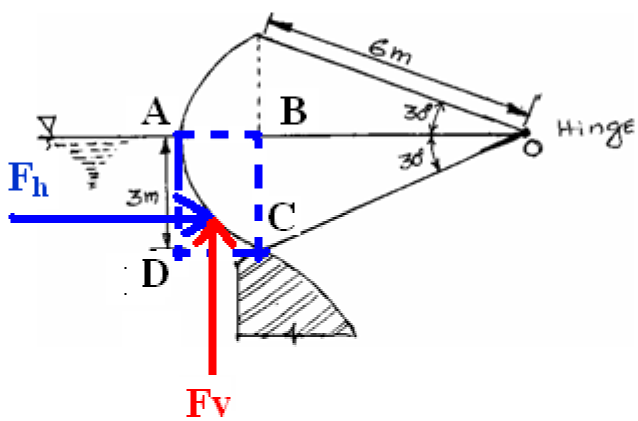
ii) Total vertical component of water pressure against gate.

iii) Resultant water pressure on gate and its inclination with horizontal.



Ans: Given L = 0.5m,
AD = BC = 3m, γW = 9.81 kN/m^3

(i) Total horizontal thrust of water on gate



$$Fh = \gamma W \times A \times \bar{h}$$

$$Fh = 9.81 \times (3.0 \times 0.5) \times \frac{3}{2}$$

$$Fh = 22.07 \text{ kN} \rightarrow \text{Rightward}$$

Acting at

$$\overline{\mathbf{h}_{c.p.}} = \mathbf{h}_c + \frac{\mathbf{I_G} \times \mathbf{Sin}^2 90°}{\mathbf{A} \times \mathbf{h}_c}$$

$$\overline{\mathbf{h}_{c.p.}} = 1.5 + \frac{\left(\frac{0.5 \times 3^3}{12}\right) \times \mathbf{Sin}^2 90°}{(3.0 \times 0.5) \times 1.5} = 1.5 + 0.5 = 2.0\mathbf{m}$$

(ii) Total vertical component of water pressure against gate = upward thrust due area ABC

Upward thrust due area ABC = Area AOC - ΔOBC

Area ABC = $\frac{\pi\times \mathbf{R}^2}{12}-\frac{1}{2}\times\mathbf{OB}\times\mathbf{BC}$

Area ABC = $\frac{\pi\times 6^2}{12}-\frac{1}{2}\times 3\cos 30°\times 3$

Area ABC = 1.636 m2

Fv = γW × Area ABC × L

Fv = 9.81 × 1.636 × 0.5 = 8.024 kN $\uparrow$ upward

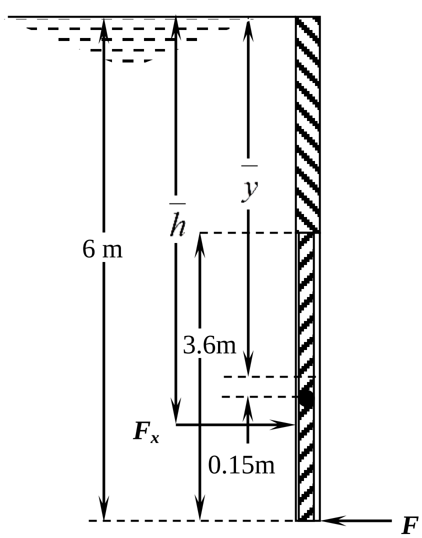
(iii) Resultant water pressure on gate and its inclination with horizontal

$$\mathbf{R}=\sqrt{\mathbf{F}_\mathbf{h}^2+\mathbf{F}_\mathbf{v}^2}=\sqrt{(22.07)^2+(8.024)^2}=23.48\quad \mathbf{kN}$$

$$\theta=\tan^{-1}\left(\frac{8.024}{22.07}\right)=0.3637$$

Inclination $\theta = 20°$

Q23. A 3.6 m x 1.5 m wide rectangular gate MN is vertical and is hinged at point 150 mm below the centre of gravity of the gate. The total depth of water is 6 m. What horizontal force must be applied at the bottom of the gate to keep the gate closed?



Solution:

Total pressure acting on the gate is F_x

$F_x = S_o\rho g A\bar{y}$

= 1000 x 10 x (3.6 x 1.5) x (6-3.6/2)

= 226.8 kN

Acting at

$$\bar{h}=\bar{y}+\frac{I_g}{A\bar{y}}$$

$$I_g = \frac{bd^3}{12} = \frac{1.5 \times 3.6^3}{12} = 5.832\ \mathrm{m}^4$$

$$\bar{h} = 4.2 + \frac{5.832}{5.4 \times 4.2} = 4.457\ \mathrm{m}$$

Let ***F*** be the force applied at the bottom of the gate required to retain the gate in equilibrium.

From the conditions of equilibrium, taking moments about the hinge, we get

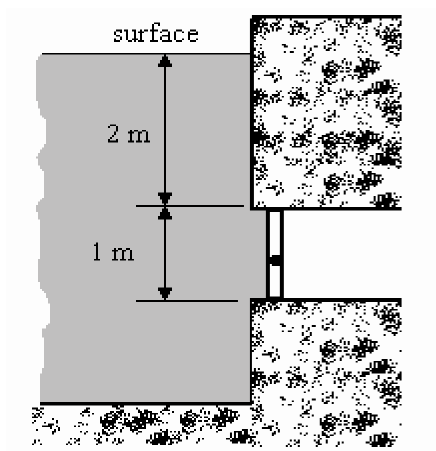
F (1.8 – 0.15) = F_x [4.457-(4.2+0.15)]

F = 14.707 kN (Ans).

Q.24 *A culvert in the side of a reservoir is closed by a vertical rectangular gate 2m wide and 1m deep as shown in figure. The gate is hinged about a horizontal axis which passes through the centre of the gate. The free surface of water in the reservoir is 2.5 m above the axis of the hinge. The density of water is 1000 kg/m^3. Assuming that the hinges are frictionless and that the culvert is open to atmosphere, determine*

(i) The force acting on the gate when closed due to the pressure of water.

(ii) The moment to be applied about the hinge axis to open the gate.



Solution: (i) The total hydrostatic force

$$F = \gamma A\ h_c$$

$$\gamma_{\text{water}} = 1000\ \text{x}9.81 = 9810\ \mathrm{N/m^3}$$

Given **A = 1m X 2m = 2m^2**

$$h_c = 2 + \frac{1}{2} = 2.5\text{m}$$

$$F = 9810\text{X}2\text{X}2.5 = 49050\text{N}$$

(ii) The moment applied about hinge axis to open the gate is say 'M'

The centre of pressure ($h_{c.p}$):

From the above $h_c = 2.5m$, $A = 2m^2$

$$(I_c)_{xx} = \frac{BD^3}{12} = \frac{2X1^3}{12} = 0.167m^4$$

$$h_{C.P.} = h_c + \frac{(I_c)_{xx}}{Ah_c} = 2.5 + \frac{0.167}{2X2.5} = 2.53334 \text{ m}$$

Distance of Hydrostatic force (F) from the water surface = 2.5334m.

Distance of hinge from free surface = 2.5m

Distance between hinge and centre of pressure of force 'F' = 2.5334 m - 2.5m = 0.0334m

Taking moment about Hinge to open the gate 'M' = F X 0.0334 = 49050 N X 0.0334 m

The moment applied about hinge axis to open the gate 'M' = 1638.27 N-m

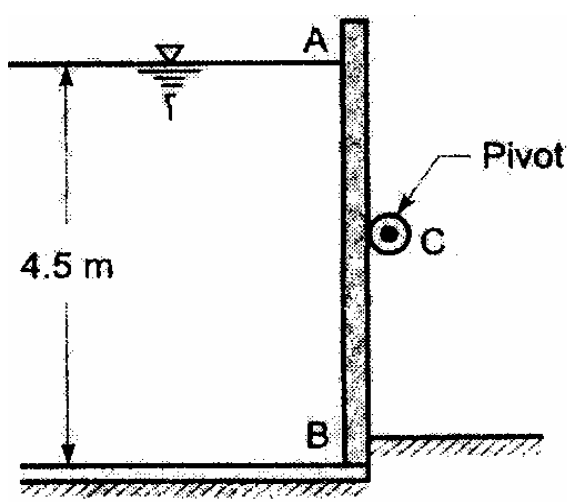
Q.25 Figure shows a rectangular flash board AB which is 4.5m high and is pivoted at C. What must be the maximum height of C above B so that the flash board will be on the verge of tipping when water surface is at A? Also determine if the pivot of the flash board is at a height h =1.5m, the reactions at B and C when the water surface is 4m above B.

Ans:

(i) The flash board would tip about the hinge point 'C' when the line of action of resultant 'R' pressure force 'F' lies from C to A anywhere on the board.

The limiting condition being the situation when the resultant force 'F' passes through 'C'

The resultant force 'F' also passes through the centroid of the pressure diagram and the centre lies at $\frac{1}{3} \times AB = \frac{4.5}{3} = 1.5m$



Hence the maximum height of 'C' from 'B' = (4.5m-3.0m) =1.5m (from bottom)

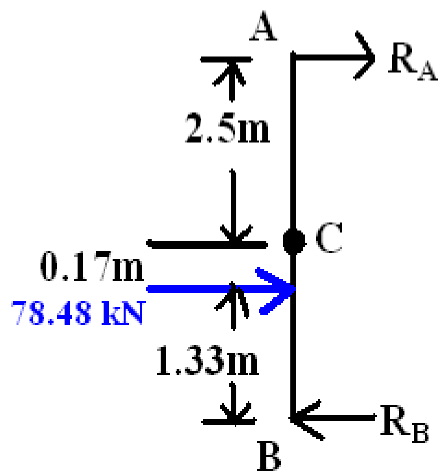
(ii) The pivot of the flash board is at a height h =1.5m from point B, the reactions at B and C when the water surface is 4m above B.

$$\bar{h} = \frac{4.0}{2} = 2.m$$

Hydrostatic force $P = \rho g A\bar{h}$ = 1000×9.81×(4.0×1.0)×2=78.48 kN acting at $\overline{h_{cp}}$

$$h_{cp} = 2.0 + \frac{1\times(4.0)^3 Sin^2 90°}{4.0\times 2.0} = 2.67m \ from \ free \ water \ surfcae$$

Or h = (4.0-2.67) = 1.33m from bottom



Let R_A and R_B be the reaction.

R_A + 78.48 = R_B
by taking moment about pivot 'C'

R_A ×2.5 + 78.48×0.17 = R_B×1.5

On solving R_A =104.38kN R_B = 182.86 kN

2.5 Gravity Dam:

A **gravity dam** is a dam constructed from concrete or stone masonry and designed to hold back water by primarily utilizing the weight of the material alone to resist the horizontal pressure of water pushing against it. Gravity dams are designed so that each section of the dam is stable, independent of any other dam section

Gravity dams generally require stiff rock foundations of high bearing strength (slightly weathered to fresh); although they have been built on soil foundations in rare cases. The bearing strength of the foundation limits the allowable position of the resultant which influences the overall stability. Also, the stiff nature of the gravity dam structure is unforgiving to differential foundation settlement, which can induce cracking of the dam structure.

Gravity dams provide some advantages over embankment dams. The main advantage is that they can tolerate minor over-topping flows as the concrete is resistant to scouring. This reduces the requirements for a cofferdam during construction and the sizing of the spillway. Large overtopping flows are still a problem, as they can scour the foundations if not accounted for in the design. A disadvantage of gravity dams is that due to their large footprint, they are susceptible to uplift pressures which act as a de-stabilising force. Uplift pressures (buoyancy) can be reduced by internal and foundation drainage systems which reduces the pressures.

2.5.1 Forces Acting on Gravity Dams:

Forces that act on a gravity dam (Fig.1) are due to:

- Water Pressure(Hydrostatic)
- Uplift Pressure
- Earthquake Acceleration
- Silt Pressure
- Wave Pressure
- Ice Pressure

>> Self Weight (W) counters the forces listed above.

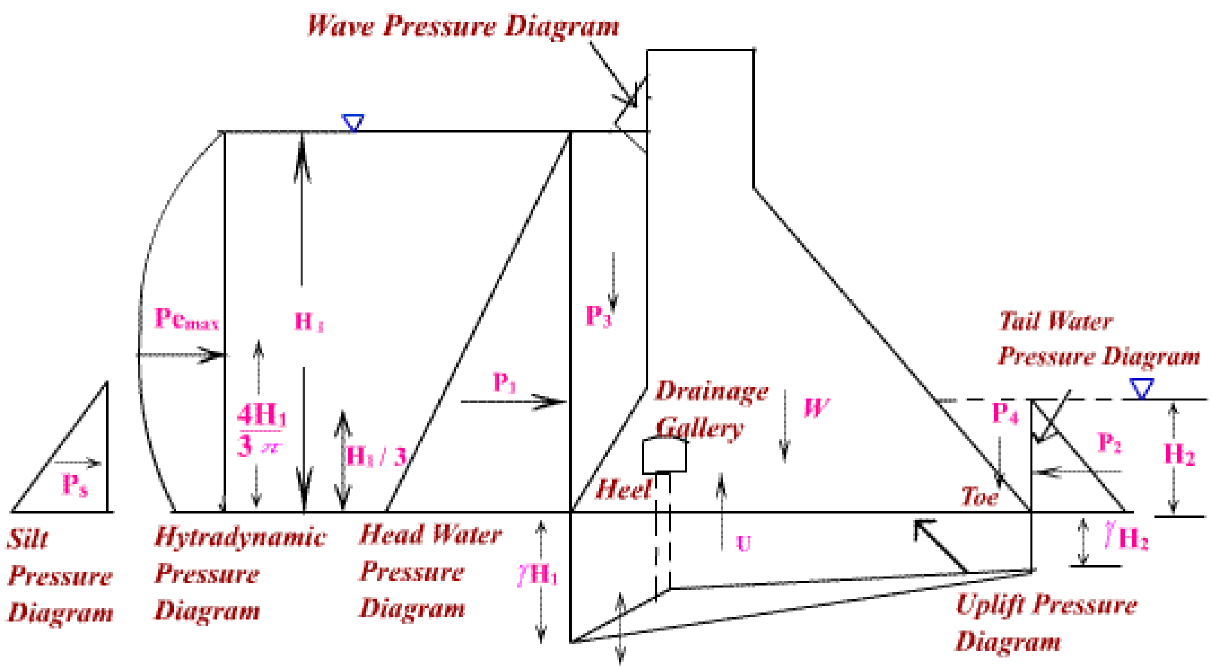


Fig. Forces on Gravity Dams

- **Force due to hydrostatic Pressure:**

Force due to hydrostatic Pressure is the major external force on a gravity dam. The intensity of pressure from zero at the water surface to the maximum (γH) at the base. The force due to this pressure is given by γH2, acting at H/3 from the base. In Fig.1, the forces P1 and P2 are due to hydrostatic pressure acting on the upstream and the downstream sides respectively. These are horizontal components of the hydrostatic force due to head water (upstream side) and tail water (downstream side) of the dam respectively.

The forces marked as P3 and P4 are the weight of water held over the inclined faces of the dam on the upstream slope and downstream slope respectively. These are the respective vertical components of the hydrostatic force on the two faces mentioned.

- **Force due to Uplift Pressure:**

Water that seeps through the pores, cracks and fissures of the foundation material and water that seeps through the body of the dam to the bottom through the joints between the body of the dam and the foundation at the base, exert an uplift pressure on the base of the dam. The force (U) due to this acts against the weight of the dam and thus contributes to destabilizing the dam.

According to the recommendation of the United States Bureau of Reclamation (USBR), the uplift pressure intensities at the heel (upstream end) and the toe (downstream end) are taken to be equal to the respective hydrostatic pressures. A linear variation of the uplift pressure is often assumed between the heel and the toe. Drainage galleries can be provided (Fig.) to relieve the uplift pressure. In such a case, the uplift pressure diagram gets modified as shown in Fig.

- **Earthquake Forces:**

The effect of an earthquake is perceived as imparting an acceleration to the foundations of the dam in the direction in which the wave travels at that moment. It can be viewed (resolved) as horizontal and vertical components of the random acceleration.

2.6 Lock Gates

Whenever a dam or a weir is constructed across a river or canal, the water levels on both the sides of the dam will be different. If it is desired to have navigation or boating in such a river or a canal, then a chamber, known as lock, is constructed between these two different water levels. Two sets of gates (one on the upstream side and the other on downstream side of the dam) are provided as shown in fig - 1.

Fig-1 : Lock Gate

(Source: http://www.codecogs.com/library/engineering/fluid_mechanics/water_pressure/lock-gate.php)

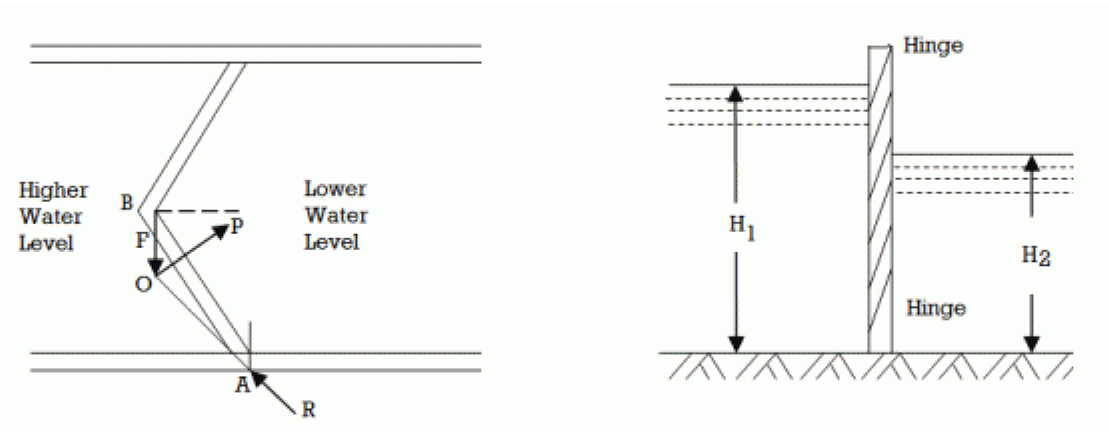


Fig-2(a) : Plan of lock gate   Fig-2(b) : Elevation of lock gate

Now consider a set of lock gates AB and BC hinged at the top and bottom at A and C respectively as shown in fig - 2(a). These gates will be held in contact at b by the water pressure, the water level being higher on the left hand side of the gates as shown in fig - 2(b).

Let,

- P = Water pressure on the gate AB or BC acting at right angles on it
- F = Force exerted by the gate BC acting normally to the contact surface of the two gates AB and BC (also known as reaction between the two gates), and
- R = Reaction at the upper and lower hinge

Since the gate AB is in equilibrium, under the action of the above three forces, therefore they will meet at one point. Let,P and F meet at O, then R must pass through this point.
Let, α = Inclination of the lock gate with the normal to the walls of the lock.
From the geometry of the figure ABO, we find that it is an isosceles triangle having its angles $\angle$ OBA and $\angle$ OAB both equal to α.

$$R\cos\alpha = F\cos\alpha$$
$$\therefore R = F \qquad (1)$$

and now resolving the force at right angles to AB

$$P = R\sin\alpha + F\sin\alpha = 2R\sin\alpha$$
$$\therefore R = \frac{P}{2\sin\alpha}$$
$$\therefore F = \frac{P}{2\sin\alpha} \qquad (2)$$

Now let us consider the water pressure on the top and bottom hinges of the gate, Let,

- H1 = Height of water to the left side of the gate.
- A1 = Wetted area (of one of the gates) on left side of the gate
- P1 = Total pressure of the water on the left side of the gate
- H2, A2, P2 = Corresponding values for right side on the gate
- RT = Reaction of the top hinge, and
- RB = Reaction of bottom hinge

Since the total reaction (R) will be shared by the two hinges (RT), therefore

$$R = R_T + R_B \qquad (3)$$

and total pressure on the lock gate,

$$P = wA\bar{x}$$
$$\Rightarrow P_1 = wA_1 \times \frac{H_1}{2} = \frac{wA_1H_1}{2}$$

Similarly, $P_1 = \frac{wA_2H_2}{2}$

Since the directions of P1 and P2 are in the opposite direction, therefore the resultant pressure,

$$P = P_1 - P_2$$

We know that the pressure P1 will act through its center of pressure, which is at a height of $\frac{H_1}{3}$ from the bottom of the gate. Similarly, the pressure P2 will also act through its center of pressure which is also at a height of $\frac{H_2}{3}$ from the bottom of the gate.
A little consideration will show, that half of the resultant pressure (i.e., P1 - P2 or P)will be resisted by the hinges of one lock gate (as the other half will be resisted by the other lock gates).

$$R_T \sin\alpha \times h = (\frac{P_1}{2} \times \frac{H_1}{3}) - (\frac{P_2}{2} \times \frac{H_2}{3}) \qquad (4)$$

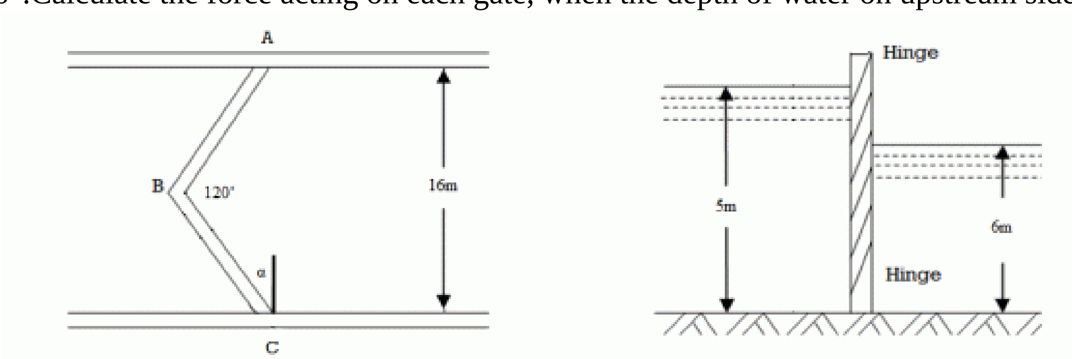
where h is the distance between the two hinges.

Also resolving the forces horizontally,

$$P_1 - P_2 = R_B \sin\alpha + R_T \sin\alpha \qquad (5)$$

From equations (4) and (5) the values of RB and RT may be found out.

Q. 26 Two lock gates of 7.5m height are provided in a canal of 16m width meeting at an angle of 120°.Calculate the force acting on each gate, when the depth of water on upstream side is 5m.



Given,

- Height of lock gates = 7.5m
- Width of lock gates = 16m
- Inclination of gates = $120°$
- H = 5m

From the geometry of the lock gate, we find that inclination of the lock gates with the walls,

$$\alpha = \frac{180° - 120°}{2} = 30°$$

and

width of each gate = $\frac{16/2}{\cos\alpha}$ = $\frac{8}{\cos 30°}$ = 9.24 m

$\therefore$ Wetted area of each gate, $A = 5 \times 9.24 = 46.2m^2$ and force acting on each gate,

$$P = wA \times \frac{H}{2} = 9.81 \times 46.2 \times \frac{5}{2} = 1133\ KN$$

15 CV 33 FLUID MECHANICS NOTES

MODULE-2

- **Module-2A : Hydrostatic forces on Surfaces**
- **Module-2B :Fundamentals of fluid flow (Kinematics)**

by

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Module-2B: Fundamentals of fluid flow (Kinematics)
Introduction. Methods of describing fluid motion. Velocity and Total acceleration of a fluid particle. Types of fluid flow, Description of flow pattern. Basic principles of fluid flow, three-dimensional continuity equation in Cartesian coordinate system. Derivation for Rotational and irroational motion. Potential function, stream function, orthogonality of streamlines and equipotential lines. Numerical problems on Stream function and velocity potential. Introduction to flow net.

2.7 Methods of Describing Fluid Motion:

Fluid kinematics refers to the features of a fluid in motion. It only deals with the motion of fluid particles without taking into account the forces causing the motion. Considerations of velocity, acceleration, flow rate, nature of flow and flow visualization are taken up under fluid kinematics.

A fluid motion can be analyzed by one of the two alternative approaches, called Lagrangian and Eulerian.

In Lagrangian approach, a particle or a fluid element is identified and followed during the course of its motion with time as demonstrated in

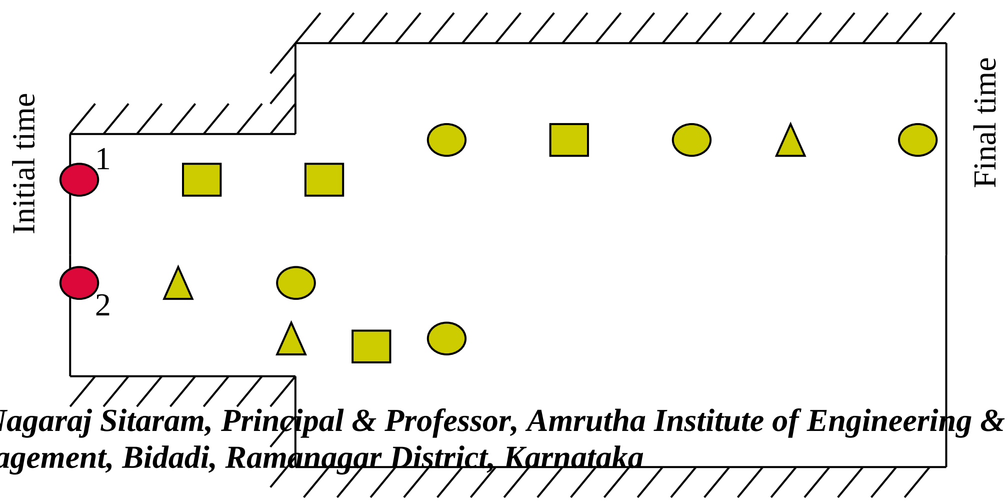


Fig. Lagrangian Approach (Study of each particle with time)

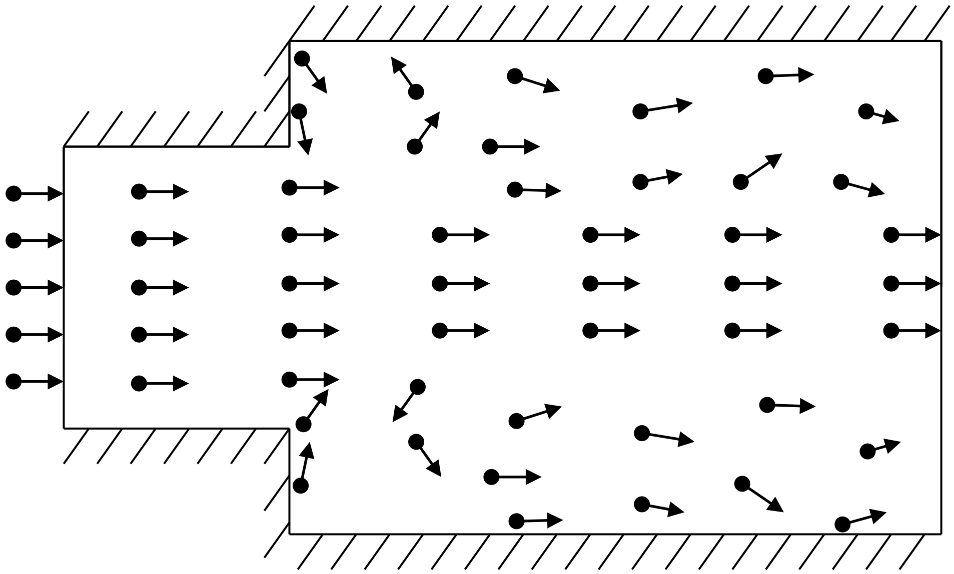


Fig. Eulerian Approach (Study at fixed station in space)

Example: To know the attributes of a vehicle to be purchased, you can follow the specific vehicle in the traffic flow all along its path over a period of time.

Difficulty in tracing a fluid particle (s) makes it nearly impossible to apply the Lagrangian approach. The alternative approach, called Eulerian approach consists of observing the fluid by setting up fixed stations (sections) in the flow field (Fig.).

Motion of the fluid is specified by velocity components as functions of space and time. This is considerably easier than the previous approach and is followed in Fluid Mechanics.

Example: Observing the variation of flow properties in a channel like velocity, depth etc, at a section.

2.8 Velocity

Velocity of a fluid along any direction can be defined as the rate of change of displacement of the fluid along that direction.

$$u = \frac{dx}{dt}$$

Where dx is the distance traveled by the fluid in time dt.

Velocity of a fluid element is a vector, which is a function of space and time.

Let V be the resultant velocity of a fluid along any direction and u, v and w be the velocity components in x, y and z-directions respectively.

Mathematically the velocity components can be written as

$u = f(x, y, z, t)$

$v = f(x, y, z, t)$

$w = f(x, y, z, t)$

and $V = ui + vj + wk = \qquad |V| = \sqrt{u^2 + v^2 + w^2}$

Where $u = \frac{dx}{dt}; v = \frac{dy}{dt}, w = \frac{dz}{dt}$

2.9 *Acceleration*

Acceleration of a fluid element along any direction can be defined as the rate of change of velocity of the fluid along that direction.

If a_x , a_y and a_z are the components of acceleration along-x, y and z- directions respectively, they can be mathematically written as

$$a_x = \frac{du}{dt}$$

But u = f (x, y, z, t) and hence by chain rule, we can write,

$$a_x = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

Similarly

$$a_y = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} + \frac{\partial v}{\partial z}\frac{dz}{dt} + \frac{\partial v}{\partial t}$$

and $$a_z = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \frac{\partial w}{\partial t}$$

But $u = \frac{dx}{dt}; v = \frac{dy}{dt}, w = \frac{dz}{dt}$

Hence

$$\left.\begin{aligned}
a_x &= \overbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}^{\text{Convective accln}} + \overbrace{\frac{\partial u}{\partial t}}^{\text{Local accln}} \\
a_y &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\
a_z &= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}
\end{aligned}\right\} \text{Total Acceleration}$$

If A is the resultant acceleration vector, it is given by

$$\boldsymbol{A} = \boldsymbol{a}_x \boldsymbol{i} + \boldsymbol{a}_y \boldsymbol{j} + \boldsymbol{a}_z \boldsymbol{k}$$

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a}_x^2 + \boldsymbol{a}_y^2 + \boldsymbol{a}_z^2}$$

For steady flow, the local acceleration will be zero

Problems

2.10 Types of fluid flow

2.10.1 Steady and unsteady flows:

A flow is said to be steady if the properties (P) of the fluid and flow do not change with time (t) at any section or point in a fluid flow.

$$\frac{\partial}{\partial t}(P) = 0$$

A flow is said to be unsteady if the properties (P) of the fluid and flow change with time (t) at any section or point in a fluid flow.

$$\frac{\partial}{\partial t}(P) \neq 0$$

Example: Flow observed at a dam section during rainy season, wherein, there will be lot of inflow with which the flow properties like depth, velocity etc.. will change at the dam section over a period of time representing it as unsteady flow.

2.10.2. Uniform and non-uniform flows:

A flow is said to be uniform if the properties (P) of the fluid and flow do not change (with direction) over a length of flow considered along the flow at any instant.

$$\frac{\partial}{\partial x}(P) = 0$$

A flow is said to be non-uniform if the properties (P) of the fluid and flow change (with direction) over a length of flow considered along the flow at any instant.

$$\frac{\partial}{\partial x}(P) \neq 0$$

Example Flow observed at any instant, at the dam section during rainy season, wherein, the flow varies from the top of the overflow section to the foot of the dam and the flow properties like depth, velocity etc., will change at the dam section at any instant between two sections, representing it as non-uniform flow.

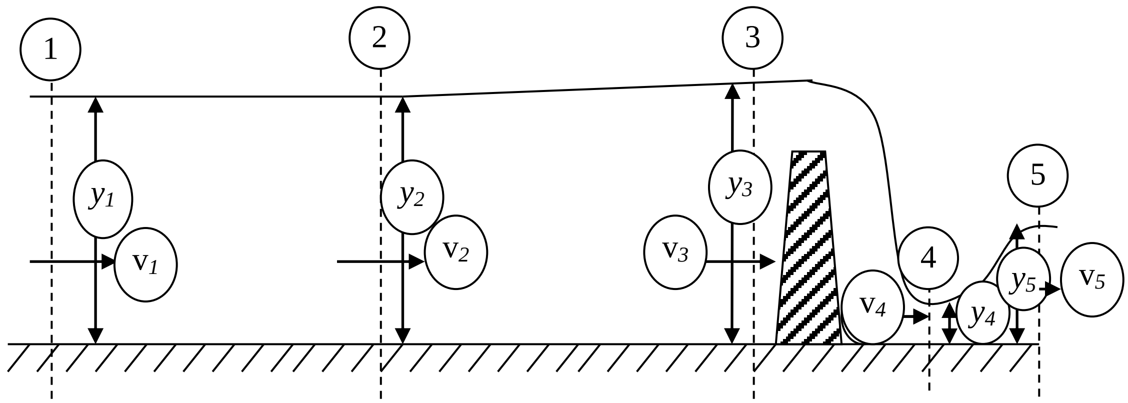


Fig. Different types of fluid flow

Consider a fluid flow as shown above in a channel. The flow is said to be steady at sections 1 and 2 as the flow does not change with respect to time at the respective sections (y_1=y_2 and v_1 = v_2)..

The flow between sections 1 and 2 is said to be uniform as the properties does not change between the sections at any instant (y_1=y_2 andv_1 = v_2).

The flow between sections 2 and 3 is said to be non-uniform flow as the properties vary over the length between the sections.

Non-uniform flow can be further classified as Gradually varied flow and Rapidly varied flow. As the name itself indicates, Gradually varied flow is a non-uniform flow wherein the flow/fluid properties vary gradually over a long length (Example between sections 2 and 3).

Rapidly varied flow is a non-uniform flow wherein the flow/fluid properties vary rapidly within a very short distance. (Example between sections 4 and 5).

Combination of steady and unsteady flows and uniform and non-uniform flows can be classified as steady-uniform flow (Sections 1 and 2), unsteady-uniform flow, steady-non-uniform flow (Sections 2 and 3) and unsteady-non-uniform flow (Sections 4 and 5).

2.10.3 One, Two and Three Dimensional flows

Flow is said to be one-dimensional if the properties vary only along one axis / direction and will be constant with respect to other two directions of a three-dimensional axis system.

Flow is said to be two-dimensional if the properties vary only along two axes / directions and will be constant with respect to other direction of a three-dimensional axis system.

Flow is said to be three-dimensional if the properties vary along all the axes / directions of a three-dimensional axis system.

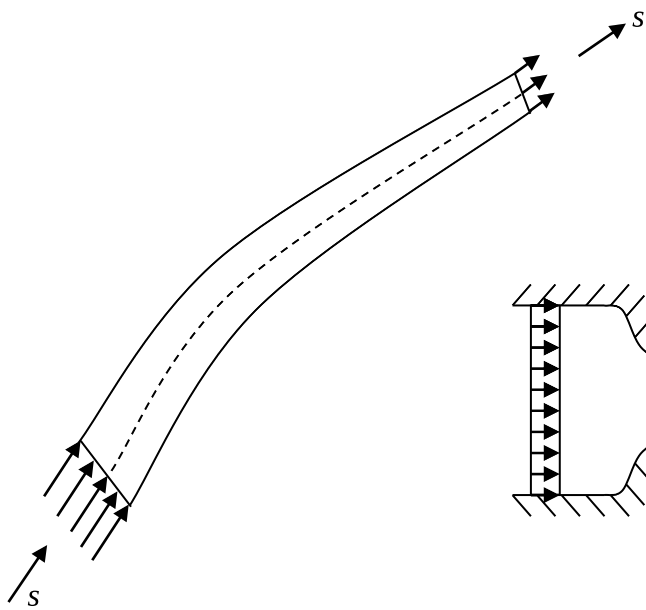


Fig. a) One- dimensional flow

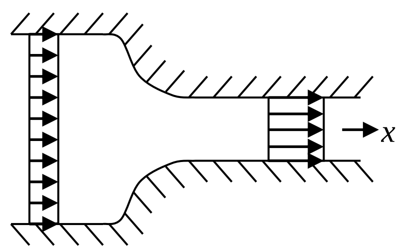


Fig. b) Two-dimensional flow

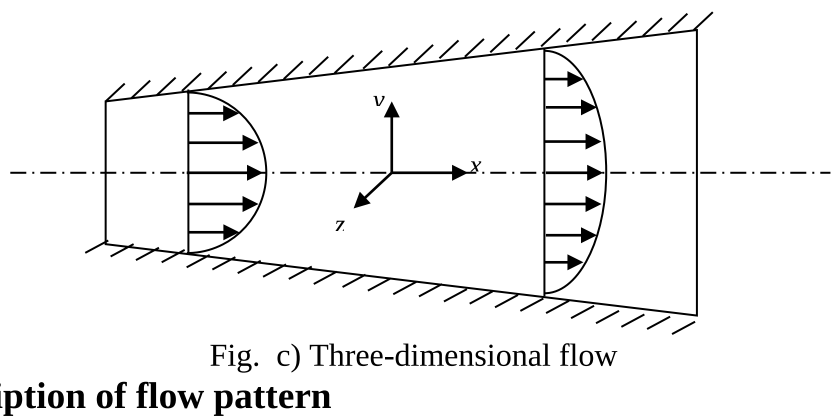
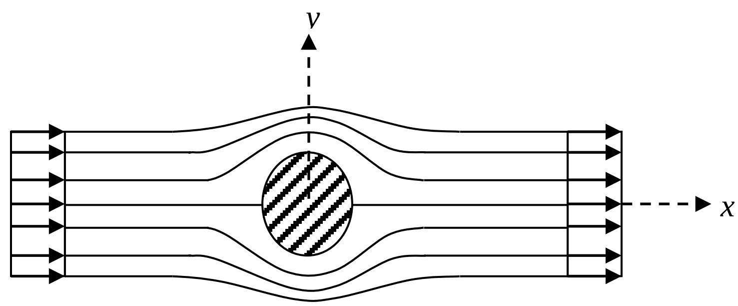


Fig. c) Three-dimensional flow

2.10.4. Description of flow pattern

Laminar and Turbulent flows:

When the flow occurs like sheets or laminates and the fluid elements flowing in a layer does not mix with other layers, then the flow is said to be laminar when the Reynolds number (Re) for the flow will be less than 2000.



$$R_e = \frac{\rho v D}{\mu}$$

Fig. 5Laminar flow

When the flow velocity increases, the sheet like flow gets mixes with other layer and the flow of fluid elements become random causing turbulence. There will be eddy currents generated and flow reversal takes place. This flow is said to be Turbulent when the Reynolds number for the flow will be greater than 4000. For flows with Reynolds number between 2000 to 4000 is said to be transition flow.

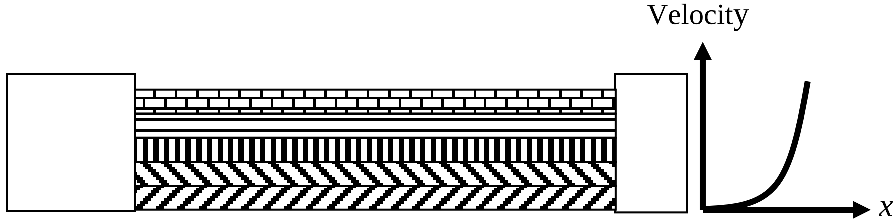


Fig. Compressible and Incompressible flows:

Flow is said to be Incompressible if the fluid density does not change (constant) along the flow direction and is Compressible if the fluid density varies along the flow direction

ρ= Constant (incompressible) and $\rho \neq$ Constant (compressible)

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2.10.5 Path line, Streamline, Streak line and Stream tube:

Path Line: It is the path traced by a fluid particle over a period of time during its motion along the fluid flow.

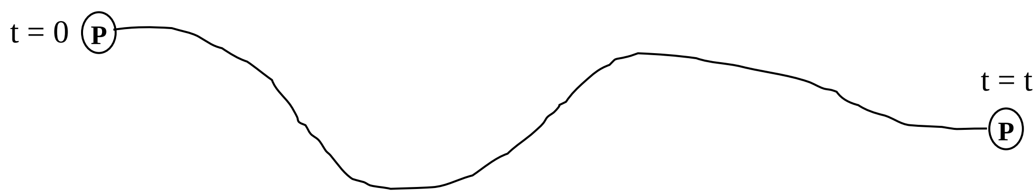


Fig. 7 Path line

Example Path traced by an ant coming out from its dwelling

Stream Lines

It is an imaginary line such that when a tangent is drawn at any point it gives the velocity of the fluid particle at that point and at that instant.

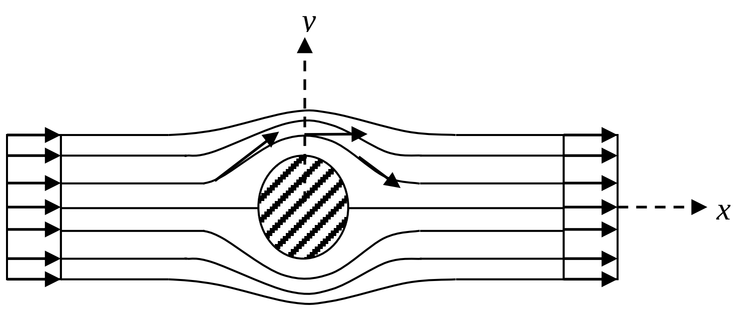


Fig. Stream lines

Example Path traced by the flow when an obstruction like a sphere or a stick is kept during its motion. The flow breaks up before the obstruction and joins after it crosses it.

Streak lines:

It is that imaginary line that connects all the fluid particles that has gone through a point/section over a period of time in a fluid motion.

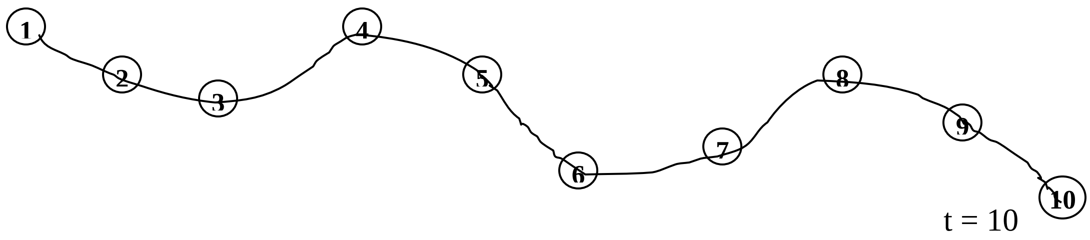


Fig. Streak lines

Stream tube:

It is an imaginary tube formed by stream line on its surface such that the flow only enters the tube from one side and leaves it on the other side only. No flow takes place across the stream tube. This concept will help in the analysis of fluid motion.

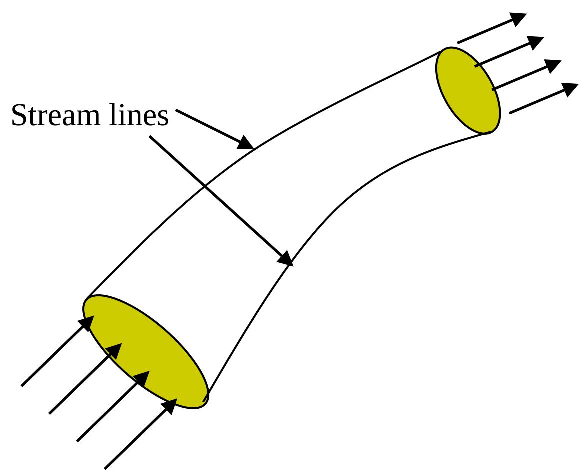


Fig. Stream tube

2.10.6. Rotational and Irrotational flows:

Flow is said to be Rotational if the fluid elements does not rotate about their own axis as they move along the flow and is Rotational if the fluid elements rotate along their axis as they move along the flow direction

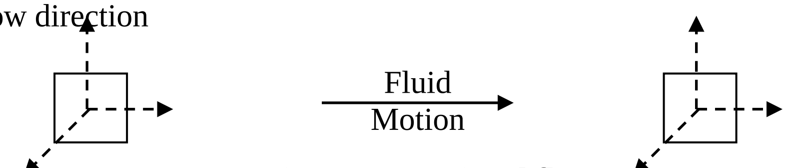
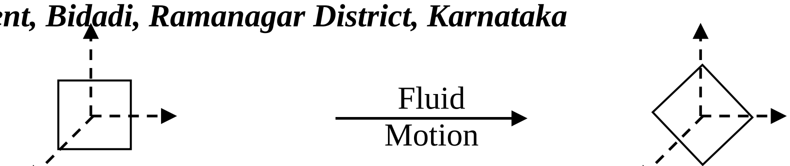


Fig. a) Irrotational flow



We know that for an irrotational two dimensional fluid flow, the rotational fluid elements about z axis must be zero.

$$w_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Substituting for *u* and v in terms of velocity potential-ϕ, we get

$$w_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right] = 0 \;\; \textbf{\textit{Laplace}} \quad \textbf{\textit{Eq.}}$$

Hence for the flow to be irrotational, the second partial derivative of Velocity potential -ϕ must be zero. This is true only when ϕ is a continuous function and exists.

Thus the properties of a velocity potential are:

1. If the velocity potential ϕ exists, then the flow should be irrotational
2. If the velocity potential ϕ satisfies the *Laplace Equation*, then it represents a possible case of a fluid flow.

Similarly for stream function ψ

$$w_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

Substituting for *u* and v in terms of stream function-ψ, we get

$$w_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right] = \frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right] = 0 \;\; \textbf{\textit{Laplace}} \quad \textbf{\textit{Eq.}}$$

The above equation is known as ***Laplace equation*** in ψ

Thus the properties of a Stream function are:

1. If the Stream function ψ exists, then it represents a possible case of a fluid flow.
2. If the Stream function ψ satisfies the ***Laplace Equation***, then the flow should be irrotational.

2.10.7 Basic principles of fluid flow:

The derivation is based on the concept of Law of conservation of mass.

Continuity Equation

Statement: The flow of fluid in a continuous flow across a section is always a constant. Consider an enlarging section in a fluid flow of fluid densityγ. Consider two sections 1 and 2 as shown in Fig. Let the sectional properties be as under

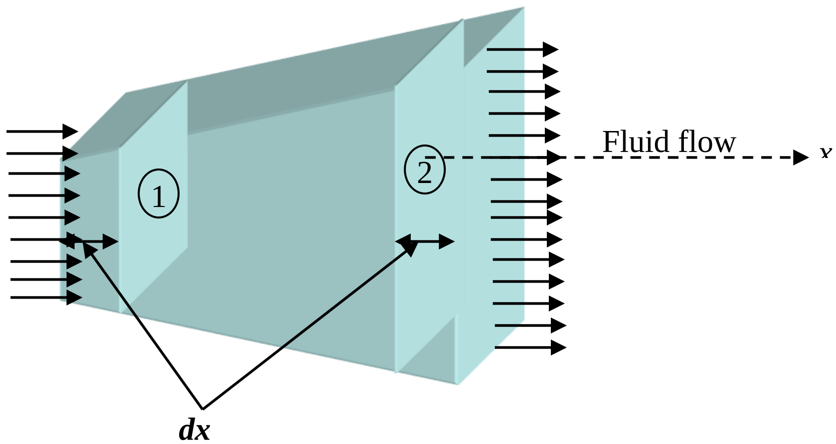


Fig. Fluid flow through a control volume

$A_1$and A_2= Cross-sectional area, $V_1$and V_2= Average flow velocity and

ρ_1 and ρ_2 = Fluid density at Section-1 and Section-2 respectively

dt is the time taken for the fluid to cover a distance dx

The mass of fluid flowing across section 1-1 is given by

m_1 = Density at section 1 x volume of fluid that has crossed section 1= $\rho_1 \times A_1 \times$ dx

Mass rate of fluid flowing across section 1-1 is given by

$$\frac{m_1}{dt} = \frac{\textbf{(Density at section - 1} \times \textbf{volume of fluid that has crossed section - 1)}}{\textbf{dt}}$$

$$\rho_1 \times A_1 \times \frac{dx}{dt} = \rho_1 \times A_1 \times V_1 \cdots\cdots Eq.1$$

Similarly Mass rate of fluid flowing across section 2-2 is given by

$$\frac{m_2}{dt} = \frac{\textbf{(Density at section - 2} \times \textbf{volume of fluid that has crossed section - 2)}}{\textbf{dt}}$$

$$\rho_2 \times A_2 \times \frac{dx}{dt} = \rho_2 \times A_2 \times V_2 \cdots\cdots Eq.2$$

From law of conservation of mass, mass can neither be created nor destroyed. Hence, from Eqs. 1 and 2, we get

$$\rho_1 \times A_1 \times V_1 = \rho_2 \times A_2 \times V_2 \qquad \text{Eq.3}$$

If the density of the fluid is same on both side and flow is incompressible then $\rho_1 = \rho_2$ the equation 3 reduces to $A_1 \times V_1 = A_2 \times V_2$

The above equations discharge continuity equation in one dimensional form for a steady, incompressible fluid flow.

Rate of flow or Discharge (Q):

Rate of flow or discharge is said to be the quantity of fluid flowing per second across a section of a flow. Rate of flow can be expressed as mass rate of flow or volume rate of flow. Accordingly

Mass rate of flow = Mass of fluid flowing across a section / time

Rate of flow = Volume of fluid flowing across a section / time

2.10.7.1 Continuity Equation in three dimensional or differential form

Consider a parallelepiped ABCDEFGH in a fluid flow of density γ as shown in Fig. Let the dimensions of the parallelepiped be dx, dy and dz along x, y and z directions respectively. Let the velocity components along x, y and z be u, v and w respectively.

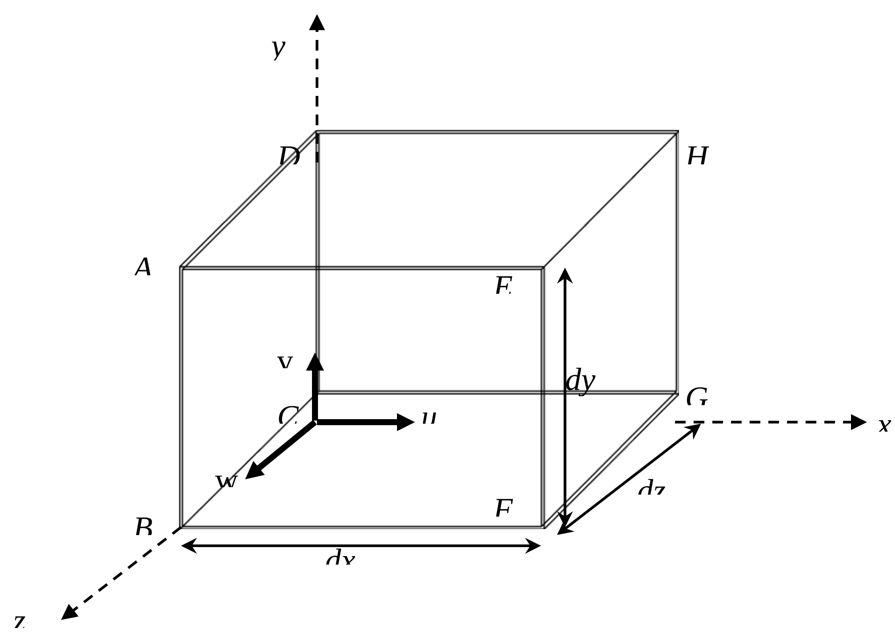


Fig. Parallelepiped in a fluid flow

Mass rate of fluid flow entering the section ABCD along x direction is given by ρ × Area × Vy

$$M_{x1} = \rho u \, dy \, dz \qquad \ldots(01)$$

Similarly mass rate of fluid flow leaving the section EFGH along × direction is given by,

$$M_{x2} = \left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right]dy\,dz \qquad \ldots(02)$$

Net gain in mass rate of the fluid along the x axis is given by the difference between the mass rate of flow entering and leaving the control volume. i.e. Eq. 1 – Eq. 2

$$dM_x = \rho u\,dy\,dz - \left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right]dy\,dz$$

$$dM_x = -\frac{\partial}{\partial x}(\rho u)dx\,dy\,dz \qquad \ldots(03)$$

Similarly net gain in mass rate of the fluid along the y and z axes are given by

$$dM_y = -\frac{\partial}{\partial y}(\rho \mathrm{v})dx\,dy\,dz \qquad \ldots(04)$$

$$dM_z = -\frac{\partial}{\partial z}(\rho \mathrm{w})dx\,dy\,dz \qquad \ldots(05)$$

Net gain in mass rate of the fluid from all the threeaxes are given by

$$dM = -\frac{\partial}{\partial x}(\rho u)dx\,dy\,dz - \frac{\partial}{\partial y}(\rho \mathrm{v})dx\,dy\,dz - \frac{\partial}{\partial z}(\rho \mathrm{w})dx\,dy\,dz$$

From law of conservation of Mass, the net gain in mass rate of flow should be zero and hence

$$\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho \mathrm{v})+\frac{\partial}{\partial z}(\rho \mathrm{w})\right]dx\,dy\,dz = 0$$

or

$$\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho \mathrm{v})+\frac{\partial}{\partial z}(\rho \mathrm{w})\right] = 0$$

This expression is known as the general Equation of Continuity in three dimensional form or differential form.

If the fluid is incompressible then the density is constant and hence

$$\left[\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}+\frac{\partial \mathrm{w}}{\partial z}\right] = 0$$

The continuity equation in two-dimensional form for compressible and incompressible flows is respectively as below

$$\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho \mathrm{v})\right] = 0$$

$$\left[\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}\right] = 0$$

2.10.8 Velocity Potential Function (ϕ) and Stream Function (ψ):

2.10.8.1 Velocity Potential (ϕ):

Velocity Potential ϕ is a scalar function of space and time such that its negative derivative with respect to any direction gives the velocity component in that direction

Thus $\phi = \phi(x,y,z,t)$ and flow is steady then,

$u = -(\partial \phi /\partial x)$; $v = -(\partial \phi /\partial y)$; $w = -(\partial \phi /\partial z)$

Continuity equation for a three dimensional fluid flow is given by

$[(\partial u/\partial x)+(\partial v/\partial y) +(\partial w/\partial z)] = 0$

Substituting for u, v and w, we get

$[(\partial /\partial x)(-\partial \phi /\partial x)+(\partial /\partial y)(-\partial \phi /\partial y) +(\partial /\partial z) (-\partial \phi /\partial z)] = 0$

i.e. $[(\partial 2\phi /\partial x2)+(\partial 2\phi/\partial y2)+(\partial 2\phi /\partial z2)] = 0$

The above equation is known as Laplace equation in ϕ

For a 2 D flow the above equation reduces to

$$[(\partial^2\phi /\partial x^2)+(\partial^2\phi/\partial y^2)] = 0$$

We know that for an irrotational two dimensional fluid flow, the rotational fluid elements about z axis must be zero. i.e. $\omega z = ½ [(\partial v/\partial x) - (\partial u/\partial y)]$

Substituting for u and v, we get

$$w_z= ½ [(\partial /\partial x)(-\partial \phi /\partial y) - (\partial /\partial y)(-\partial \phi /\partial x)]$$

For the flow to be irrotational, the above component must be zero

$\omega z = ½ [(-\partial^2\phi /\partial x \partial y) - (-\partial^2\phi / \partial y \partial x)] = 0$

i.e. $(-\partial^2\phi /\partial x \partial y) = (-\partial^2\phi / \partial y \partial x)$

This is true only when ϕ is a continuous function and exists.

Thus the properties of a velocity potential are:

1. If the velocity potential ϕ exists, then the flow should be irrotational.
2. If the velocity potential ϕ satisfies the Laplace Equation, then it represents a possible case of a fluid flow.

Equi-potential lines:

It is an imaginary line along which the velocity potential ϕ is a constant

i.e. ϕ = Constant

$\therefore \quad d\phi = 0$

But $\phi = f(x,y)$ for a two dimensional steady flow

$\therefore \quad d\phi = (\partial \phi / \partial x)dx + (\partial \phi / \partial y)dy$

Substituting the values of u and v, we get

$d\phi = -u\, dx - v\, dy \Rightarrow 0$

or $u\, dx = -v\, dy$

or $(dy/dx) = -u/v$ … (01)

Where dy/dx is the slope of the equi-potential line.

2.10.8.2 Stream Function (ψ)

Stream Function ψ is a scalar function of space and time such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction.

Thus $\psi = \psi(x,y,z,t)$ and flow is steady then,

$u = -(\partial \psi / \partial y)$; $v = (\partial \psi / \partial x)$

Continuity equation for a two dimensional fluid flow is given by

$$[(\partial u/\partial x)+(\partial v/\partial y)] = 0$$

Substituting for u and v, we get

$$[(\partial /\partial x)(-\partial \psi /\partial y)+(\partial /\partial y)(\partial \psi /\partial x)] = 0$$

i.e. $[(-\partial^2\psi /\partial x \partial y) + (\partial^2\psi / \partial y \partial x)] = 0$

or $(\partial^2\psi /\partial x \partial y) = (\partial^2\psi / \partial y \partial x)$

This is true only when ψ is a continuous function.

We know that for an irrotational two dimensional fluid flow, the rotational fluid elements about z axis must be zero.*i.e.* $\boldsymbol{\omega_z} = ½ [(\partial v/\partial x) - (\partial u/\partial y)]$

Substituting for *u* and v, we get

$\boldsymbol{\omega_z} = ½ [(\partial/\partial x)(\partial \psi /\partial x) - (\partial/\partial y)(-\partial \psi /\partial y)]$

For the flow to be irrotational, the above component must be zero

i.e. $[(\partial^2\phi /\partial x^2)+(\partial^2\phi/\partial y^2)] = 0$

The above equation is known as ***Laplace equation*** in $\boldsymbol{\psi}$

Thus the properties of a Stream function are:

1. If the Stream function ψ exists, then it represents a possible case of a fluid flow.
2. If the Stream function ψ satisfies the Laplace Equation, then the flow should be irrotational.

Line of constant stream function or stream line

It is an imaginary line along which the stream function ψ is a constant

i.e. ψ = Constant

$d\psi = 0$

But $\psi = f(x,y)$ for a two dimensional steady flow

$d\psi = (\partial \psi / \partial x)dx + (\partial \psi / \partial y)dy$

Substituting the values of u and v, we get

$d\psi = v\,dx - u\,dy \Rightarrow 0$

or $v\,dx = u\,dy$

or $(dy/dx) = v/u$ … (02)

Where dy/dx is the slope of the Stream line.

From Eqs. 1 and 2, we get that the product of the slopes of equi-potential line and stream line is given by -1. Thus, the equi-potential lines and stream lines are orthogonal to each other at all the points of intersection.

2.10.8.3 Relationship between Stream function (ψ) and Velocity potential (ϕ)

We know that the velocity components are given by

$\boldsymbol{u} = -(\partial\phi/\partial x)$ $\quad$ $\mathbf{v} = -(\partial\phi/\partial y)$

and $\boldsymbol{u} = -(\partial\psi/\partial y)$ $\quad$ $\mathbf{v} = (\partial\psi/\partial x)$

Relation between (ϕ and ψ):

$$\boldsymbol{u} = -\frac{\partial\phi}{\partial \boldsymbol{x}} = -\frac{\partial\psi}{\partial \boldsymbol{y}}$$

$$\boldsymbol{v} = -\frac{\partial\phi}{\partial \boldsymbol{y}} = \frac{\partial\psi}{\partial \boldsymbol{x}}$$

Thus $\boldsymbol{u} = -(\partial\phi/\partial x) = -(\partial\psi/\partial y)$ and $\mathbf{v} = -(\partial\phi/\partial y) = (\partial\psi/\partial x)$

Hence $(\partial\phi/\partial x) = (\partial\psi/\partial y)$ and $(\partial\phi/\partial y) = -(\partial\psi/\partial x)$

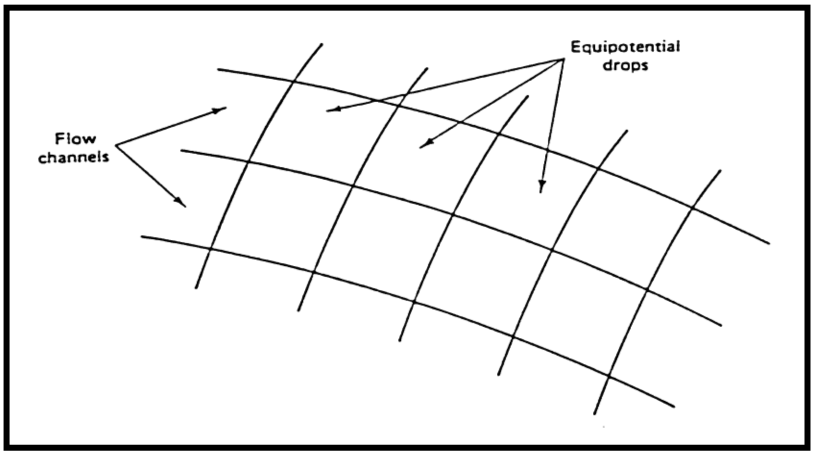
ϕ-lines and ψ-lines intersect orthogonally

2.11 Flow net & its Applications:

A grid obtained by drawing a series of equi-potential lines and stream lines is called a Flow net. The flow net is an important tool in analysing two dimensional flow irrotational flow problems.

A grid obtained by drawing a series of streamlines (ψ) and equipotential (ϕ) lines is known as flow net. The construction of flow net (ϕ-ψ lines) is restricted by certain conditions

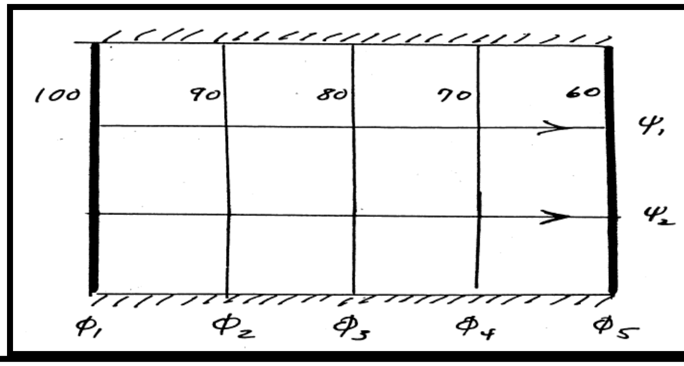
- ✓ The flow should be two dimensional
- ✓ The flow should be steady
- ✓ The flow should be Irrotational
- ✓ The flow is not governed by gravity force

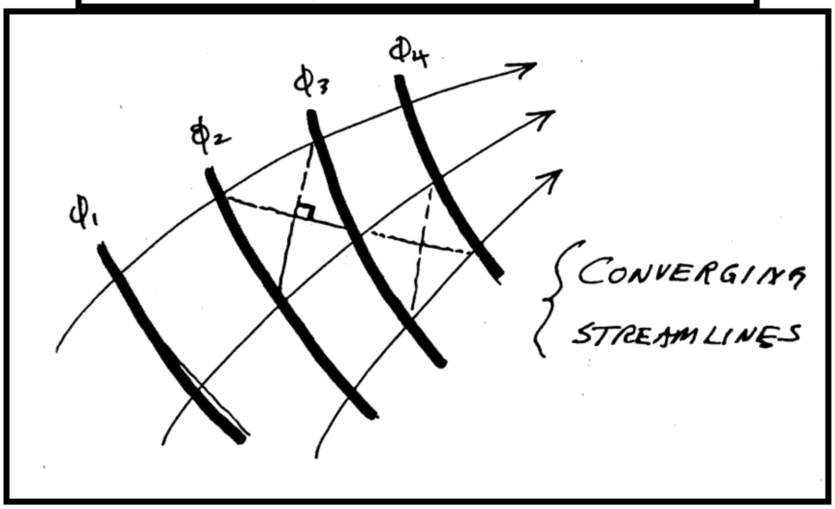


Uses of Flow net

To determine

- The streamlines and equipotential lines
- Quantity of seepage, upward lift pressure below the hydraulic structures (dam, gate, locks etc.)
- Velocity and pressure distribution, for given boundaries of flow
- To design streamlined structure
- Flow pattern near well





Methods of Drawing flow net

- Analytical Method
- Graphical Method
- Electrical Analogy Method
- Hydraulic Models
- Relaxation Method
- Hele Shaw or Viscous Analogy Method

The practical use of streamlines and velocity potential lines are:

(i) Quantity of seepage

(ii) Upward lift pressure below the hydraulic structures (dam, gate, locks etc.)

(iii) Velocity and pressure distribution, for given boundaries of flow

(iv) To design streamlined structure flow pattern near well

Solved Problems:

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Q.1. The velocity field in a fluid is given by,

$$\boldsymbol{V}_s = (3\boldsymbol{x} + 2\boldsymbol{y})\boldsymbol{i} + (2\boldsymbol{z} + 3\boldsymbol{x}^2)\boldsymbol{j} + (2\boldsymbol{t} - 3\boldsymbol{z})\boldsymbol{k}$$

i. What are the velocity components *u*, v, and w?

ii. Determine the speed at the point (1,1,1).

iii. Determine the speed at time t=2 s at point (0,0,2)

$$u = (3x + 2y), \mathrm{v} = (2z + 3x^2), \mathrm{w} = (2t - 3z)k$$

Solution: The velocity components at any point *(x, y, z)* are

Substitute *x=1, y=1, z=1 in the above expression*

*u = (3*1+2*1) = 5,* v = *(2*1+3*1) = 5,* w = *(2t-3)*

$V^2 = u^2 + \mathrm{v}^2 + \mathrm{w}^2$

$= 5^2 + 5^2 + (2t-3)^2$

$$V_{(1,1,1)} = \sqrt{(4t^2 - 12t + 59)}$$

$= 4t^2 - 12t + 59$

Substitute *t = 2 s, x=0, y=0, z=2 in the above expression for u, v and w*

u = 0, v = *(4 + 0) = 4,* w = *(4 – 6) = -2*

$V^2_{(0,0,2,2)} = (0 + 15 + 4) = 20$

V = 4.472 units

Q. 2. The velocity distribution in a three-dimensional flow is given by:

u = - x, v = *2y* and w = *(3-z)*. Find the equation of the stream line that passes through point *(1,1,1).*

$$\frac{dx}{u} = \frac{dy}{\mathrm{v}} = \frac{dz}{\mathrm{w}} \text{ or } \frac{dx}{-x} = \frac{dy}{2y} = \frac{dz}{(3-z)}$$

$$\frac{dx}{-x} = \frac{dy}{2y}$$

Solution: The stream line equation is given by

Integrating we get

Where *A* is an integral constant. Substituting *x=1 & y=1, A = 0*

Considering the *x* and *z* components,

$$-\log_e x = \frac{1}{2}\log_e y + A,$$

$$\therefore \log_e x = \log_e y^{-1/2} \text{ or } x = \frac{1}{\sqrt{y}}$$

$$\frac{dx}{-x} = \frac{dy}{(3-z)}$$

$$-\log_e x = -\log_e(3-z) + B,$$

Integrating we get

Where *B* is an integral constant. Substituting *x=1 & z=1, B* = log *e 2*

$$\therefore -\log_e x = -\log_e(3 - z) + \log_e 2 = -\log_e\left(\frac{3-z}{2}\right)$$

$$\textbf{or } x = \left(\frac{3-z}{2}\right)$$

From Eqs. 1 and 2, the final equation of the stream line that passes through the point (1,1,1) is

$$x = \frac{1}{\sqrt{y}} = \left(\frac{3-z}{2}\right)$$

Q3. A fluid particle moves in the following flow field starting from the points *(2,1,0)* at *t=0.* Determine the location of the fluid particle at *t = 3* s

$$u = \frac{t^2}{2x}, v = \frac{ty^2}{18}, w = \frac{z}{2t}$$

Solution

Integrating we get

$$u = \frac{dx}{dt} = \frac{t^2}{2x} \textbf{ or } 2x dx = t^2 dt$$

$$x^2 = \frac{t^3}{3} + A$$

$$x^2 = \frac{t^3}{3} + 4$$

$$x^2 = \frac{3^3}{3} + 4 = \sqrt{13}$$

Where *A* is an integral constant. Substituting *x=2, t=0, A = 4*

Integrating we get

$$\mathrm{v} = \frac{dy}{dt} = \frac{ty^2}{18} \text{ or } \frac{dy}{y^2} = \frac{tdt}{18} \qquad -\frac{1}{y} = \frac{t^2}{36} + B$$

Where *B* is an integral constant.

$$\frac{1}{y} = 1 - \frac{t^2}{36} \qquad \frac{1}{y} = 1 - \frac{3^2}{36} = \frac{3}{4} \text{ or } y = \frac{4}{3}$$

$$\mathrm{w} = \frac{dz}{dt} = \frac{z}{2t} \text{ or } \frac{2dz}{z} = \frac{dt}{t}$$

Substituting *y=1, t=0, B = -1*

At *t = 3 s,*

Integrating we get

$$2\log_e z = \log_e t + C$$

Where *C* is an integral constant.

Substituting *z=0, t=0, C = 0*

$$2\log_e z = \log_e t \text{ or } z^2 = t$$

At *t = 3 s*,

$$z^2 = 3 \text{ or } z = \sqrt{3}$$

From Eqs. 1, 2 and 3, at the end of 3 seconds the particle is at a point

$$\left(\sqrt{13}, \frac{4}{3}, \sqrt{3}\right)$$

Q.4. The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation:

(i) ***u*** **=** $\boldsymbol{x^2 + y^2 + z^2}$ **; v =** $\boldsymbol{xy^2 - yz^2 + xy}$**;** ***(ii)*** **v =** $\boldsymbol{2y^2}$ **; w =** ***2xyz.***

Solution:

The continuity equation for incompressible flow is given by

$[(\partial u/\partial x)+(\partial \mathrm{v}/\partial y) +(\partial \mathrm{w}/\partial z)] = 0$ …*(01)*

$u = x^2 + y^2 + z^2;$ $(\partial u/\partial x) = 2x$

$\mathrm{v} = xy^2 - yz^2 + xy;$ $(\partial \mathrm{v}/\partial y) = 2xy - z2 + x$

Substituting in Eq. 1, we get

$2x + 2xy - z^2 + z + (\partial w/\partial z) = 0$

Rearranging and integrating the above expression, we get

$\mathbf{w = (-3xz - 2xyz + z^3/3) + f(x,y)}$

Similarly, solution of the second problem

$\mathbf{u = -4xy - x^2y^2 + f(y,z).}$

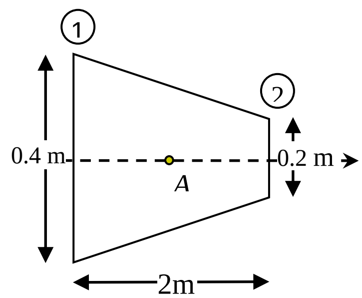
Q.5. Find the convective acceleration at the middle of a pipe which converges uniformly from 0.4 m to 0.2 m diameter over a length of 2 m. The rate of flow is 20 lps. If the rate of flow changes uniformly from 20 lps to 40 lps in 30 seconds, find the total acceleration at the middle of the pipe at 15th second.

Solution: $D_1 = 0.4$ m, $D_2 = 0.2$ m, L = 2 m, Q = 20 lps = 0.02 m^3/s.

$Q_1 = 0.02\ m^3/s$ and $Q_2 = 0.04\ m^3/s$

Case (i): Flow is one dimensional and hence the velocity components $v = w = 0$

$\therefore$ Convective acceleration $= u(\partial u/\partial x)$



$A_1 = (\pi/4)(D_1^2) = 0.1257\ m^2$

$A_2 = (\pi/4)(D_2^2) = 0.0314\ m^2$

$u_1 = Q/A_1 = 0.02/0.1257 = 0.159\ m/s$

and $u_2 = Q/A_2 = 0.02/0.0314 = 0.637\ m/s$

As the diameter changes uniformly, the velocity will also Change uniformly. The velocity u at any distance x from inlet is given by

$$u = u_1 + (u_2 - u_1)/(x/L) = 0.159 + 0.2388\,x$$

$$(\partial u/\partial x) = 0.2388$$

$\therefore$ Convective acceleration $= u(\partial u/\partial x) = (0.159 + 0.2388\,x)\,0.2388$

At A, $x = 1\ m$ and hence

(Convective accln) $_{x=1} = 94.99\ mm/s^2$

Case (ii): Total acceleration = (convective + local) acceleration at ***t* =*15* seconds**

Rate of flow $Q_{t=15} = Q_1 + (Q_2 - Q_1)(15/30) = 0.03\ m3/s.$

$u_1 = Q/A_1 = 0.03/0.1257 = 0.2386\ m/s$

and $u_2 = Q/A_2 = 0.03/0.0314 = 0.9554$ ***m/s***

The velocity u at any distance x from inlet is given by

$$u = u_1 + (u_2 - u_1)/(x/L) = 0.2386 + 0.3584\, x$$

$$(\partial u/\partial x) = 0.3584$$

$\therefore$ Convective acceleration $= u(\partial u/\partial x) = (0.2386 + 0.3584\, x)\, 0.3584$

At **A,** $x = 1$ ***m*** and hence

(Convective accln) $_{x=1} = 0.2139\ m/s^2$

Local acceleration

Diameter at **A** is given by $D = D_1 + (D_1 - D_2)/(x/L) = 0.3\ m$

and $A = (\pi/4)(D^2) = 0.0707\ m^2$

When $Q_1 = 0.02\ m^3/s$, $u_1 = 0.02/0.0707 = 0.2829\ m/s$

When $Q_2 = 0.04\ m^3/s$, $u_2 = 0.02/0.0707 = 0.5659\ m/s$

Rate of change of velocity = Change in velocity/time

$= (0.5629-0.2829)/30 = 9.43 \times 10^{-3} m/s^2$

$\therefore$ **Total acceleration** $= 0.2139 + 9.43 \times 10^{-3} = 0.2233\ m/s^2$

Q.6. In a flow the velocity vector is given by V = 3xi + 4yj -7zk. Determine the equation of the stream line passing through a point M (1, 4, 5).

Ans: Given the Velocity vector V = 3xi+4yj -7zk

$\Rightarrow$ u = 3x ; v = 4y; w = -7z

To determine the equation of the stream line passing through a point M (1, 4, 5)

The 3-D equation of streamline is given by,

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\frac{dx}{3x} = \frac{dy}{4y} = \frac{dz}{-7z} \cdots Eq.1$$

The streamline equation at point **M (1, 4, 5),** x =1, y = 4, z = 5

Substituting the values of x, y, and z in Eq.1

$$\frac{dx}{3} = \frac{dy}{16} = \frac{dz}{-35}$$

The equation of a streamline ds = 3i +16k – 35k

Q.7. A 250 mm diameter pipe carries oil of specific gravity 0.9 at a velocity of 3 m/s. At another section the diameter is 200 mm. Find the velocity at this section and the mass rate of flow of oil.

Solution:

$D_1 = 0.25\ m$; $D_2 = 0.2\ m$; $S_o = 0.9$; $V_1 = 3\ m/s$; $\rho = 1000\ kg/m^3$ *(assumed);*

$V2 = ?$; Mass rate of flow = ?

From discharge continuity equation for steady incompressible flow, we have

$$Q = A_1V_1 = A_2V_2 \qquad (01)$$

$$A_1 = (\pi/4)D_1^2 = (\pi/4)0.25^2 = 0.0499\ m^2$$

$$A_2 = (\pi/4)D_2^2 = (\pi/4)0.20^2 = 0.0314\ m^2$$

Substituting in Eq. 1, we get

$$Q = 0.0499 \text{ x } 3 = 0.1473\ m^3/s$$

Mass rate of flow = $\rho Q = 0.1479 \text{ x } 1000 = 147.9\ kg/m^3$ ***(Ans)***

$V_2 = (A_1 / A_2) \text{ x } V_1 = (D_1 / D_2)^2 \text{ x } V_1 = (0.25/0.2)^2 \text{ x } 3 = 4.6875\ m/s$ ***(Ans)***

Q.8. In a two dimensional incompressible flow the fluid velocity components are given by

$u = x - 4y$ and $v = -y - 4x$

Where u and v are x and y-*c*omponents of velocity of flow. Show that the flow satisfies the continuity equation and obtain the expression for stream function. If the flow is potential, obtain also the expression for the velocity potential.

Solution:

$u = x - 4y$ and $v = -y - 4x$

$(\partial u / \partial x) = 1$ and $(\partial v / \partial y) = -1$

$(\partial u / \partial x) + (\partial v / \partial y) = 1 - 1 = 0.$

Hence it satisfies continuity equation and the flow is continuous and velocity potential exists.

Let ϕ be the velocity potential.

Then $(\partial \phi / \partial x) = -u = -(x - 4y) = -x + 4y$ (1)

and $(\partial \phi / \partial y) = -v = -(-y - 4x) = y + 4x$ (2)

Integrating Eq. 1, we get

$$\phi = (-x^2/2) + 4xy + C \qquad (3)$$

Where C is an integral constant, which is independent of x and can be a function of y.

Differentiating Eq. 3 w.r.t. y, we get

$$(\partial \phi / \partial y) = 0 + 4x + (\partial C / \partial y) \Rightarrow y + 4x$$

Hence, we get $(\partial C / \partial y) = y$

Integrating the above expression, we get $C = y^2/2$

Substituting the value of C in Eq. 3, we get the general expression as

$$\phi = (-x^2/2) + 4xy + y^2/2$$

Stream Function

Let ψ be the velocity potential.

Then $(\partial \psi / \partial x) = \mathbf{v} = (-y - 4x) = -y - 4x$ (4)

and $(\partial \psi / \partial y) = u = -(x - 4y) = -x + 4y$ (5)

Integrating Eq. 4, we get

$$\psi = - y\,x - 4\,(x2/2) + K \qquad (6)$$

Where K is an integral constant, which is independent of x and can be a function of y.

Differentiating Eq. 6 w.r.t. y, we get

$$(\partial \psi / \partial y) = - x - 0 + (\partial K / \partial y) \Rightarrow -x + 4\,y$$

Hence, we get $(\partial K / \partial y) = 4\,y$

Integrating the above expression, we get $C = 4\,y^2/2 = 2\,y^2$

Substituting the value of K in Eq. 6, we get the general expression as

$$\psi = - y\,x - 2\,x^2 + 2\,y^2$$

Q.9. The components of velocity for a two dimensional flow are given by

$$u = x\,y; \qquad \mathbf{v} = x^2 - \frac{y^2}{2}$$

Check whether (i) they represent the possible case of flow and (ii) the flow is irrotational.

Solution:

$$u = x\,y; \qquad \text{and} \qquad \mathbf{v} = x^2 - \frac{y^2}{2}$$

$(\partial u/\partial x) = y \qquad (\partial v/\partial y) = -y$

$(\partial u/\partial y) = x \qquad (\partial v/\partial x) = 2x$

For a possible case of flow the velocity components should satisfy the equation of continuity.

i.e. $$\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial v}{\partial y}\right)=0$$

Substituting, we get ***y – y = 0.***

Hence it is a possible case of a fluid flow.

For flow to be irrotational in a two dimensional fluid flow, the rotational component in ***z*** direction (ωz) must be zero, where

$$w_z = \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)-\left(\frac{\partial u}{\partial y}\right)\right]=\frac{1}{2}[2x-x]\neq 0$$

Hence, the flow is not irrotational.

Q.10. Find the components of velocity along ***x*** and ***y*** for the velocity potential $\phi = a \text{ Cos } xy$**.** Also calculate the corresponding stream function.

Solution:

$\phi = a \text{ Cos } xy.$

$$\left(\frac{\partial \phi}{\partial x}\right) = -u = -aySin(xy) \qquad (1)$$

and $$\left(\frac{\partial \phi}{\partial y}\right) = -v = -axSin(xy) \qquad (2)$$

Hence $u = ay \text{ Sin } xy$ and $v = ax \text{ Sin } xy.$

Q.11. The stream function and velocity potential for a flow are given by,

$\psi = 2xy$ and $\phi = x^2 - y^2$

Show that the conditions for continuity and irrotational flow are satisfied

Solution:

From the properties of Stream function, the existence of stream function shows the possible case of flow and if it satisfies Laplace equation, then the flow is irrotational.

(i) $\psi = 2xy$

$(\partial \psi / \partial x) = 2y$ and $(\partial \psi / \partial y) = 2x$

$(\partial^2 \psi / \partial x^2) = 0$ and $(\partial^2 \psi / \partial y^2) = 0$

$(\partial^2 \psi / \partial x\, \partial y) = 2$ and $(\partial^2 \psi / \partial y\, \partial x) = 2$

$(\partial^2 \psi / \partial x\, \partial y) = (\partial^2 \psi / \partial y\, \partial x)$

Hence the flow is Continuous.

$(\partial^2 \psi / \partial x^2) + (\partial^2 \psi / \partial y^2) = 0$

As it satisfies the Laplace equation, the flow is irrotational.

From the properties of Velocity potential, the existence of Velocity potential shows the flow is irrotational and if it satisfies Laplace equation, then it is a possible case of flow

(ii) $\phi = x^2 - y^2$

$(\partial \phi / \partial x) = 2x$ and $(\partial \phi / \partial y) = -2y$

$(\partial^2 \phi / \partial x^2) = 2$ and $(\partial^2 \phi / \partial y^2) = -2$

$(\partial^2 \phi / \partial x\, \partial y) = 0$ and $(\partial^2 \phi / \partial y\, \partial x) = 0$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$$

Hence the flow is irrotational

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

As it satisfies the Laplace equation, the flow is Continuous.

Q.12. In a 2-D flow, the velocity components are ***u* = 4*y*** and **v = -4*x***

i. Is the flow possible?
ii. if so, determine the stream function
iii. What is the pattern of stream lines?

Solution:

For a possible case of fluid flow, it has to satisfy continuity equation.

i.e. $$\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) = 0 \qquad (1)$$

$u = 4y$ and $v = -4x$

$(\partial u / \partial x) = 0$ $\qquad$ $(\partial v / \partial y) = 0$

Substituting in Eq. 1, we get ***0.***

Hence the flow is possible.

Stream function

We know that $(\partial \psi / \partial x) = v = -4x$ (2)

and $(\partial \psi / \partial y) = -u = -4y$ (3)

$\psi = -2x^2 + C(y)$ (4)

Where ***C*** is an integral constant and a function of ***y.***

Differentiating Eq. 4, w.r.t. ***y***, we get

$$(\partial \psi / \partial y) = 0 + \partial C(y) / \partial y = -u = -4y$$

Integrating the above expression w.r.t. ***y*** we get

$$C(y) = -2y^2.$$

Substituting the above value in Eq. 4, we get the general expression as

$$\psi = -2x^2 - 2y^2 = -2(x^2 + y^2)$$

The above equation is an expression of concentric circles and hence the stream function is concentric circles.

Q.13. A stream function in a two dimensional flow is $\psi = 2xy$. ***D***etermine the corresponding velocity potential.

Solution:

Given $\psi = 2xy.$

$u = -(\partial \phi / \partial x) = -(\partial \psi / \partial y) = -2x$ (01)

$v = -(\partial \phi / \partial y) = (\partial \psi / \partial x) = 2y$ (02)

Integrating Eq. 1, w.r.t. x, we get

$\phi = 2x^2/2 + C = x^2 + C(y)$ (03)

Where ***C(y)*** is an integral constant independent of ***x***

Differentiating Eq. 3 w.r.t. y, we get

$$(\partial \phi / \partial y) = 0 + (\partial C(y) / \partial y) = -2y$$

Integrating the above expression w.r.t. y, we get

$$C(y) = -y^2$$

Substituting for ***C(y)*** in Eq. 3, we get the general expression for ϕ as

$\boldsymbol{\phi = x^2 + C = x^2 - y^2}$ ***(Ans)***

Q.14. The velocity potential for a flow is given by the function $\boldsymbol{\phi = x^2 - y^2}$**.** Verify that the flow is incompressible.

Solution:

From the properties of velocity potential, we have that if $\boldsymbol{\phi}$ satisfies Laplace equation, then the flow is steady incompressible continuous fluid flow.

Given $\boldsymbol{\phi = x^2 - y^2}$

$(\partial\phi/\partial x) = 2x$ $\qquad$ $(\partial\phi/\partial y) = -2y$

$(\partial^2\phi/\partial x^2) = 2$ $\qquad$ $(\partial^2\phi/\partial^2 y) = -2$

From Laplace Equation, we have $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial^2 y) = 2 - 2 = 0$

Q.15. If for a two dimensional potential flow, the velocity potential is given by ϕ = x (2y-1). Determine the velocity at the point P (4, 5). Determine also the value of stream function ψ at the point 'P'.

Ans:

(i) The velocity at the point P (4, 5), x =4, y = 5

$$\phi = x\,(2y-1).$$

$$\frac{\partial\phi}{\partial x} = -u = (2y-1), \quad u = (1-2y)$$

$$\frac{\partial\phi}{\partial y} = -v = x\times 2, \qquad v = -2x$$

u at 'P'(4,5) = -9 Units/s

v(4,5) at 'P' = -8 Units/s

Velocity at P = -9i-8j, Velocity $\sqrt{(-9)^2 + (-8)^2} = 12.04 \quad Units$

(ii) Stream function $\psi_{P\,(4,\,5)}$

Given ϕ = x (2y-1)

$$\frac{\partial\phi}{\partial x} = -u = (2y-1) = \frac{\partial\psi}{\partial y}$$

$$\frac{\partial\phi}{\partial y} = -v = x\times 2 = -\frac{\partial\psi}{\partial x}$$

$$\frac{\partial\psi}{\partial y} = -u = (2y-1)\cdots Eq.1$$

$$\frac{\partial\psi}{\partial x} = u = -2x\cdots Eq.2$$

Integrating Eq.1 with respect ‘y’ we get

$$\int d\psi = \psi = \frac{2\times y^2}{2} - y + C(f(x))\cdots Eq.3$$

Differentiating Eq.3 with respect to ’x’

$$\frac{\partial\psi}{\partial x} = \frac{\partial C}{\partial x} \qquad from \quad Eq.2 \quad \frac{\partial\psi}{\partial x} = -2x$$

$$\frac{\partial C}{\partial x} = -2x \quad Integrating \rightarrow C = -x^2$$

Substituting value of C in Eq.3

$$\boldsymbol{\psi = \left(y^2 - y - x^2\right)}$$

Q.16. A stream function is given by ψ = $2x^2$-$2y^2$. Determine the velocity and velocity potential function at (1, 2)

Ans: Given: ψ = $2x^2$-$2y^2$

$$\frac{\partial\boldsymbol{\psi}}{\partial\boldsymbol{x}} = 4\boldsymbol{x} = -\boldsymbol{v};\ \boldsymbol{v} = -4\boldsymbol{x} \Rightarrow \boldsymbol{Velocity\ at}\,(1{,}2), \quad \boldsymbol{v} = -4 \quad \boldsymbol{Units}$$

$$\frac{\partial\boldsymbol{\psi}}{\partial\boldsymbol{y}} = -4\boldsymbol{y} = \boldsymbol{u};\ \boldsymbol{u} = -4\boldsymbol{y} \Rightarrow \boldsymbol{Velocity\ at}\,(1{,}2), \quad \boldsymbol{u} = -8 \quad \boldsymbol{Units}$$

Resultant velocity $V_{(1,2)}$ = $\sqrt{(-4)^2 + (-8)^2} = 8.94\,\boldsymbol{Units}$

$$\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{x}} = -\boldsymbol{u} \quad \Rightarrow \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{x}} = -(-4\boldsymbol{y}) = 4\boldsymbol{y} \Rightarrow \boldsymbol{\phi} = 4\times\boldsymbol{x}\times\boldsymbol{y} + \boldsymbol{C}(\boldsymbol{f}(\boldsymbol{y})\boldsymbol{only})\cdots\boldsymbol{eq}1$$

$$\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{y}} = -\boldsymbol{v} \quad \Rightarrow \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{x}} = -(-4\boldsymbol{x}) = 4\boldsymbol{x} \Rightarrow \boldsymbol{\phi} = 4\times\boldsymbol{x}\times\boldsymbol{y} + \boldsymbol{C}(\boldsymbol{f}(\boldsymbol{x})\boldsymbol{only})\cdots\boldsymbol{eq}2$$

From Eq.1 $$\frac{\partial \phi}{\partial y} = (4x + \frac{\partial C}{\partial y}) \Rightarrow \frac{\partial C}{\partial y} = 4x - \frac{\partial \phi}{\partial y} \Rightarrow \frac{\partial C}{\partial y} = 4x - \left(\frac{\partial \psi}{\partial x}\right) \Rightarrow \frac{\partial C}{\partial y} = 4x - 4x = 0$$
$$\frac{\partial C}{\partial y} = 0 \ \textit{Integrating} \quad C = 0$$

$$\therefore \phi = 4 \times x \times y \qquad \Rightarrow \phi = 4 \times 1 \times 2 = 8 \ \textit{Units}$$

Q.17. The velocity potential ϕ for a two dimensional flow is given by $\mathbf{(x^2 - y^2) + 3xy}$. Calculate: (i) the stream function ψ and (ii) the flow rate passing between the stream lines through (1, 1) and (1, 2).

Ans: Given $\phi = (x^2 - y^2) + 3xy$

(i) To determine the ψ function

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \cdots Eq.(1)$$

$$d\psi = -v\,dx + u\,dy \cdots Eq.(2)$$

As per definition of velocity potential (ϕ) and stream function (ψ);

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = u \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = v$$

$$u = \frac{\partial \phi}{\partial x} = (2x + 3y) = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = (-2y + 3x) = \left(-\frac{\partial \psi}{\partial x}\right) = v$$

Substituting the value of u and v in terms of x and y in equation 2, we obtain

$$d\psi = -v\,dx + u\,dy = -(-2y + 3x)dx + (2x + 3y)dy$$
$$d\psi = (2y + 3x)dx + (2x + 3y)dy \cdots Eq.3$$

Integrating the equation-3 (partially w.r.t 'x' the 'dx-term' and w.r.t 'y' the 'dy-term')

$$\psi = \left(2xy + \frac{3}{2}x^2\right) + \left(2xy + \frac{3}{2}y^2\right) = 4xy + \frac{3}{2}\left(x^2 + y^2\right)$$

$$\boxed{\psi = 4xy + \frac{3}{2}\left(x^2 + y^2\right)}$$

(ii) The flow rate passing between the stream lines through (1, 1) and (1, 2).

The equation of stream function is given by $\boxed{\psi = 4xy + \frac{3}{2}\left(x^2 + y^2\right)}$

The value of Point streamline at (1, 1) is obtained by substituting x = 1, y = 1

$$\boldsymbol{\psi}_{(1,1)} = 4\boldsymbol{xy} + \frac{3}{2}\left(\boldsymbol{x}^2 + \boldsymbol{y}^2\right) = 4 \times 1 \times 1 + \frac{3}{2}(1^2 + 1^2) = 7\,\boldsymbol{Units}$$

The value of Point streamline at (1, 2) is obtained by substituting x = 1, y = 2

$$\boldsymbol{\psi}_{(1,2)} = 4\boldsymbol{xy} + \frac{3}{2}\left(\boldsymbol{x}^2 + \boldsymbol{y}^2\right) = 4 \times 1 \times 2 + \frac{3}{2}(1^2 + 2^2) = 15.5\,\boldsymbol{Units}$$

The flow rate passing between the stream lines through (1, 1) and (1, 2)

$$\mathbf{q = \psi_{(1,2)} - \psi_{(1,1)} = (15.5\text{-}7)}$$

$$\boxed{\mathbf{q = 8.5\ m^2/s/unit\ width}}$$

Q.18. The velocity components in a 2-dimensional incompressible flow field are expressed as

$$u = \left(\frac{y^3}{3} + 2x - x^2 \times y\right), \quad v = \left(x \times y^2 - 2y - \frac{x^3}{3}\right)$$

Is the flow irrotational? If so determine the corresponding stream function.

Ans: Given the components of velocity

$$u = \left(\frac{y^3}{3} + 2x - x^2 \times y\right), \quad v = \left(x \times y^2 - 2y - \frac{x^3}{3}\right)$$

The condition for Irrorational flow

$$\left(\frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial y}\right)$$

$$LHS \quad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(x \times y^2 - 2y - \frac{x^3}{3}\right) \text{ and } RHS \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y^3}{3} + 2x - x^2 \times y\right)$$

i.e. LHS = $(y^2 – x^2)$ and RHS = $(y^2 – x^2)$

Hence the flow is Irrorational

The corresponding stream function 'ψ'can be obtained by using following relationship

$$\frac{\partial \psi}{\partial x} = v = \left(x \times y^2 - 2y - \frac{x^3}{3}\right) \cdots Eq.1$$

$$\frac{\partial \psi}{\partial y} = -u = -\left(\frac{y^3}{3} + 2x - x^2 \times y\right) \cdots Eq.2$$

Integrating Eq.1 with respect to 'x'

$$\psi = \frac{x^2 \times y^2}{2} - 2 \times x \times y - \frac{x^4}{12} + C_1\left(f(y)\right) \cdots Eq.3$$

Differentiating Eq.3 with respect to 'y'

$$\frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{y}} = \boldsymbol{x}^2 \times \boldsymbol{y} - 2\boldsymbol{x} + \frac{\partial \boldsymbol{C}_1}{\partial \boldsymbol{y}}$$

$$\frac{\partial \boldsymbol{C}_1}{\partial \boldsymbol{y}} = -\frac{\boldsymbol{y}^3}{3}$$

$$\textbf{\textit{Integrating}}, \boldsymbol{C}_1 = -\frac{\boldsymbol{y}^4}{12} + \boldsymbol{C}; \quad (\textbf{\textit{assu}} \min \boldsymbol{g}\ \boldsymbol{C} = 0)$$

$$\boldsymbol{C}_1 = -\frac{\boldsymbol{y}^4}{12}$$

The stream function 'ψ' is given by

$$\psi = \frac{x^2 \times y^2}{2} - 2 \times x \times y - \frac{x^4}{12} + -\frac{y^4}{12}$$

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