**MODULE - I**

# COMPLEX VARIABLES

## Complex number:

- The Real and Imaginary part of a complex number $z = x + iy$ are $x$ and $y$ respectively, and we write

$$\operatorname{Re} z = x \text{ and } \operatorname{Im} z = y$$

- We may represent the complex number $z$ in polar form:

$$z = r[\cos\theta + i\sin\theta]$$

- Where x= $r\cos\theta$, y= $r\sin\theta$ ,r is called the absolute value and $\theta$ is the argument of Z.

Now

$$z = r\, e^{i\theta}$$

$$|z| = r\left|e^{i\theta}\right|$$

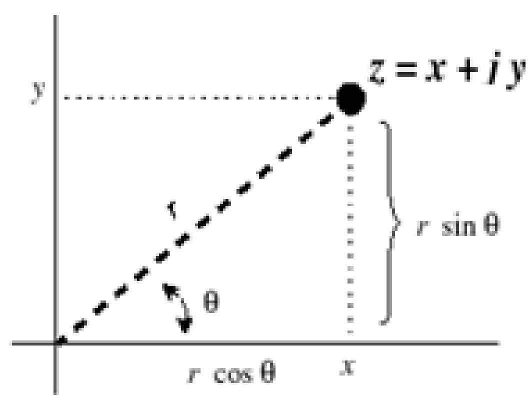
$$|z| = r \quad and \quad \arg z = \theta$$

- Geometrically $|z|$ is the distance of the point z from the origin. For any complex number

$$z = x + i\,y$$

$$|z| = \sqrt{x^2 + y^2}$$

$$r = \sqrt{x^2 + y^2}$$



- Distance between two points, $z_1 = x_1 + iy_1 \quad and \quad z_2 = x_2 + iy_2$

  Now $z_1 - z_2 = (x_1 - x_2) + i\,(y_2 - y_1)$ is a complex number.

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

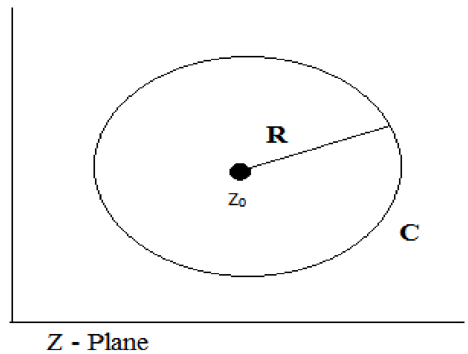
- **Equations and inequalities of curves and regions in the complex plane:**

- Consider $|z - z_0| = R$ ---(1)
- Where $z = x+iy$ is any point and $z_0=x_0+iy_0$ is a fixed point, R is a given real constant.

$$|z - z_0| = R \quad \textbf{OR} \quad z - z_0 = R\,e^{i\theta} \quad 0 \le \theta \le 2\pi$$

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = R$$

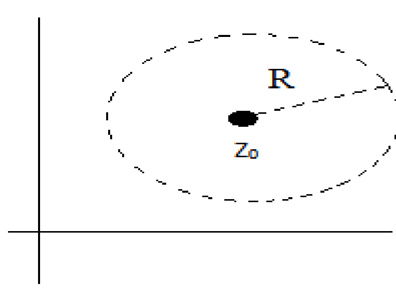
$$(x - x_0)^2 + (y - y_0)^2 = R^2 \text{ ----(2)}$$




Equation (2) represents a circle C of radius R with the center at a point $(x_0, y_0)$. Hence equation (1) represents a circle C center at $z_0$ with radius R in the complex plane

**Consequently we have,**

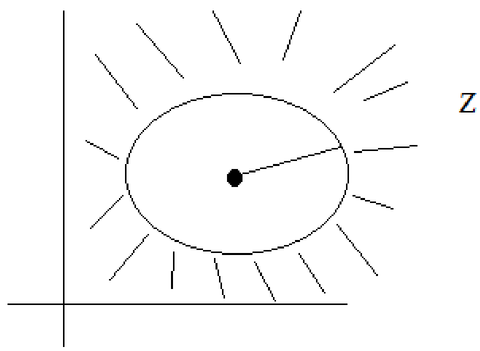
1. The inequality $|z - z_0| < R$, holds for any point $z$ inside C; ie. $|z - z_0| < R$ represents set of complex points lies inside C or interior points of C. such a region is called a circular disk or more precisely open circular disk or open set.



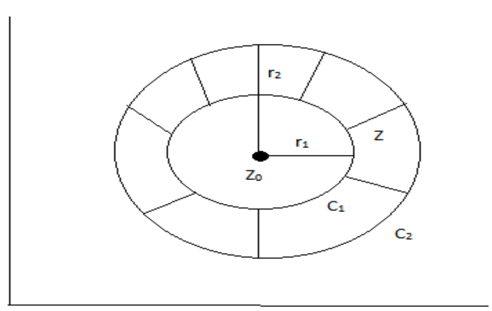
**Note:** If R is very small say δ>0 (no matter, how small but not zero) then $|z-z_0|<\delta$ is called a nhd of the point $z_0$.

2. The inequality $|z-z_0|\leq R$ , holds for any $z$ inside and on the C. such a region is called circular disk or closed set [ $|z-z_0|\leq R$ consists interior of C and C itself].

3. The inequality $|z-z_0|>R$ represents exterior of the circle C.




4. The inequality $r_1<|z-z_0|<r_2$ represents a region between two concentric circles $C_1$ and $C_2$ of radii $r_1$ and $r_2$ respectively. Where $z_0$ is the center of circles. Such a region is called an open circular ring or annular region.




5. Suppose $z_0$ = 0, then $|z|=R$ represents a circle C of radius R with center at the origin in the complex plane.

**Consequently we have the following:**

The equation $|z| = 1$ represents the unit circle of radius 1 with center at the origin.

a) $|z| < 1$ : represents the open unit disk.

b) $|z| \leq 1$ **:** represents the closed unit disk.

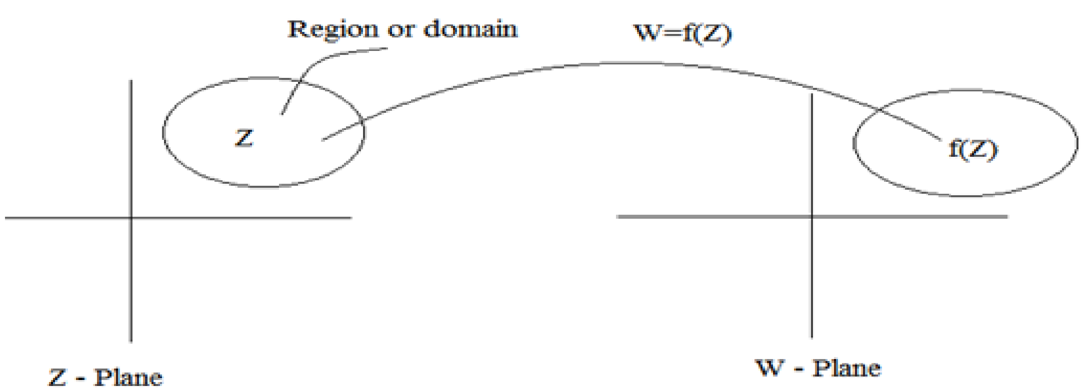
[Students become completely familiar with representations of curves and regions in the complex plane]

**Complex variable:**

- If $x$ and $y$ are real variables, then $z=x+iy$ is said to be a complex variable.

**Complex Function:**

- If, to each value of a complex variable $z$ in some region of the complex plane or z-plane there corresponds one or more values of $W$ in a well defined manner, then $W$ is a function of $z$ defined in that region (domain), and we write $W=f(z)$.

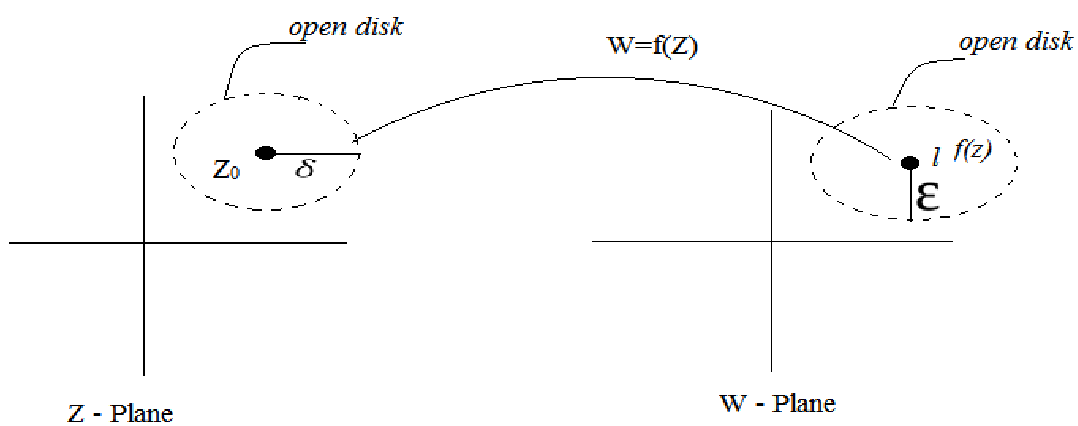


**Observation:**

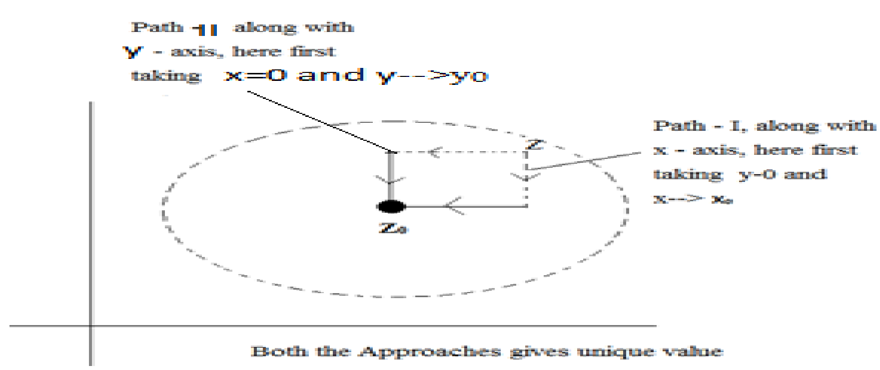
- The real and imaginary part of a complex function $W = f(z) = u + iv$ are $u$ and $v$ which are depends on:
    i. *x,y* in Cartesian form.
    ii. r, θ in polar form.

**Limit:**

A complex valued function *f(z)* is said to have the limit *l* as *z* approaches to $z_0$ (except perhaps at $z_0$) and if every positive real number ∈>0 (no matter, how small but not zero) we can find a positive real number δ>0 such that $|f(z) - l| < \varepsilon$ whenever $|z - z_0| < \delta$ for all values $z \neq z_0$ or $\lim\limits_{Z \to Z_0} f(z) = f(z_0)$




- $z \to z_0$ means that, **z** approaches to $z_0$ through independent of path.




$z = z_0$

**Continuity of :** A complex function $W = f(z)$ is said to be continuous at a point $z_o$ if

i) $f(z_0)$ is exists.

ii) $\lim_{Z \to Z_0} f(z) = f(z_0)$

**Note:** If $f(z)$ is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

## Derivative of f(z):

A complex function f(z) is said to be differentiable at $z=z_0$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is unique. This limit is then called the derivative of f(z) at $z=z_0$ and denoted by $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ or $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ where $\delta z = z - z_0$.

**Theorem:** The necessary conditions for the derivative of the function w = f(z) to exist for all values of z in a region R,

i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous function of x and y in R.

ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$ the relation (ii) are known as Cauchy- Riemann equations or briefly C-R Equations.

**Proof:** If f(z) possesses a unique derivative at any point z in R, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

In Cartesian form $f(z) = u(x, y) + i\, v(x, y)$

$$\delta z = \delta x + i\, \delta y, \quad and$$
$$f(z + \delta z) = u(x + \delta x, y + \delta y) + i\, v(x + \delta x, y + \delta y)$$

$$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{[u(x + \delta x, y + \delta y) + i\, v(x + \delta x, y + \delta y)] - [u(x, y) + iv(x, y)]}{\delta x + i\delta y} \right\}$$

$$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{u(x + \delta x, y + \delta y) - u(x, y)}{\delta x + i\delta y} + i\, \frac{v(x + \delta x, y + \delta y) - v(x, y)}{\delta x + i\delta y} \right\} --- (1)$$

Let us consider the limit $\delta z \to 0$ along the path parallel to the x-axis (for which $\delta y = 0$), then

RHS of (1) becomes $f'(z) = \lim_{\delta z \to 0} \left\{ \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \frac{v(x+\delta x, y) - v(x,y)}{\delta x} \right\}$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} --- (2)$$

Let us consider the limit $\delta z \to 0$ along the path parallel to the y-axis (for which $\delta x = 0$), then RHS of (1)

$$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \frac{v(x, y+\delta y) - v(x,y)}{i\delta y} \right\}$$

$$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + \frac{v(x, y+\delta y) - v(x,y)}{\delta y} \right\}$$

$$f'(z) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} --- (3)$$

Now existence of $f'(z)$ requires equality of (2) and (3)

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary part from both the sides.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = - \quad \frac{\partial v}{\partial x}.$$

## Analytic function:

A complex function $f(z)$ is said to be analytic at a point $z = z_0$ if it is differentiable at $z_0$ as well as in a nhd of the point $z_0$. An analytic function is also called a regular function or an holomorphic function.

**Theorem (2):** If $f(z)=u+iv$ is analytic at a point $z=x+iy$, then u and v satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$ at that point.

**Proof:**

$f(z)$ is analytic means that $f(z)$ possesses a unique derivative at a point $z=x+iy$. (proof of theorem(1) follows)

### Cauchy-Riemann equations in Polar form:

**Property:** show that the polar form of Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad and \quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

**Solution:**

Complex variable z in polar form is

$$z=r\,e^{i\theta}---(1)$$

W=f(z)

$$u+i\,v=f(re^{i\theta})----(2)$$

where u and v are functions of r $\theta$

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=f'(re^{i\theta}).\quad e^{i\theta}---(3)$$

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=f'(re^{i\theta}).\quad rie^{i\theta}---(4)$$

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}= ri\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right]$$

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}= \left[ri\frac{\partial u}{\partial r}-r\frac{\partial v}{\partial r}\right]$$

Equating real and imaginary parts we get

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad and \quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

**Note-1:** The necessary conditions for *f(z )* to be analytic are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

these two relations are called Cauchy-Riemann Equations.

**Note-2**: The sufficient conditions for *f(z)* to be analytic are, four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ must exist and must be continuous at all points of the region.

## Example-1:

- Show that *f(z)=Re z* is not analytic.

Solution: $f(z) = Re\, z = x$

$$u = x \ \ and\ v = 0$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x}$$

C-R equation $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ $\qquad$ $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$, are not satisfied

Hence *f(z)=Re z=x* is not analytic similarly *f(z)=Im z=y* is not analytic

**Property-1:** The real and imaginary parts of an analytic functions *f(z)=u+iv* in some region of the z-plane are solutions of Laplace's equations in two variables.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0 \quad and \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

**Solution:** *f(z)=u+iv* is an analytic function, then

$$(By\ C-R\ Equation) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} ----(1)$$

Consider $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} ----(2)$, $\qquad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} ----(3)$

Diff *(2)* with respect to *x*

Diff *(3)* with respect to *y*

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x \partial y}----(4)$$

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y \partial x}----(5)$$

Adding *(4)* and *(5)*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0----(6)$$

Diff *(2)* with respect to y $\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u}{\partial y \partial x}----(7)$

Diff *(2)* with respect to *x* $\frac{\partial^2 v}{\partial x^2}=-\frac{\partial^2 u}{\partial x \partial y}----(8)$

Adding *(7)* and *(8)* we get $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0----(9)$

- Thus both functions *u(x,y)* and *v(x,y)* satisfy the Laplace's equations in two variables. For this reasons, they are known as Harmonic functions or Conjugate Harmonic function.

**Polar form:** If *f(z)=u(r, θ)+i v(r, θ)* is an analytic function, then show that *u* and *v* satisfy Laplace's equation in polar form.

- Laplace equation in Polar form in two variables,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \quad \text{and} \quad \frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0$$

We have C-R equation in polar form

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}----(1)$$

$$\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}----(2)$$

Differentiate *(1)* with respect to *r*, $\frac{\partial^2 u}{\partial r^2}=-\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}----(3)$

Differentiate *(2)* with respect to *θ*, $\frac{\partial^2 u}{\partial \theta^2}=-r\frac{\partial^2 v}{\partial \theta \partial r}----(4)$

using *(4)* and *(1)* on *RHS* of Equation *(3)*, we get

$$\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{r}\left(-\frac{1}{r}\frac{\partial^2 u}{\partial \theta}\right)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

- *Hence* ***u*** is Harmonic

From *(1)* we get, $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial \theta}$

Differentiate with respect to $\theta$ $\frac{\partial^2 v}{\partial \theta^2} = r\frac{\partial^2 u}{\partial \theta \partial r} ----(5)$

From *(2)* we get $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} ----(6)$

Differentiate with respect to $r$ $\frac{\partial^2 v}{\partial r^2} = +\frac{1}{r^2}\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta} ----(7)$

using *(5)*,*(6)* on RHS of *(7)*

$$\frac{\partial^2 v}{\partial r^2} = \frac{1}{r}\left(-\frac{\partial v}{\partial r}\right) - \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 v}{\partial \theta^2}\right) = 0$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

*Hence* **v** is Harmonic

**Orthogonal System:**

- Two curves are said to be orthogonal to each other when they intersect at right angles at each of their point of intersections.

**Property:** If *w=f(z)=u+iv* be an analytic function then the family of curves $u(x,y)=c_1$ and $v(x,y)=c_2$ form an orthogonal system.

**Solution:** *f(z)=u+iv* is an analytic functions.

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x} \end{array}\right\} ----C-\text{R e}quation$$

$$u(x,y)=c_1$$

differentiate with respect to x, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\;\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}}=m_1----(2)$$

differentiate w.r.t, $x$ we get

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\;\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}}=m_2---(3)$$

$$\therefore \quad m_1.m_2 = \frac{+\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}} \times \frac{+\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}}$$

$$= \frac{\dfrac{\partial v}{\partial y}}{-\dfrac{\partial v}{\partial x}} \times \frac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}} \quad \text{(By C-R Equations)}$$

$m_1.m_2 = -1$, form an orthogonal system

Polar form: Consider $u(r,\theta) = c_1 ---(1) \quad and \quad v(r,\theta) = c_2 ---(2)$

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} &= -r \frac{\partial v}{\partial r} \end{aligned}\right\} ----(3) \text{ C-R Equations}$$

differentiate (1) w.r.t. $\theta$

$$\frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial r} \frac{dr}{d\theta} = 0$$

$$\frac{dr}{d\theta} = \frac{-\dfrac{\partial u}{\partial \theta}}{\dfrac{\partial u}{\partial r}} -----(4)$$

$\tan \phi_1 = \dfrac{r}{\dfrac{dr}{d\theta}}$ where $\phi_1$ being the angle between the radius vector and the tangent to the curve(1)

$$\tan\phi_1 = \frac{r}{\dfrac{-\dfrac{\partial u}{\partial \theta}}{\dfrac{\partial u}{\partial r}}}$$

$$\tan\phi_1 = -\frac{r\dfrac{\partial u}{\partial r}}{\dfrac{\partial u}{\partial \theta}} -----(5)$$

Differentiate (2) w. r. t. $\theta$

$$\frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial r}\frac{dr}{d\theta} = 0$$

$$\frac{dr}{d\theta} = \frac{-\dfrac{\partial v}{\partial \theta}}{\dfrac{\partial v}{\partial r}}$$

$\tan\phi_2 = \dfrac{r}{\dfrac{dr}{d\theta}}$ , where $\phi_2$ being the angle between the radius and the tangent to the curve(2)

$$\tan\phi_1 \times \tan\phi_2 = \frac{r\dfrac{\partial u}{\partial r}}{\dfrac{\partial u}{\partial \theta}} \times \frac{r\dfrac{\partial v}{\partial r}}{\dfrac{\partial v}{\partial \theta}}$$

$$= \frac{r.\dfrac{1}{r}\dfrac{\partial v}{\partial \theta}}{-r\dfrac{\partial \theta}{\partial r}} \times \frac{r\dfrac{\partial v}{\partial r}}{\dfrac{\partial v}{\partial \theta}}$$

$$= -1 \text{ form an orthogonal system}$$

**Note:** We have $z = x + iy \quad and \quad \overline{z} = x - iy$

$$Now \quad x = \frac{1}{2}(z + \overline{z})$$

$$y = \frac{1}{2i}(z - \overline{z})$$

$$Consider \quad f(z) = u(x, y) + i \quad v(x, y) ----(1)$$

$$f(z) = u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) + i \quad v\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$$

$put \quad z = \overline{z}$ we get

$$f(z) = u(z, 0) + i \quad v(z, 0) ----(2)$$

$\therefore$ (2) is same as (1) if we replace $x$ by $z$ and $y$ by 0

Similarly in polar form if we replace $r$ by $z$ and $\theta$ by 0 in $f(z) = u(r, \theta) + i \ v(r, \theta)$

This is due to Milne-Thomson

$$Note:\ (i)\ \sin(i\,x) = i\sin\text{h}\,x \quad or \quad \sin hx = \frac{1}{i}\left[\sin(i\,x)\right]$$

$$(ii)\ \cos(i\,x) = \cos hx$$

Example:1

Show that *f(z)=sin z* is analytic and hence find, $f'(z)$

**Solution:** *f(z) = sin(z)*

*=sin(x+iy)*

*=sin(x)cos(iy)+cos(x)sin(iy)*

*f(z)=sin x cos hy + i cos x sin hy*

Equating *real* and *imaginary* parts *u=sinx coshy* and *v=cosx sin hy--(1)*

*u* and *v* satisfies necessary conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \cos x \cos\text{h}\,y + i(-\sin x)\sin\text{h}\,y) ----(*)$$

$$= \cos(x)\cos(iy) - i\sin x.\frac{1}{i}\sin(iy)$$

$$= \cos(x)\cos(iy) - \sin x\sin(iy)$$

$$= \cos(x+iy)$$

$$f'(z) = \cos(z) \qquad \therefore \frac{d[\sin z]}{dz} = \cos z$$

or By Milne's Thomson method replace $x$ by $z$ and $y$ by 0 in (*)

$$f'(z) = \cos(z).1 - 0 \qquad \therefore f'(z) = \cos(z) \quad or \quad \frac{d[\sin z]}{dz} = \cos z$$

2) *Show that* $w = z + e^z$ is analytic, hence find $\dfrac{dw}{dz}$

*Solution*: Let $w = f(z) = u + iv$.

$w = (x + e^x \cos y) + i(y + e^x \sin y)$

Equating real and imaginary parts

$u = (x + e^x \cos y),\ v = (y + e^x \sin y)$

$u$ and $v$ satisfies C-R equations

consider

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= (1 + e^x \cos y) + i(e^x \sin y)$$

$$= 1 + e^x[\cos y + i \sin y] - - - (1)$$

$$= 1 + e^x . e^{iy}$$

$$= 1 + e^z$$

$$\frac{d[z + e^z]}{dz} = 1 + e^z$$

Or By Milne's-Thomson method replace $x$ by $z$ and y by 0 in (1), we get derivative of $z + e^z$

## Example-3:

show that $w = \log(z)$ *is* analytic, hence find $f'(z)$

$w = \log[re^{i\theta}]$

$w = \log(r) + i\theta$ equating real and imaginary parts

$u = \log(r)$ and $v = \theta$, u and v satisfies C-R equation in polar form.

consider

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$= e^{-i\theta}\left[\frac{1}{r}\right]$$

$$= \frac{e^{-i\theta}}{r}$$

$$f'(z) = \frac{1}{re^{1\theta}} - - - - (1)$$

$$f'(z) = \frac{1}{z}$$

$$\therefore \frac{d[\log z]}{dz} = \frac{1}{z}$$

or by Milne's Thomson method replace $r$ by $z$ and $\theta$ by 0 in RHS of (1), we get $\dfrac{d[\log z]}{dz} = \dfrac{1}{z}$

**Cauchy's-Riemann equations in Cartesian form**

**Statement:** The real and imaginary part of an analytic function *f(z)=u(x,y)+iv(x,y)* satisfies Cauchy's-Riemann equations.

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ at each point

Note: A function f(z) is analytic, then

$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ limit approaches along the x-axis

and $f'(z)=\frac{\partial u}{\partial y}-i\frac{\partial u}{\partial y}$ limit approaches along the y-axis

**Example:** The function $f(z)=z^2$ is analytic for all z, and $f'(z)=2z$

**Solution:**

$f(z)=(x^2-y^2)+i\,2xy$ is analytic every in the complex plane.

$u=x^2-y^2$ and $v=2xy$

$$\frac{\partial u}{\partial x}=2x,\quad \frac{\partial u}{\partial y}=-2y,\quad \frac{\partial v}{\partial x}=2y,\quad \frac{\partial v}{\partial y}=2x$$

$$\begin{aligned} f'(z)&=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\\ &=2x+i2y\\ &=2(x+iy)\\ &=2z \end{aligned}$$

$$\therefore \frac{d(z^2)}{dz}=2z$$

**Note:** If $f(z)=u(r,\theta)+iv(r,\theta)$ then Cauchy-Riemann equation in polar form:

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

$where\ f'(z)=e^{-i\theta}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right]$ limit approaches along the radial line and

$f'(z)=\frac{e^{-i\theta}}{r}\left[\frac{\partial v}{\partial \theta}-i\frac{\partial u}{\partial \theta}\right]$ a limit approach along angular path.

**Construction of Analytic Function:**

Construction of analytic function $f(z)=u+iv$ when $u$ or $v$ or $u\pm v$ is given.

**Example1**: Find the Analytic Function *f(z)*, whose real part is $e^{2x}[x\cos 2y-y\sin 2y]$.

**Solution:**

$$\text{Given} \quad u=e^{2x}[x\cos 2y-y\sin 2y]----(1)$$

*Differentiate* (1) w.r.t. $x$

$$\frac{\partial u}{\partial x}=e^{2x}\left[\cos 2y\right]+2e^{2x}\left[x\cos 2y-y\sin 2y\right]----(2)$$

*Differentiate* (1) w.r.t. $y$

$$\frac{\partial u}{\partial y}=e^{2x}\left[-2.x.\sin 2y-y.2\cos 2y-\sin 2y\right]----(3)$$

$$Consider \quad f'(z)=\frac{\partial u}{\partial x}+i\,\frac{\partial v}{\partial x}-----(4)$$

$By$ C-R Equations replace $\dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y}$

$$f'(z)=\frac{\partial u}{\partial x}-i\,\frac{\partial u}{\partial y}-----(5)$$

using (2) and (3) on RHS (5)

$$f'(z)=e^{2x}\left[\cos 2y+2x\cos 2y-2y\sin 2y\right]+i\,e^{2x}\left[2x\sin 2y+2y\cos 2y+\sin 2y\right]$$

By Milne's Method replace $x$ by $z$ and $y$ by 0

$f'(z)=e^{2z}[1+2z]$

$f'(z)=e^{2z}+2e^{2z}.z$

int $egrate$ we get

$$f(z)=\frac{1}{2}e^{2z}+2\left[\frac{e^{2z}}{2}.z-\frac{e^{2z}}{4}\right]+c$$

$$f(z)=\frac{1}{2}e^{2z}+ze^{2z}-\frac{1}{2}e^{2z}+c$$

$$f(z)=ze^{2z}+c$$

$Note: \quad u+iv=(x+iy)e^{2x}.e^{i2y}+c$

$=e^{2x}(x+iy)(\cos 2y+i\sin 2y)$

$u+iv=e^{2x}\left[(x\cos 2y-y\sin 2y)+i(y\cos 2y+x\sin 2y)\right]+c$

$\therefore u=e^{2x}[x\cos 2y-y\sin 2y]+c$

$v=e^{2x}(y\cos 2y+x\sin 2y)$

$Taking \quad c=0$ *we get*

$u=e^{2x}[x\cos 2y-y\sin 2y]$ which is real part

and $v=e^{2x}[y\cos 2y+x\sin 2y]$ is imaginary part of a required analytic function $f(z)$

2) Find the Analytic function whose real part is $\dfrac{\sin 2x}{\cos 2y - \cos 2x}$

$$Solution: u = \frac{\sin 2x}{\cosh 2y - \cos 2x} ------(1)$$

Differentiate w.r.t. $x$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x).2\cos 2x - \sin 2x[+2\sin 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cosh 2y \cos 2x - 2[\cos^2(2x) + \sin^2 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} ------(2)$$

Differentiate (1) w.r.t. $y$

$$\frac{\partial u}{\partial y} = \frac{\sin 2x[-(2\sinh 2y)]}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} ------(3)$$

Consider $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

By C-R equation replace $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ $\qquad f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$

$$f'(z)=\frac{\left[2\cos 2x\cosh 2y-2\right]+i2\sin 2x\sinh 2y}{(\cosh 2y-\cos 2x)^2}$$

By Milne's Thomson method replace $x$ by $z$ and $y$ by 0

$$f'(z)=\frac{2[\cos 2z-1]+i.0}{(1-\cos 2z)^2}$$

$$f'(z)=\frac{-2[1-\cos 2z]}{(1-\cos 2z)^2}$$

$$f'(z)=\frac{-2}{[1-\cos 2z]}$$

$$f'(z)=\frac{-2}{2\sin^2 z}$$

$$f'(z)=-\cos ec^2 z$$

$\operatorname{int} ergate$

$$f(z)=+\cot z+c$$

3) Construct the analytic function whose imaginary part is $\left(r-\frac{1}{r}\right)\sin\theta,\ r\neq 0.$

Hence find the Real part.

Solution: Given $v=\left(r-\frac{1}{r}\right)\sin\theta----(1)$

*Differentiate* (1) $w.r.t.\ \theta$

$$\frac{\partial u}{\partial\theta}=\left(r-\frac{1}{r}\right)\cos\theta-----------(2)$$

*Differentiate* (1) $w.r.t.\ r$

$$\frac{\partial u}{\partial r}=\left(1+\frac{1}{r^2}\right)\sin\theta------------(3)$$

$$Consider\ \ f'(z)=e^{-i\theta}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right]-----(4)$$

By C-R Equation replace $\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ $\quad on$

$RHS\ of\ (4)\ we\ get$

$$f'(z)=e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta}+i\frac{\partial v}{\partial r}\right]$$

$$f'(z)=e^{-i\theta}\left[\frac{1}{r}\left(r-\frac{1}{r}\right)\cos\theta+i\left(1+\frac{1}{r^2}\right)\sin\theta\right]$$

By Milne's method replace $r$ by $z$ and $\theta$ by 0

$$f'(z)=e^{0}\left[\frac{1}{z}\left(z-\frac{1}{z}\right).1+i.0\right]$$

$$f'(z)=\left(1-\frac{1}{z^2}\right)$$

Integrate we get

$$f(z)=z+\frac{1}{z}+ic$$

To find real part: Consider $f(z)=re^{i\theta}+\dfrac{1}{re^{i\theta}}+ic$

$$u+iv=\left(r\cos\theta+ir\sin\theta\right)+\frac{1}{r}\left(\cos\theta-i\sin\theta\right)+ic$$

$$u+iv=\left(r+\frac{1}{r}\right)\cos\theta+i\left[\left(r-\frac{1}{r}\right)\sin\theta+c\right]$$

$E$quating real and imaginary parts

$$u = \left( r + \frac{1}{r} \right) \cos\theta$$

$$v = \left( r - \frac{1}{r} \right) \sin\theta + c$$ to get actual imaginary part of an analytical function

$f(z) = u + iv$ *taking* $c = 0$

$$\therefore v = \left( r - \frac{1}{r} \right) \sin\theta$$

4) Find an analytic function $f(z)$ as a function of $z$

given that the sum of real and imaginary part is $x^3 - y^3 + 3xy(x - y)$

*Solution* : The sum of real and imaginary part is given by

$$u + v = x^3 - y^3 + 3xy(x - y) - - - - - - - - (1)$$

*Differentiate* (1) *w.r.t.* $x$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 - 0 + 3xy + 3y(x - y)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 + 3xy + 3y(x - y) - - - - (2)$$

*Differentiate* (1) *w.r.t.* $y$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0 - 3y^2 + 3xy(-1) + 3x(x - y)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -3y^2 - 3xy + 3x(x - y) - - - - - - - (3)$$

By C-R Equation replace $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad in(3)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = -3y^2 - 3xy + 3x(x-y) ------(4)$$

*Consider*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 + 3xy + 3y(x-y)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = -3y^2 - 3xy + 3x(x-y)$$

$$2\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + (x-y)3(x+y)$$

$$2\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 -----------(5)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 + 3xy + 3y(x-y)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = -3y^2 - 3xy + 3x(x-y)$$

$$2\frac{\partial v}{\partial x} = 3x^2 + 3y^2 + 6xy + (x-y).3(y-x)$$

$$= 3x^2 + 3y^2 + 6xy - 3(x-y)^2$$

$$= 3x^2 + 3y^2 + 6xy - 3x^2 - 3y^2 + 6xy$$

$$= 12xy$$

$$\frac{\partial u}{\partial x} = 6xy --------(6)$$

$$Consider\ f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$=\left(3x^2-3y^2\right)+i6xy\left[by\ (5)\&(6)\right]$$

$By$ Milne's Thomson method replace $x$ by $z$ and $y$ by 0

$$f'(z)=3z^2$$

$\text{int}\,egrat$

$$f(z)=z^3+c$$

5) Find an analytic function f(z)-u+iv, given that u+v=$\frac{1}{r^2}\left[\cos 2\theta-\sin 2\theta\right], r\neq 0$

$$Solution: u+v=\frac{1}{r^2}\left[\cos 2\theta-\sin 2\theta\right]-----(1)$$

$Differentiate$ (1) $w.r.t.\, r$

$$\frac{\partial u}{\partial r}+\frac{\partial v}{\partial r}=-\frac{2}{r^3}\left[\cos 2\theta-\sin 2\theta\right]-----(2)$$

$Differentiate$ (1) $w.r.t.\, \theta$

$$\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial \theta}=\frac{2}{r^2}\left[-2\sin 2\theta-2\cos 2\theta\right]-----(3)$$

By C-R Equations

$$\left.\begin{aligned}\frac{\partial u}{\partial r}&=\frac{1}{r}\frac{\partial v}{\partial \theta}\\ \frac{\partial u}{\partial \theta}&=-r\frac{\partial v}{\partial r}\end{aligned}\right\}\text{in LHS of (3)}$$

$$-r\frac{\partial v}{\partial r}+r\frac{\partial u}{\partial r}=\frac{-2}{r^2}[\sin 2\theta+\cos 2\theta]$$

$$\frac{\partial u}{\partial r}-\frac{\partial v}{\partial r}=\frac{-2}{r^3}[\sin 2\theta+\cos 2\theta]------(4)$$

*Consider*

$$\begin{aligned}\frac{\partial u}{\partial r}+\frac{\partial v}{\partial r}&=\frac{-2}{r^3}[\cos 2\theta-\sin 2\theta]\\ \frac{\partial u}{\partial r}-\frac{\partial v}{\partial r}&=\frac{-2}{r^3}[\cos 2\theta+\sin 2\theta]\\ \hline 2\frac{\partial u}{\partial r}&=\frac{-2}{r^3}[2\cos 2\theta]\end{aligned}$$

$$\frac{\partial u}{\partial r}=\frac{-2}{r^3}\cos 2\theta----------------(5)$$

Subtract (3)-(4) we get

$$2\frac{\partial u}{\partial r}=-\frac{2}{r^3}[-2\sin 2\theta]$$

$$\frac{\partial u}{\partial r}=\frac{2}{r^3}\sin 2\theta----------(6)$$

$$Consider\ \ f'(z)=e^{-i\theta}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right]$$

$$f'(z)=e^{-i\theta}\left[-\frac{2}{r^3}\cos 2\theta+i\frac{2}{r^3}\sin 2\theta\right]$$

*By* Milne's Thomson method replace $r$ by $z$ and $\theta$ by 0

$$f'(z)=-\frac{2}{r^3}$$

int *egrate*

$$f(z)=-2\left(-\frac{1}{2z^2}\right)+\text{c}$$

$$f(z)=\frac{1}{z^2}+\text{c}$$

6) Show that $u = \left( r + \frac{1}{r} \right) \cos\theta$ is harmonic. find its harmonic conjugate and also corresponding analytic function.

Solution: Given $u = \left( r + \frac{1}{r} \right) \cos\theta --------(1)$

we shall show that u is a solution of Laplace's equation in two variables in polar form.

i.e $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - 0 ----------(2)$

Differentiate (1) w.r.t. $r$

$$\frac{\partial u}{\partial r} = \left( 1 - \frac{1}{r^2} \right) \cos\theta ---------------(3)$$

Differentiate (3) w.r.t. $r$

$$\frac{\partial^2 u}{\partial r^2} = +\frac{2}{r^3} \cos\theta -----------------(4)$$

Differentiate (1) w.r.t. $\theta$

$$\frac{\partial u}{\partial \theta} = \left( 1 + \frac{1}{r} \right) (-\sin\theta) ---------------(5)$$

Differentiate (5) w.r.t. $\theta$

$$\frac{\partial^2 u}{\partial \theta^2} = -\left( r + \frac{1}{r} \right) \cos\theta --------(6)$$

*Consider*

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{2}{r^3}\cos\theta + \frac{1}{r}\left( 1 - \frac{1}{r^2} \right)\cos\theta - \frac{1}{r^2}\left( r + \frac{1}{r} \right)\cos\theta$$

$$= \frac{2}{r^3}\cos\theta + \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta - \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta$$

$$= \frac{2}{r^3}\cos\theta - \frac{2}{r^3}\cos\theta$$

$$= 0$$

$\therefore u$ is solution of equation(2)

Hence $u$ is harmonic function.

*Consider*

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] ------- (7)$$

By C-R Equation $\frac{\partial u}{\partial \theta}$=-r $\frac{\partial v}{\partial r}$

$$\therefore\ replace\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}\ in\ (7)$$

$$f'(z) = e^{-i\theta}\left[\left(1 - \frac{1}{r^2}\right)\cos\theta - \frac{i}{r}\left(r + \frac{1}{r}\right)\sin\theta\right]$$

By Milne's Thomson method replace $r$ by $z$ and $\theta$ by 0

$$f'(z) = \left(1 - \frac{1}{z^2}\right) - i.o$$

$$f'(z) = \left(1 - \frac{1}{z^2}\right)$$

Integrate

$$f(z) = z + \frac{1}{z}$$

*To* find harmonic Conjugate

consider $\quad u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Equating real and imaginary parts

$$\therefore u = \left(r + \frac{1}{r}\right)\cos\theta$$

$$v = \left(r - \frac{1}{r}\right)\sin\theta$$

which is required conjugate harmonic

7) If *f(z)* is a regular function of z show that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

**Solution:**

We have $f(z) = u + iv$

$$\therefore |f(z)| = \sqrt{u^2+v^2} \text{ --------(1)}$$

$$|f(z)|^2 = u^2 + v^2 \text{ ---------(2)}$$

and $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$\therefore |f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \text{ -----(3)}$$

Differentiate (2) w.r.t. *x*

$$\frac{\partial |f(z)|^2}{\partial x} = \frac{\partial}{\partial x}(u^2+v^2)$$

$$= 2u\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x}$$

Again differentiate w.r.t. *x*

$$\frac{\partial^2 |f'(z)|^2}{\partial x^2} = 2\left\{\frac{\partial}{\partial x}\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right]\right\}$$

$$= 2\left\{u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial x}\right\}$$

$$2\left\{u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right\} \text{ ------(4)}$$

Similarly Differentiate (2) w.r.t. $y$ we get

$$\frac{\partial^2 |f(z)|^2}{\partial y^2} = 2\left\{u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right\} ------(5)$$

Adding (4) and (5) we get

$$\frac{\partial^2 |f(z)|^2}{\partial x^2} + \frac{\partial^2 |f(z)|^2}{\partial y^2} = 2\left\{u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + v\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} ---(6)$$

w. k. t. if $f(z)=u+iv$ is regular or analytic function then real part $u$ and imaginary part $v$ satisfies Laplace equation in two variables or two dimensional Laplace equation.

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Using these on RHS of (6)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

By C-R Equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$= 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\}$$

$$= 2\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2\right\}$$

$$= 4\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

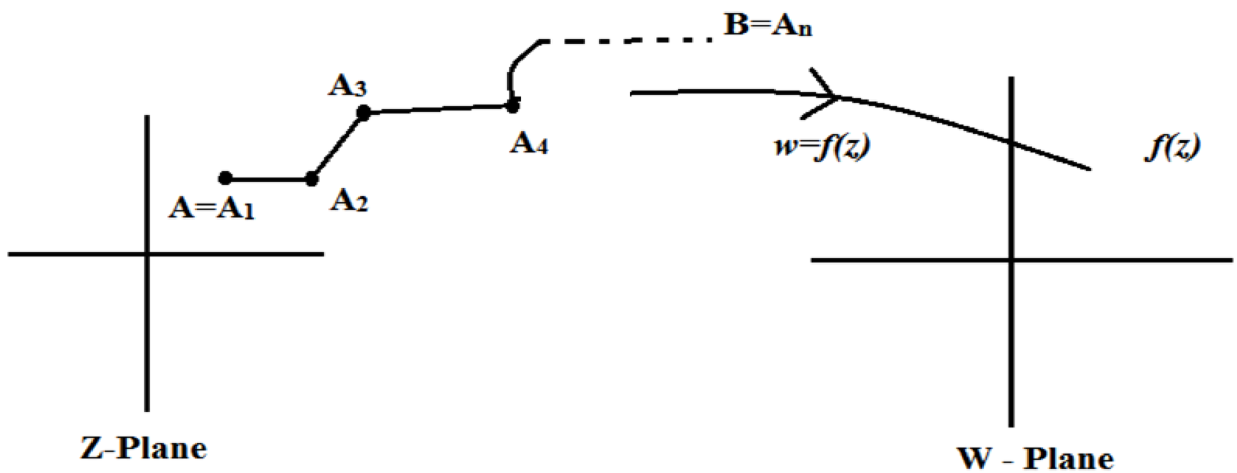
$$= 4|f'(z)|^2 \quad \text{[from (3)]}$$

**Complex integration:**

**Line Integral:**

Let *f(z)* be a single valued complex function and continuous defined at each point on a curve C between end points A and B, in the z-plane. Then the line integral of *f(z)* along the curve C traversed from A to B is denoted by

$$\int_A^B f(z)dz \quad \text{or} \quad \int_C f(z)dz$$




**Note:** Now, we divide this curve C into n parts between the points $A = A_1(z_1), A_2(z_2), ------ A_n(z_n) = B$

We get *n* line segments say $C_1 : A_1 \; to \; A_2, C_2 : A_2 \; to \; A_3 -------- C_n : A_{n-1} \; to \; A_n$

$\therefore C : C_1 \cup C_2 \cup C_3 \cup --------- \cup C_{n,}$ is union of $C_1 C_2 --- C_n$

$$\int_C f(z)dz = \int_{C_1 \cup C_2 \cup C_3 \cup ---- \cup C_n} f(z)dz$$

$$= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + ------ \int_{C_n} f(z)dz$$

or

$$\int_A^B f(z)dz = \int_{A_1}^{A_2} f(z)dz + \int_{A_2}^{A_3} f(z)dz + \int_{A_3}^{A_4} f(z)dz + ------- \int_{A_{n-1}}^{A_n} f(z)dz$$

Note: If curve C is traversed from B to A then line integral of $f(z)$ along C is

$$\int_{-C} f(z)dz = -\int_{C} f(z)dz$$

$$ie.\quad \int_{B}^{A} f(z)dz = -\int_{A}^{B} f(z)dz$$

Note: Now setting $z = x + iy$

$$f(z) = u(x, y) + i\, v(x, y)$$

$$\text{or}\quad f(z) = u + iv$$

$$\therefore dz = dx + idy$$

$$\int_{C} f(z)dz = \int_{C} (u + iv)(dx + idy)$$

$$= \int_{C} (udx - vdy) + i(vdx + udy)$$

$$\int_{C} f(z)dz = \int_{C} (udx - vdy) + i\int_{C} (vdx + udy)$$

This shows that evaluationof the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

$$\int_{A(x_1,y_1)}^{B(x_2,y_2)} f(z)dz = \int_{(x_1,y_1)}^{(x_2,y_2)} (udx - vdy) + i\int_{(x_1,y_1)}^{(x_2,y_2)} (vdx + udy)$$

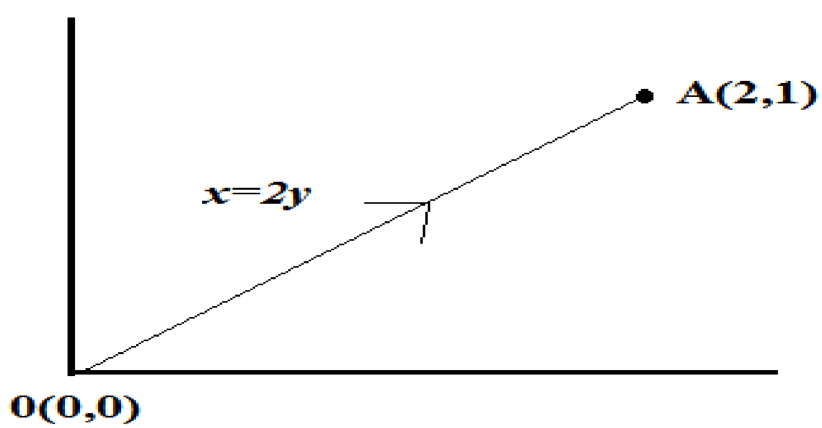
Example: Evaluate $\int_{0}^{2+i} (\bar{z})^2\, dz$ along

(i) The line $y = \dfrac{x}{2}$, (ii) The real axis upto 2 and then vertically to $2 + i$

Solution: We have $\bar{z} = x - iy$

$dz = dx + idy$

$(i)$ Line integral of $f(z)=(\bar{z})^2$ along the curve $x=2y$ between the points $z_1 = 0$ and $z_2 = 2 + i$



Along *oA:* $x=2y \quad \therefore dx=2dy$

$$\int_0^{2+i} (\bar{z})^2\, dz = \int_{(0,0)}^{(2,1)} (x-iy)^2(dx+idy)$$

Replace $x=2y$ and $dx=2dy$

$$\int_0^{2+i} (\bar{z})^2\, dz = \int_{(0,0)}^{(2,1)} (2y-iy)^2(2dy+idy)$$

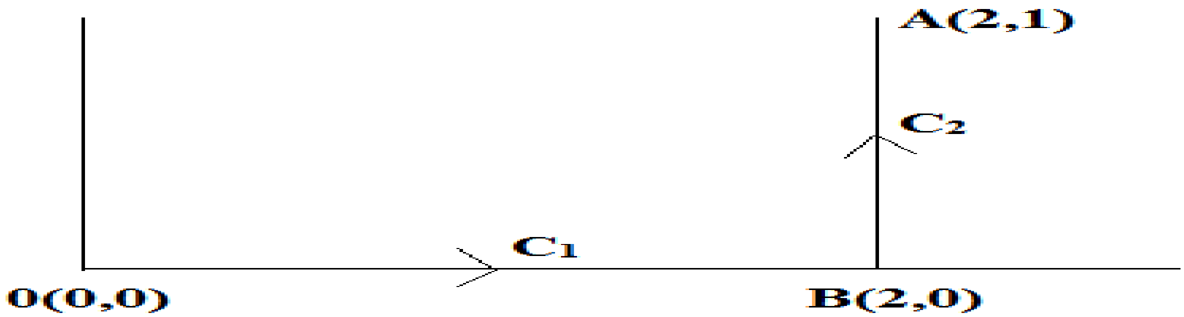
Here integral is a function of y alone and y varies from 0 to 1

$$= \int_{y=0}^{1} (2-i)^2(2+i)y^2.dy$$

$$= \frac{5}{3}(2-i)\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{5}{3}(2-i)$$

(ii) Line intergral along the real axis upto *2* and then vertically *(2+i)*




Here Curve $C:\ C_1 \cup C_2$

where $C:$ O *to* $A$ divided into $C_1 : O\ to\ B$ and $C_2 : B\ to\ A$

$$\int_0^{2+i} (\bar{z})^2\, dz = \int_{(0,0)}^{(2,0)} (x-iy)^2(dx+idy) + \int_{(2,0)}^{(2,1)} (x-iy)^2(dx+idy)$$

In the first integral x is varies from 0 to 2 and y=0 $\therefore$ dy=0

In the second integral y is varies from 0 to 1 and x=2 $\therefore$ dx=0

Using these on RHS of the above integral

$$= \int_{x=0}^{2} x^2 dx + \int_{y=0}^{1} (2-iy)^2 .idy$$

$$= \frac{x^3}{3}\Bigg|_0^2 + i\left.\frac{(2-iy)^3}{-3i}\right]_0^1$$

$$= \frac{8}{3} - \frac{1}{3}\left[(2-i)^3 - 8\right]$$

$$= \frac{8}{3} - \frac{1}{3}[(3-4i)(2-i)-8]$$

$$= \frac{8}{3} - \frac{1}{3}[-6-11i]$$

$$= \frac{8}{3} + \frac{1}{3}(6+11i)$$

$$= \frac{1}{3}[14+11i]$$

$Example - 2: Evalute \int_C z^3 dz$ ,along the circle $|z| = 1.$

Solution: The given Curve C is $|z| = 1$

Complex variable $z$ in polar form

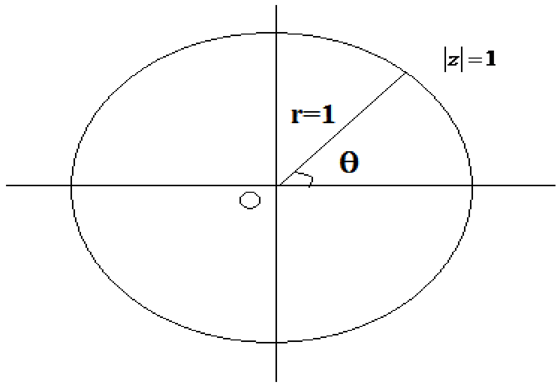
$$z = re^{i\theta}$$

$r = 1$ and $\theta$ varies from 0 to $2\pi$

$$z = e^{i\theta}$$

$$dz = e^{i\theta}.id\theta$$

$$dz = ie^{i\theta}d\theta$$




$$\therefore \int_C z^3 dz = \int_{|z|=1} z^3 dz$$

Along $|z|=1$, r=1, $\theta$=0 to $2\pi$

$$= \int_{\theta=0}^{2\pi} (e^{i\theta})^3 . i \, e^{i\theta} d\theta$$

$$= i \int_{\theta=0}^{2\pi} e^{4i\theta} d\theta$$

$$= i \left[ \frac{e^{4i\theta}}{4i} \right]_0^{2\pi}$$

$$= \frac{1}{4} \left[ e^{8\pi i} - 1 \right]$$

$$\int_C z^3 dz = \frac{1}{4} [\cos 8\pi + i \sin 8\pi - 1]$$

$$= \frac{1}{4} [1 + 0 - 1]$$

$$= 0$$

Example-3: Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy$ along

(i) The parabola $x = 2t$ and $y = t^2 + 3$

(ii) The straight line from (0,3) to (2,4)

*Solution*:

(i) Along x=2t and $y = t^2 + 3$, from the given limit, x $\rightarrow$ 0 to 2 and $y \rightarrow 3$ *to* 4. Compute limit for $t$ ie.

| $x$ | $t=\frac{x}{2}$ | $y$ | $t=\sqrt{y-3}$ |
|---|---|---|---|
| 0 | 0 | 3 | 0 |
| 2 | 1 | 4 | 1 |

Here t varies from 0 to 1, as x varies from 0 to 2 and y varies from 3 to 4

$\therefore x = 2t \qquad dx = 2dt$

$y = t^2 + 3 \qquad$ dy=2tdt

Let $I = \int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy$

$$I = \int_{t=0}^{1} \left[2(t^2+3) + 4t^2\right]2dt + \left[6t - t^2 - 3\right]2t\,dt$$

$$= 2\int_{t=0}^{1} \left[6t^2 + 6\right]dt + \left[6t^2 - t^3 - 3t\right]dt$$

$$= 2\int_{t=0}^{1} \left[6t^2 + 6 + 6t^2 - t^3 - 3t\right]dt$$

$$= 2\int_{t=0}^{1} \left[12t^2 - 3t - t^3 + 6\right]dt$$

$$= 2\left[\frac{12t^3}{3} - \frac{3t^2}{2} - \frac{t^4}{4} + 6t\right]_0^1$$

$$= 2\left[\frac{12}{3} - \frac{3}{2} - \frac{1}{4} + 6\right]$$

$$= 2\left[10 - \frac{7}{4}\right]$$

$$= 2\frac{[40-7]}{4}$$

$$= \frac{33}{2}$$

(ii) Along straight line from $(0,3)$ to $(2,4)$.

Equation of line joining the points $(0,3)$ to $(2,4)$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{(x_2-x_1)}$$

$$\frac{y-3}{x-0}=\frac{4-3}{(2-0)}$$

$$\frac{y-3}{x}=\frac{1}{2}$$

$$x=2y-6 \qquad \text{or} \qquad y=\frac{1}{2}[x+6]$$

Let $\text{I}=\int_{(0,3)}^{(2,4)}(2y+x^2)dx+(3x-y)dy------(1)$

Taking $y=\frac{1}{2}(x+6)$ $\quad \therefore dy=\frac{dx}{2}$ and $x$ varies from 0 to 2

$$\text{I}=\int_0^2\left[2.\frac{1}{2}(x+6)+x^2\right]dx+\left[3x-\frac{1}{2}(x+6)\right]\frac{dx}{2}$$

$$=\int_0^2(x^2+x+6)dx+(6x-x-6)\quad\frac{dx}{4}$$

$$= \frac{1}{4}\int_0^2 \left[4x^2 + 4x + 24 + 5x - 6\right] dx$$

$$= \frac{1}{4}\int_0^2 \left(4x^2 + 9x + 18\right) dx$$

$$= \frac{1}{4}\left[4\frac{x^3}{3} + 9\frac{x^2}{2} + 18x\right]_0^2$$

$$= \frac{1}{4}\left[4 \times \frac{8}{3} + 9 \times \frac{4}{2} + 36\right]$$

$$= \frac{1}{4}\left[\frac{32}{3} + 18 + 36\right]$$

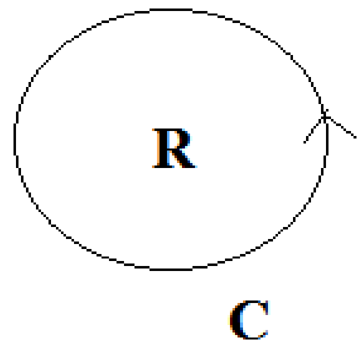
$$= \frac{1}{4}\left[\frac{32 + 54 + 108}{3}\right]$$

$$= \frac{194}{12}$$

$$= \frac{97}{6}$$

## Cauchy's Theorem

Statement: If f(z) is analytic function and $f'(z)$ is continuous at all points inside and on a simple closed curve C then $\int_C f(z)dz = 0$




Proof : Let $f(z)=u+iv$ and $z=x+iy$,
$dz = dx + idy$ as usual.
Then

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy) --------(1)$$

The given curve in the complex plane is a simple closed curve C

Greens Theorem states that

$$\int_C Mdx + Ndy = \iint_A \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy,$$ Where A is a region bounded by A

Applying this theorem on RHS of (1) we obtain

$$\int_C f(z)dz = \iint_A \left[ \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} \right] dx\, dy + i\iint_A \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] dx\, dy$$

Since $f(z)$ is analytic, we have Cauchy Riemann Equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\int_C f(z)dz = \iint_A \left[ -\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \right] dx\, dy + i\iint_A \left[ \frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \right] dx\, dy$$

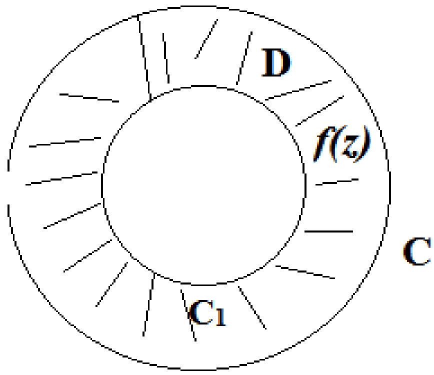
$=0$ This proves Cauchy's Theorem

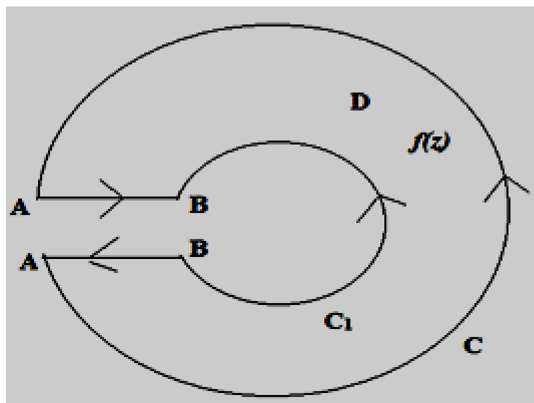
## Extension of Cauchy's Theorem:

If $f\left(z\right)$ is analytic in the region $D$ between two simple closed curve $C$ and $C_1$, then

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

To Prove this, we need to introduced the cross cut AB, say




Now $f(z)$ is analytic at all points inside and on a simple closed curve

$\square : C \cup AB \cup C_1 \cup BA,$ By Cauchy's Theorem

$$\int_{\square} f(z)dz = 0$$

$$\int_{C \cup AB \cup C_1 \cup BA} f(z)dz = 0$$

$$\int_C f(z)dz + \int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{BA} f(z)dz = 0$$

$$\int_C f(z)dz + \int_{AB} f(z)dz + \int_{-C_1} f(z)dz + \int_{-AB} f(z)dz = 0$$

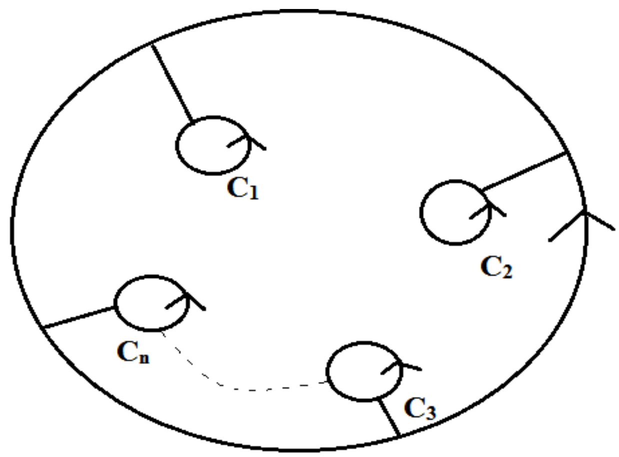
$$\int_C f(z)dz + \int_{AB} f(z)dz - \int_{C_1} f(z)dz - \int_{AB} f(z)dz = 0$$

$$\int_C f(z)dz - \int_{C_1} f(z)dz = 0$$

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

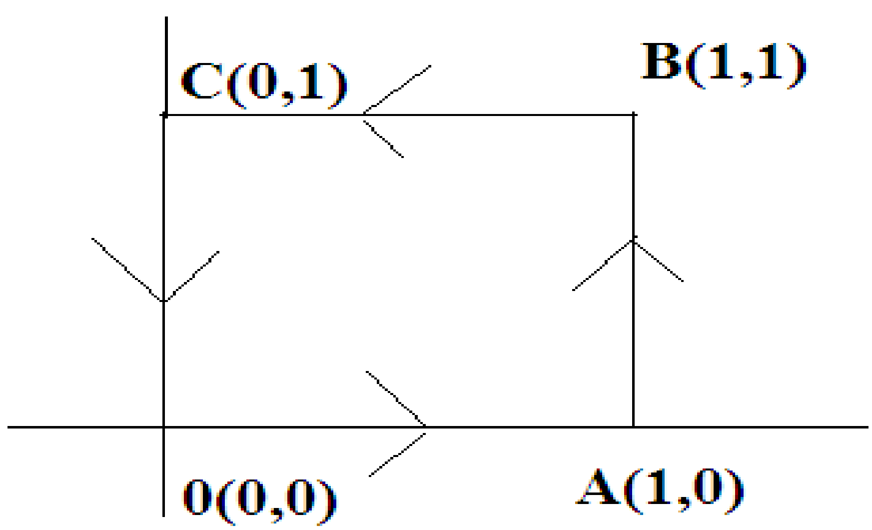
If $C_1, C_2, C_3 \ldots\ldots\ldots\ldots C_n$ be any $n$ number of closed curves with in C then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \int_{C_n} f(z)dz$$




**Example:** Verify Cauchy's Theorem for the function $f(z) = z^2$ where C is the square having vertices (0,0), (1,0), (1,1),(0,1).

**Solution:**



Here the given curve C is the square in the Complex plane as shown in the above figure.

Since $f(z) = z^2$ is analytic everywhere in the complex plane, it is analytic at all points inside and on the curve C.

By Cauchy's Theorem

$$\int_C f(z)dz = 0$$

$$\int_C z^2 dz = 0.................(*)$$

$$\int_C z^2 dz = \int_{OA} z^2 dz + \int_{AB} z^2 dz + \int_{BC} z^2 dz + \int_{CO} z^2 dz$$

$$\int_C z^2 dz = \int_{(0,0)}^{(1,0)} z^2 dz + \int_{(1,0)}^{(1,1)} z^2 dz + \int_{(1,1)}^{(0,1)} z^2 dz + \int_{(0,1)}^{(0,0)} z^2 dz.................(1)$$

Consider $\int_{(0,0)}^{(1,0)} z^2 dz = \int_{(0,0)}^{(1,0)} (x+iy)^2 (dx+idy)$

Here $y = 0 \quad \therefore dy = 0$ and $x$ varies from 0 to 1

$$= \int_{x=0}^{1} (x+io)^2 (dx+o)$$

$$= \int_{x=0}^{1} x^2 dx$$

$$= \frac{1}{3}...............(2)$$

Consider $\int_{(1,0)}^{(1,1)} z^2 dz = \int_{(1,0)}^{(1,1)} (x+iy)^2 (dx+idy)$

Here $x = 1,\ dx = 0$ and y varies from 0 to 1

$$= \int_{y=0}^{1} (1+iy)^2 (idy)$$

$$= i \int_{y=0}^{1} (1+iy)^2$$

$$= i\left[\frac{(1+iy)^3}{3i}\right]_0^1 \qquad\qquad (1+i)^2 = 2i$$

$$= \frac{1}{3}\left[(1+i)^3 - 1\right]$$

$$= \frac{1}{3}\left[(1+i)(2i) - 1\right]$$

$$= \frac{1}{3}\left[2i - 2 - 1\right]$$

$$= \frac{1}{3}\left[2i - 3\right]$$

$$= \frac{2}{3}i - 1 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Consider $\int_{(1,1)}^{(0,1)} (x+iy)^2 (dx+idy)$

Here $y = 1,\ dy = 0$ and $x$ varies from 1 to 0

$$= \int_{x=1}^{0} (x+i)^2\, dx$$

$$= \left.\frac{(x+i)^3}{3}\right|_1^0$$

$$= \frac{1}{3}\left[i^3 - (1+i)^3\right]$$

$$= \frac{1}{3}\left[-i - (1+i)2i\right]$$

$$= \frac{1}{3}\left[-i - 2i + 2\right]$$

$$= \frac{1}{3}\left[-3i + 2\right]$$

$$= -i + \frac{2}{3} \ldots\ldots\ldots\ldots(4)$$

Consider $\int_{(0,1)}^{(0,0)} z^2 dz = \int_{(0,1)}^{(0,0)} (x+iy)^2 (dx+idy)$

Here $x=0$, $dx=0$ and $y$ varies from 1 to 0

$= \int_{y=1}^{0} (iy)^2 i dy$

$= -i\left[\frac{y^3}{3}\right]_1^0$

$= -i\left[0-\frac{1}{3}\right]$

$= \frac{i}{3}$...............(5)

Substitute 2,3,4&5 on RHS of (1)

$\int_C z^2 dz = \frac{1}{3}+\frac{2i}{3}-1+\frac{2}{3}-i+\frac{i}{3}$

$= -\frac{2}{3}+\frac{2i}{3}+\frac{2}{3}-\frac{2i}{3}$

$= 0$ Hence Cauchy's Theorem verified

If $C$ is the circle $|z|=1$ verify Cauchy's Theorem for $f(z)=z^3$

## Example-2:

Show that $\int_C |z|^2 dz = i-1,$ where C is the square having vertices (0,0)(1,0)(1,1)(0,1).

Give the reason for Cauchy's theorem not being satisfied.

Solution:-

$$\int_C |z|^2 dz = \int_{0A} |z|^2 dz + \int_{AB} |z|^2 dz + \int_{BC} |z|^2 dz + \int_{C0} |z|^2 dz$$

$$= \int_{(0,0)}^{(1,0)} (x^2+y^2)(dx+idy) + \int_{(1,0)}^{(1,1)} (x^2+y^2)(dx+idy) + \int_{(1,1)}^{(0,1)} (x^2+y^2)(dx+idy) + \int_{(0,1)}^{(0,0)} (x^2+y^2)(dx+idy)$$

$$= \int_{x=0}^{1} x^2 dx + \int_{y=0}^{1} (1+y^2) i dy + \int_{x=1}^{0} (x^2+1) dx + \int_{y=1}^{0} y^2 . i dy$$

$$= \frac{1}{3} + i\left(\frac{4}{3}\right) - \frac{4}{3} - \frac{i}{3}$$

$$= -1 + i$$

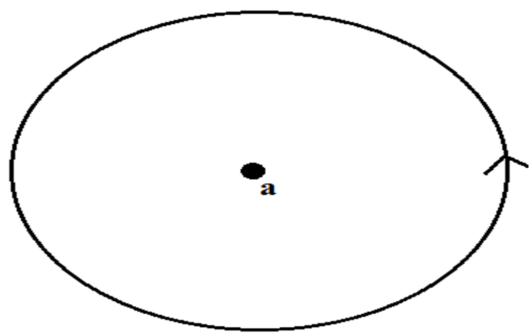
$\therefore \int_C |z|^2 = i - 1 \neq 0$. Hence Cauchy's Theorem is not verified since $f(z) = |z|^2 = x^2 + y^2$

ie. $u + iv = x^2 + y^2$ is not analytic. The necessary conditions $u_x = v_y$, $u_y = -v_x$ are not satisfied. This is the reason for Cauchy's Theorem not being satisfied.

## Cauchy's Integral formula:

**Statement:** If *f(z)* is analytic within and on a closed curve *C* and if *a* is any point within *C*, then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz$

**Proof:** Consider a closed curve *C* with '*a*' is a point within *C*




Consider function $\frac{f(z)}{(z-a)}$ which is a analytic at all points within $C$ except at $z = a$.

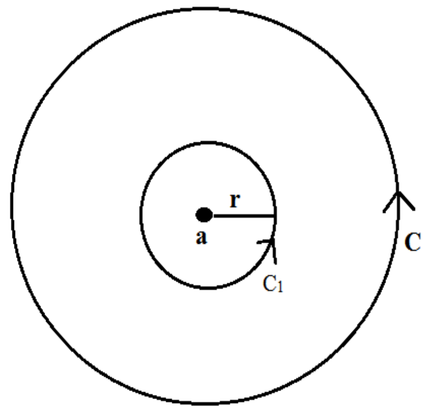
with the point '$a$' as centre and radius r, draw a small circle $C_1$

lying entirely within $C$

Now $\frac{f(z)}{(z-a)}$ being analytic in the region

enclosed by $C_1$ and $C$, we have by Cauchy's Theorem

$$\int_C \frac{f(z)}{z-a} dz = \int_{C_1} \frac{f(z)}{(z-a)} dz$$




For any point $z$ on $C_1$, $z - a = re^{i\theta}$

and $dz = i\,re^{i\theta}d\theta \qquad \therefore z = a + re^{i\theta}$

Where $\theta$ varies from 0 to $2\pi$

$$\int_C \frac{f(z)}{(z-a)}dz = \int_0^{2\pi} \frac{f(a+r\,e^{i\theta})}{r\,e^{i\theta}}.ire^{i\theta}d\theta$$

$$= i\int_0^{2\pi} f(a+r\,e^{i\theta})d\theta$$

in the limiting form, as the circle $C_1$ shrinks to the point 'a' ie as $r \to 0$,

The above line integral approach to

$$\int_C \frac{f(z)}{(z-a)}dz = i\int_0^{2\pi} f(a)d\theta$$

$$= i\,f(a)\int_0^{2\pi} d\theta$$

$$= 2\pi i.\,f(a)$$

$$\therefore\ f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz,$$

which is the desired Cauchy's Integral formula

Note:- Generalized the Cauchy's Integral formula:

(i) $f'(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-a)^2}dz$

(ii) $f''(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-a)^3}dz$ and so on

$$f^n(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Note:- In view of solving problems we consider Cauchy's integral formula as

$$\int_C \frac{f(z)}{(z-a)}dz = \begin{cases} 2\pi i\ f(a) & \text{if } a \text{ is inside } C \\ 0 & \text{if } a \text{ is outside C} \end{cases}$$

**Problems on Cauchy's Integral formula:**

Example-1:

Evaluate $\displaystyle\int_C \frac{e^z}{(z+i\pi)}dz$ over each of the following regions C:

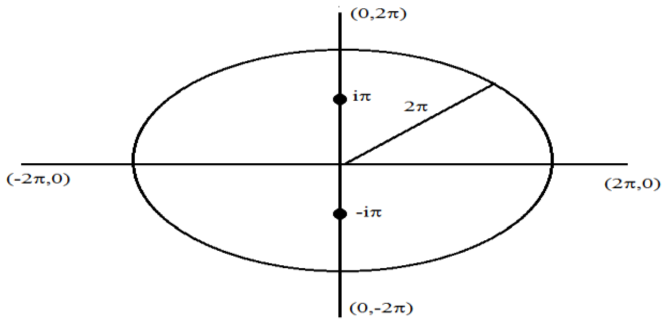
$(i)$ $|z| = 2\pi$ (ii) $|z| = \dfrac{\pi}{2}$ (iii) $|z-1| = 1$

Solution:

$$\int_C \frac{e^z}{(z+i\pi)}dx = \int_C \frac{f(z)}{\left[z-(-)i\pi\right]}dz$$

where f(z)=e$^z$, which is analytic everywhere in the complex plane

(i) $|z| = 2\pi$ is a circle centre at the origin and radius $2\pi$




$$\int_C \frac{e^z}{(z+i\pi)}dz = \int_C \frac{f(z)}{[z-(-i\pi)]}dz$$

Here the point $a = -i\pi$ lies inside the circle $|z| = 2\pi$ and $f(z) = e^z$

is analytic within and on the circle $|z| = 2\pi$. By Cauchy's Integral Formula

$= 2\pi i\ f(-i\pi)$

$= 2\pi i\ e^{-i\pi}$

$= 2\pi i[\cos\pi - i\sin\pi]$

$= -2\pi i$

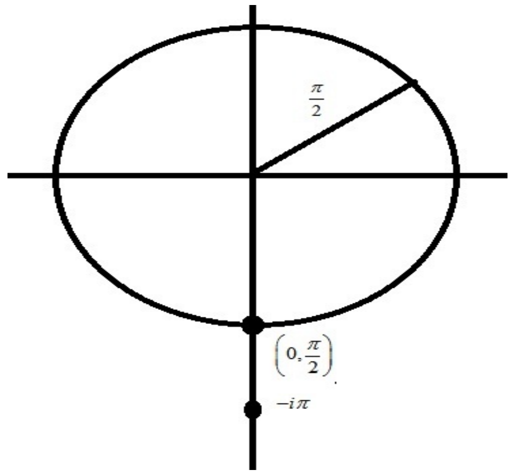
(ii) $|z| = \dfrac{\pi}{2}$ is a circle centre at the origin and radius $\dfrac{\pi}{2}$

$$\int_C \frac{e^z}{(z+i\pi)}dz = \int_C \frac{f(z)}{[z-(-i\pi)]}dz$$

Here point a=-i$\pi$ lies outside the circle

circle $|z| = \dfrac{\pi}{2}$, by Cauchy,s Integral

formula $\displaystyle\int_C \frac{e^z}{(z+i\pi)} = 0$



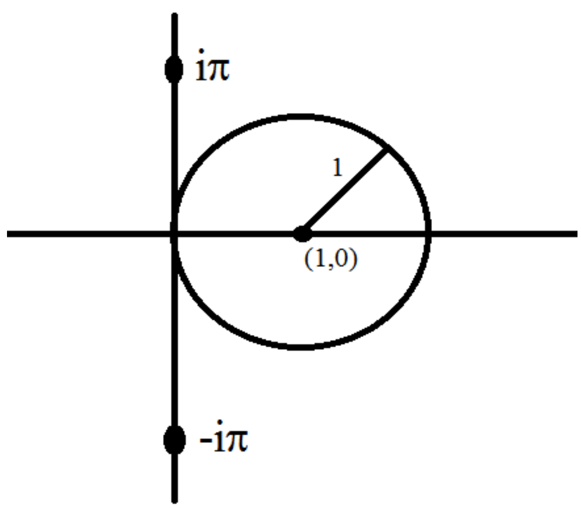

(iii) $|z-1|=1$ is a circle centre at the point (1.0) and radius 1.

$$\int_C \frac{e^z}{(z+i\pi)}dz = \int_C \frac{f(z)}{[z-(-i\pi)]}dz$$

Here point a=-i$\pi$ lies outside the circle

$|z-1|=1$ by Cauchu's Integral formula

$$\int_C \frac{e^z}{(z+i\pi)}dz = 0$$



Evaluate using Cauchy's integral formula:

(i) $\int_C \frac{e^{2z}}{(z+1)(z-2)}dz$ where C represents the circle $|z|=3$.

Solution: $\int_C \frac{e^{2z}}{(z+1)(z-2)}dz = \int_C \frac{f(z)}{(z+1)(z-2)}dz$..................(1)

Where $f(z)=e^{2z}$ which is analytic every where in the complex plane.

Consider $\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$

$$1 = A(z-2) + B(z+1)$$

put z = 2, $\quad B = \frac{1}{3}$

put z = -1 $\quad A = -\frac{1}{3}$

$$\frac{1}{(z+1)(z-2)} = \frac{-\frac{1}{3}}{(z+1)} + \frac{\frac{1}{3}}{(z-2)}$$

$$= \frac{1}{3}\left[\frac{1}{(z-2)} - \frac{1}{(z+1)}\right] \text{................(2)}$$

using (2) in (1) we get

$$\int_C \frac{e^{2z}}{(z+1)(z-2)}dz = \int_C f(z).\frac{1}{3}\left[\frac{1}{(z-2)} - \frac{1}{(z+1)}\right]dz$$

$$= \frac{1}{3}\left\{\int_C \frac{f(z)}{(z-2)}dz - \int_C \frac{f(z)}{[z-(1)]}dz\right\} \text{....................(*)}$$

$|z|=3$ is a circle centre at the origin and radius 3

$$=\frac{1}{3}\left\{\int_C \frac{f(z)}{(z-2)}dz-\int_C \frac{f(z)}{[z-(-1)]}dz\right\}$$

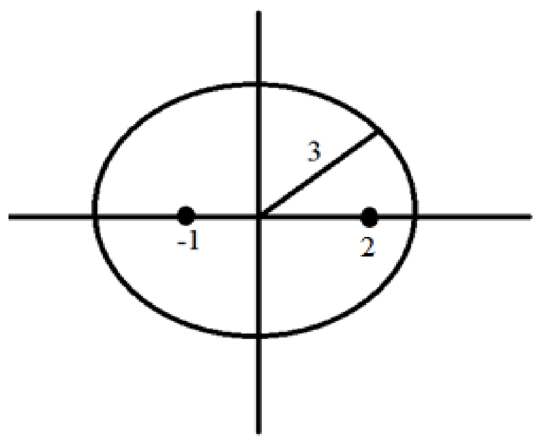
here point $a$=2, $a$=-1 both lies inside the circle $|z|=3$

$$=\frac{1}{3}\ 2\pi i f(2)-\frac{1}{3}\ 2\pi i f(-1)$$

$$=\frac{1}{3}\ 2\pi i\, e^{4}-\frac{1}{3}\ 2\pi i\, e^{-2}$$

$$=\frac{1}{3}\ 2\pi i\left[e^{4}-e^{-2}\right]$$

$$=\frac{2\pi i}{3}\left[e^{4}-e^{-2}\right]$$




**Singular point, Poles and Residues**:

(i) A point $z=a$ at which the complex function *f(z)* is fails to be analytic is called a singular point or singularity of *f(z)*.

**Example:**

(1) $f(z)=\dfrac{1}{z}$, $\quad z=0$ is a singular point

(2) $f(z)=\dfrac{1}{z-2}$, $\quad z=2$ is a singular point

(ii) A singular point $z=a$ is said to be an isolated singular point of $f(z)$ if there exists a neighborhood of $\boldsymbol{a}$ which encloses no other singular point of $f(z)$.

**Example:**

$f(z)=\frac{1}{z}$, $z=0$ is an isolated singylar point of $f(z)=\frac{1}{z}$,
since nhd of '0' which encloses no other singular point of
$f(z)=\frac{1}{z}$ or $\frac{1}{z}$ is analytic everywhere in the complex plane except at $z=0$

**Note:** If $a$ is an isolated singular point of a function $f(z)$ then we can expand $f(z)$ by Laurent's series given by

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \text{...............(1)}$$

in the domain $0<|z-a|<R$
Here the first term involving positive power series
of $(z - a)$ is called analytic part of $f(z)$ and
second part involving negative power series of $(z - a)$ is called
principle part of $f(z)$.

**Note:** The nature of the isolated singularity depends upon the number of terms in principle part. Hence we have the following cases.

(i) Removable Singularity: If all the negative powers of

$(z - a)$ in (1) are completely absent then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$.

Here the singularity can be removed by defining $f(z)$ at point $z = a$ in such a way that it becomes analytic at $z = a$. such singularity is called a removable singularity.

Example: $f(z) = \dfrac{z - \sin z}{z^2}$

Here $z = 0$ is a singularity

$$\therefore \frac{z - \sin z}{z^2} = \frac{1}{z^2}\left[ z - \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots\dots\dots\dots\dots \right) \right]$$

$$= \frac{1}{z^2}\left[ \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - + \dots\dots\dots\dots\dots \right]$$

$$= \frac{z}{3!} + \frac{z^3}{5!} + \frac{z^5}{7!} + \dots\dots\dots\dots\dots\dots\dots$$

Since there is no negative powers of $z$ in the expansion $z = 0$ is a removable singularity

$(\text{ii})$ Poles: If all the negative powers of $(z - a)$ in $(1)$ after the $m^{\text{th}}$ term are missing, then the singularity at $z = a$ is called a pole of order '$m$'

ie. $f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n + \dfrac{b_1}{(z-a)} + \dfrac{b_2}{(z-a)^2} + \dots\dots\dots\dots\dots + \dfrac{b_m}{(z-a)^m} + 0 + \dots.$

Note: A pole order one is called a simple pole.

Note: Poles of $f(z)$ can be determine by equating the denominator to zero

Example: $f(z)=\dfrac{e^z}{(z-1)^4}$

Here $z=1$ is a singularity and put $z-1=t$

$\therefore z=t+1$

$$\frac{e^z}{(z-1)^4}=\frac{e^{t+1}}{t^4}$$

$$=\frac{e}{t^4}.e^t$$

$$=\frac{e}{t^4}\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...............+\right]$$

$$=e\left[\frac{1}{t^4}+\frac{1}{t^3}+\frac{1}{2t^2}+\frac{1}{6t}+\frac{t}{4!}+\frac{t^2}{5!}+.........................\right]$$

$$=e\left[\frac{1}{(z-1)^4}+\frac{1}{(z-1)^3}+\frac{1}{2(z-1)^2}+\frac{1}{6(z-1)}+\frac{(z-1)}{4!}+\frac{(z-1)^2}{5!}+............\right]$$

Here there are four terms containing negative powers of (z-1) thus z=1 is a pole of order four.

(iii) Essential Singularity: If the number of negative powers of $(z-a)$ in (1) is infinite, then $z=a$ is called an essential Singularity.

Example: $f(z)=ze^{\frac{1}{z^2}}$

$$=z\left[1+\frac{1}{z^2}+\frac{1}{2!z^3}+\frac{1}{3!z^4}+..............\right]$$

$$=z+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^4}+..........................$$

$$f(z)=z+z^{-1}+\frac{1}{2!}z^{-2}+\frac{1}{3!}z^{-3}+.........................(*)$$

Here there are infinite number of terms in the negative powers of z, therefore $z = 0$ is an essential singularity of $f(z)$.

Expansion given by (*) is expansion of $f(z)$ around an isolated singularity $z = 0$.

**Residues:**

The coefficient of $(z - a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at that point.

The residue of $f(z)$ at $z = a$ is given by

$$\text{Res } f(a) = \frac{1}{2\pi i}\int_C f(z)dz \qquad \text{or} \int_C f(z)dz = 2\pi i \ \text{Res } f(a)$$

(1) If $f(z)$ has a simple pole at $z=a$ then

$$\text{Res } f(a) = \lim_{z\to a}[(z-a)f(z)]$$

(2) If $f(z)$ has a pole of order m at $z=a$ then

$$\text{Res } f(a) = \lim_{z\to a}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]\right\}$$

**Example:**

Determine the poles of the function $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ and the residues at each pole.

**Solution:**

$$f(z)=\frac{z^2}{(z-1)^2(z+2)}$$

Here $z=1$ is a pole of order 2

$Z=-2$ is a pole of order 1 or simple pole

$$\therefore Res\ f(1)=\lim_{z\to 1}\left\{\frac{1}{1!}\frac{d[(z-1)^2.\frac{z^2}{(z+2)(z-1)^2}}{dz}\right\}$$

$$=\lim_{z\to 1}\frac{d\left[\frac{z^2}{(z+2)}\right]}{dz}$$

$$=\lim_{z\to 1}\frac{z^2+4z}{(z+2)^2}$$

$$Res\, f(1)=\frac{5}{9}$$

$$Res\, f(2) = \lim_{z \to -2} (z+2).\frac{z^2}{(z+2)(z-1)^2}$$

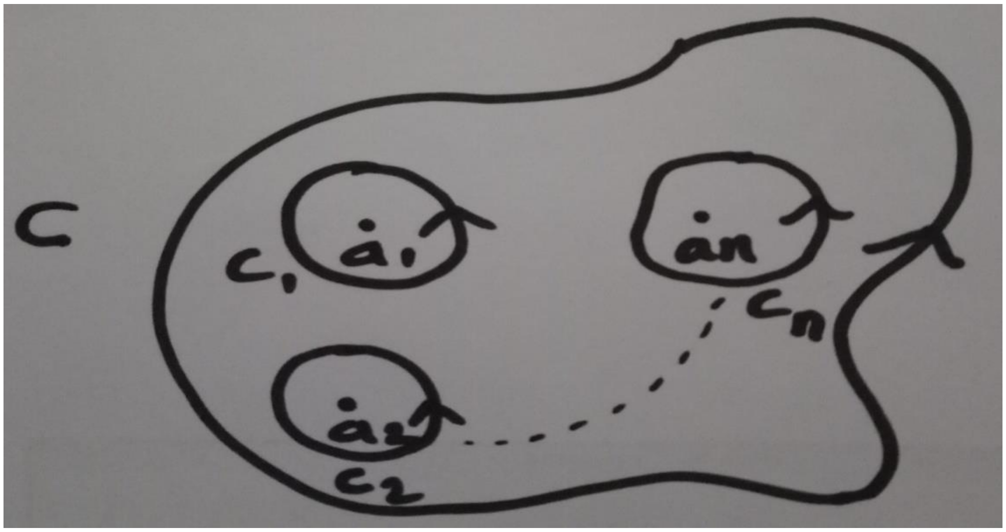
$$= \lim_{z \to -2} \frac{z^2}{(z+1)^2}$$

$$= \frac{4}{9}$$

**Cauchy's Residue Theorem:**

**Statement:** If *f(z)* is analytic within and on a closed curve C except at a finite number of singular points $a_1\ a_2 \ldots\ldots\ldots\ldots\ldots\ldots a_n$ all are within C, then

$$\int_C f(z)dz = 2\pi i\left[\operatorname{Res} f(a_1) + \operatorname{Res} f(a_2) + \ldots\ldots\ldots\ldots\ldots + \operatorname{Res} f(a_n)\right]$$




**Example:**

Using Cauchy's residue theorem, Evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)^2}dz$, Where C is the circle $|z| = 2.5$

**Solution:**

Clearly $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ is analytic within and on a given circle $|z| = 2.5$, except at $z = 1$, and $z =$ -2.

$z$=1 is a pole of order 2.

$\therefore \operatorname{Res} f(1) = \frac{5}{9}$...................(1)

$z = -2$ is a simple pole

$\therefore \operatorname{Res} f(-2) = \dfrac{4}{9}$

By Cauchy's residue Theorem

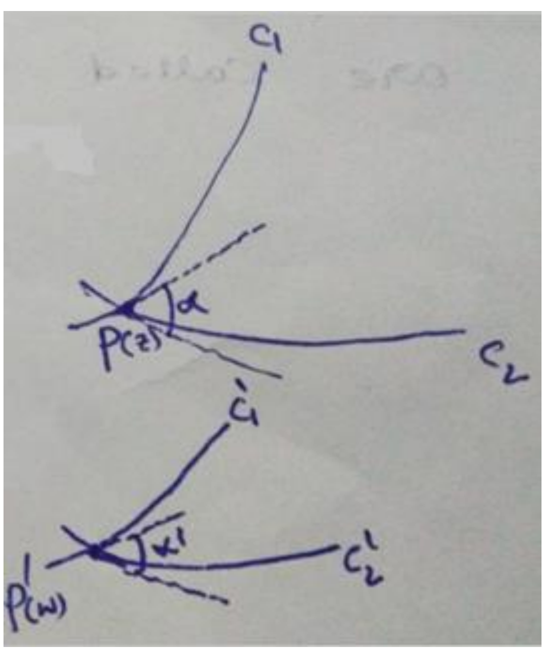
$$\int_C \frac{z^2}{(z-1)^2(z+2)}dz = 2\pi i\left\{\operatorname{Res} f(1) + \operatorname{Res} f(-2)\right\}$$

$$= 2\pi i\left\{\frac{5}{9} + \frac{4}{9}\right\}$$

$$= 2\pi i$$

**Conformal Transformation:**

**Definition:** Suppose two curves $C_1$ and $C_2$ in the Z – plane intersect at the point P and the corresponding curves and in the W – Plane intersect at . If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at in magnitude and sense.Then the transformation is said to be conformal.



**Note:** If *w=f(z)* is an analytic function of *z* in a region of the z – plane then *w=f(z)* is conformal at all points of that regions where $f'(z) \neq 0$

**Note:** To investigate the specific properties of a mapping *w=f(z)*. We may consider the images of

i) Straight line *x*= constant

ii) Straight line *y*=constant

III) $|z| =$ constant and the lines through the origin

**Note:** The curves defined by *u(x , y)* =constant and *v(x , y)* = constant, the pre images in the *z* –plane can be investigated. These curves are called the level curves of *u* and *v*.

1) Discuss the transformation $w = z^2$

**Solution:**

$w = z^2$................(1)

$w = (x+iy)^2$

$u + iv = x^2 - y^2 + i\,2xy$

Equating real and imaginary parts

$u = x^2 - y^2$ and $v = 2xy$

$\frac{dw}{dz} = 2z = 0$ for *z*=0 therefore it is a critical point of the mapping

**Case (i)** Determine the images of the straight line *x*=constant.

$\therefore$ The line $x = c_1$ has the image

$u = c_1^2 - y^2$ and $v = 2c_1 y$

Now eliminate y from the above relali

$v^2 = 4c_1^2\left[c_1^2 - u^2\right]$.......................(3)

Equation given by (3) is a parabola with focus at the origin and opening to the left.

**Case (ii)** Determine the images of the straight line *y*=constant.

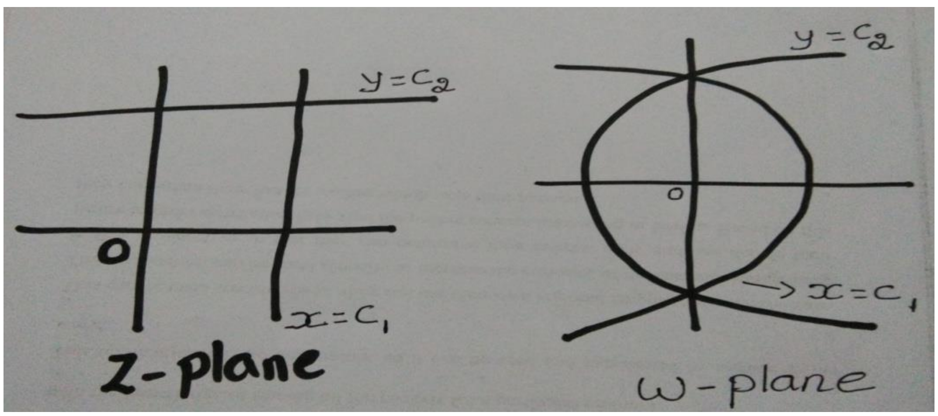
The line $y=c_2$ has the image

$u = x^2 - c_2^2$ and $v = 2c_2 x$

now eliminate x from the above relations.

$v^2 = 4c_2^2(u^2 + c_2^2)$..................(4)

Equation given by (4) is a parabola with focus at the origin opening to the right




Here the pairs of lines $x = c_1$ and $y = c_2$ in the z- plane map into parabolas in the w- plane as shown in the above figure

**Case (iii)**

Determine the images of $|z| = r$

Taking $z = re^{i\theta}$

$\therefore w = r^2 e^{2i\theta}$

$w = R\mathrm{e}^{i\phi}$ ............(5) Where $R=r^2$

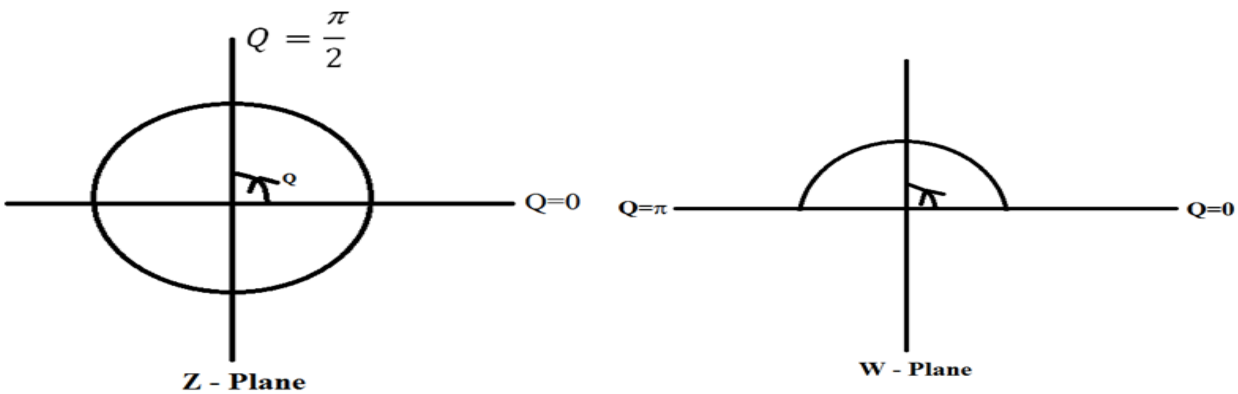
$\phi=2\theta$

$|w| = R$

$\therefore$ The angles at the origin are doubled under the mapping $w=z^2$.

The first quadrant of the z-plane $0 \le \theta \le \frac{\pi}{2}$ is

mapped upon the entire upper half of the w-plane




2) Discuss the transformation $w = z + \frac{1}{z}, z \neq 0$

Solution: The given transformation is conformal except at the points $z=\pm 1$.

since $\frac{dw}{dz} = 1 - \frac{1}{z^2} = 0$ for $z=\pm 1$

$w = re^{i\theta} + \frac{1}{r}e^{-i\theta}$

$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + \left(r - \frac{1}{r}\right)\sin\theta$

Equating real and imaginary parts we get

$$\left.\begin{aligned} u &= \left(r + \frac{1}{r}\right)\cos\theta \\ v &= \left(r - \frac{1}{r}\right)\sin\theta \end{aligned}\right\} \text{..................(1)}$$

Case (i):- Find the images of circle, r=constant ie. r=c, represents a circle with constant Radius.

$\cos\theta = \frac{u}{a}$ where a=constant

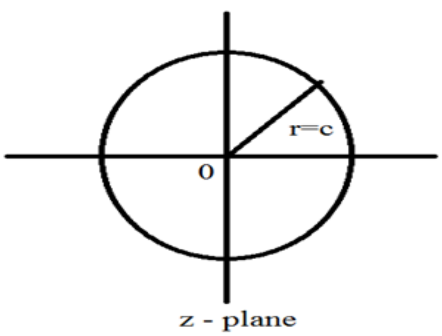
$\sin\theta = \frac{v}{b}$ where b=constant

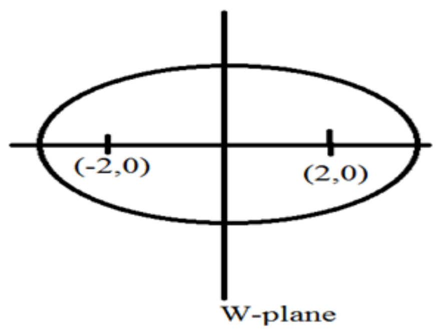
$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$........................(2)

Equation given by (2) represents ellipses whose principal axes lie in $u$ and $v$ axes and have the length 2a and 2b respectively with foci$(\pm 2,0)$

Thus the circle $r$=constant is mapped onto ellipses under the transformation

$w = z + \frac{1}{z}$




Case (ii) Find the images of line $\theta$=constant, passing through origin, ie. $\theta$=C

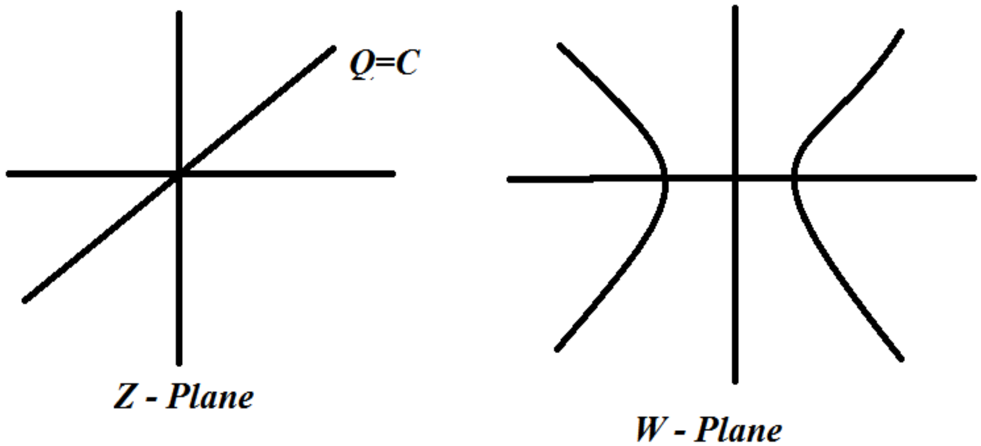
From (1) $\frac{u}{a} = \left(r + \frac{1}{r}\right)$ Where $a = \cos c$

$\frac{v}{b} = \left(r - \frac{1}{r}\right)$ where $b = \sin c$

$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$....................(3) where A=2a and B=2b

Equation given by (3) represents hyperbalas in the $w$- plane.

Thus lines $\theta$=constant is mapped onto hyperbolas under $w=z+\frac{1}{z}$




3) Discuss transformatin of $w = e^z$

Solution: $w = e^z$

$$u+iv=e^{x+iy}$$

$$=e^x.e^{iy}$$

$$=e^x\left[\cos y + i\sin y\right]$$

$$u+iv=e^x\cos y + ie^x\sin y$$

Equating real and imaginary perts

$$\left.\begin{array}{l} u = e^x\cos y \\ v = e^x\sin y \end{array}\right\}\text{....................}(1)$$

Case (i): Find the images of x=constant ie. x=c

from (1) we have $u = e^c\cos y$

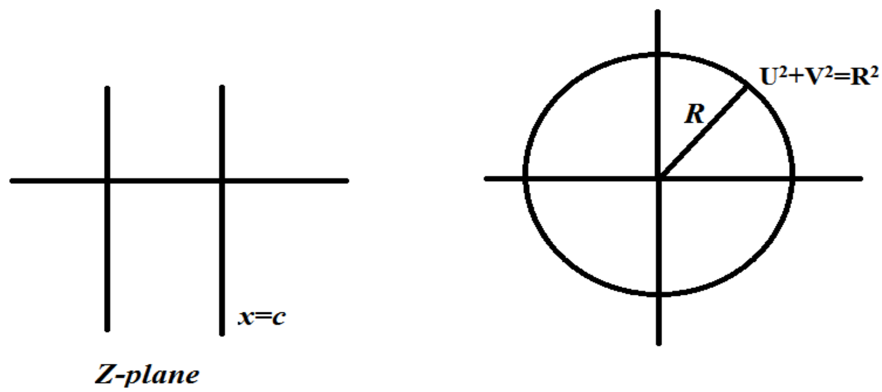
$$v = e^c\sin y$$

$$u^2+v^2=e^{2c}$$

$$u^2+v^2=R^2 \text{............................}(2) \quad \text{where } R=e^c$$

Equations given by (2) represents a circle centre at the origin with radius $R$

Thus the line $x$=constant in the $z$ -Plane is mapped onto circle in the $w$- plane under the transformation $w=e^z$




Case (ii): Find the images of a line y=constant ie. y=c

From (1) we have

$$\left.\begin{aligned} u &= e^x \cos(c) \\ v &= e^x \sin(c) \end{aligned}\right\}$$

$$\frac{v}{u} = \frac{e^x \sin(c)}{e^x \cos(c)}$$

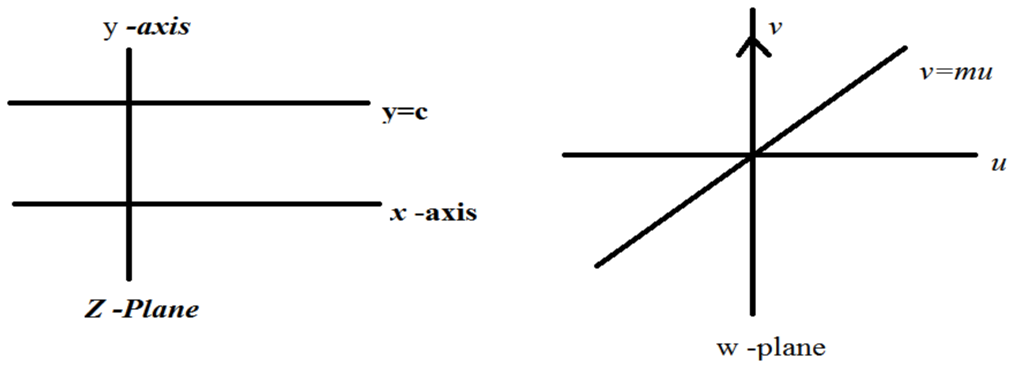
$$\tan(c) = \frac{v}{u}$$

$$\therefore v = \tan(c).u$$

$v = mu$.............(2)  where m=tan(c) slope

Equations given by (3) represents a straight line passing through the origin with slope m=tan c in the w – plane.

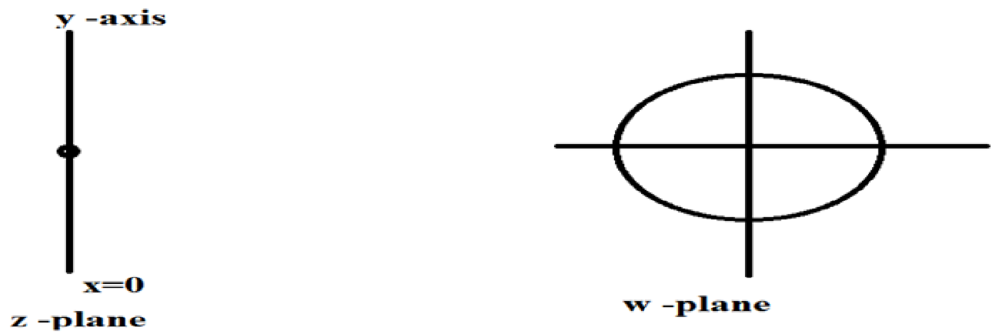
Thus the line y=constant in the *z* – plane is mapped onto straight line passing through the origin in the w – plane under the transformation $W=e^z$



Observation:

(1) Since $e^z \neq 0$, for all $z$, the point $w=0$ is not an image of any point $z$.

(2) Suppose $c=0$ ie. $x=0$ means that the $y$ - axis in the $z$ - plane is mapped onto the unit circle $u^2+v^2=1$




**Bilinear Transformation:**

- Let a,b,c and d be complex constant such that *ad-bc≠0.* Then the transformation defined by

$$w = \frac{az+b}{cz+d} \text{.......................(1)}$$

is called Bilinear Transformation

- from (1) we find

$$z = \frac{b-wd}{cw-a} \text{.................(2)}$$

is also calleda Bilinear Transformation

Note: The condition ad-bc $\neq 0$ ensures that $\frac{dw}{dz} \neq 0$

ie. The transformation is conformal if $ad - bc \neq 0$

**Invariant Point:**

Invariant points of bilinear transformation,

If z maps into itself in the w-plane ie w=z

z= $\dfrac{az+b}{cz+d}$ or $cz^2+(d-a)z-b=0$..........(3)

- Equation given by (3) is a quadratic equation in z, the roots of the equation are which are $Z_1$, $Z_2$ invariant points or fixed points of the Bilinear transformation.

- **Cross Ratio**: Bilinear transformation preserves cross ratio of three points say points $Z_1$, $Z_2$ $Z_3$ of the z-plane maps onto the points $W_1$ $W_2$ $W_3$ of the w-plane.

this cross ration is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Solving this equation for $w$ interms of $z$

we obtain the unique bilinear transformation

that transforms $z_1$ $z_2$ $z_3$ onto $w_1$ $w_2$ $w_3$ respectively

Example: Find the bilinear transformation that transforms the points $z_1=i$ $z_2=1$, $z_3=-1$ onto the points $w_1=1$, $w_2=0$, $w_3=\infty$ respectively. Also find the invarient points and the images of region $|z|<1$ under this transformation.

Solution: The required bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)w_3}{w_3\left(\frac{w}{w_3}-w_3\right)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \because \frac{w}{w_3}\to 0 \quad \mathrm{w}_3\to\infty$$

$$\frac{(w-1)(0-1)}{(0-1)(0-1)}=\frac{(z-i)(1+1)}{(z+1)(1-i)}$$

$$-(w-1)=\frac{(z-i)(1+1)}{(z+1)(1-i)}$$

$$-w+1=\frac{2(z-i)}{(z+1)(1-i)}$$

$$w=1-\frac{2(z-i)}{(z+1)(1-i)}$$

$$w=\frac{(z+1)(1-i)-2(z-i)}{(z+1)(1-i)}$$

$$=\frac{z-iz+1-i-z-iz+1+i}{(z+1)(1-i)}$$

$$w=\frac{(z+1)(1-i)-2(z-i)}{(z+1)(1-i)}$$

$$=\frac{z-iz+1-i-2z+2i}{(z+1)(1-i)}$$

$$= \frac{-z - iz + 1 + i}{(z+1)(1-i)}$$

$$= \frac{(1-z) + i(1-z)}{(z+1)(1-i)}$$

$$= \frac{(1-z)(1+i)}{(z+1)(1-i)}$$

$$= \frac{(1-z)}{(1+z)} \frac{(1+i)}{(1-i)} \frac{(1+i)}{(1+i)}$$

$$w = \frac{(1-z)}{(1+z)} \cdot \frac{2i}{2} \qquad (1+\mathrm{i})^2 = 2i$$

$$1 - i^2 = 2$$

$$w = \frac{i(1-z)}{1+z} \text{............(*)} \text{ is a required bilinear transform}$$

To find the invariant points of bilinear transform

Taking $w$=2 in equation (*)

$$z = \frac{i(1-z)}{(1+z)}$$

$$z^2 + z = i - iz$$

$$z^2 + (1+i)z - i = 0$$

$$z = \frac{-(1+i) \pm \sqrt{(1+i)^2 - 4(-i)}}{2}$$

$$= \frac{-(1+i) \pm \sqrt{2i + 4i}}{2}$$

$$= \frac{-(1+i) \pm \sqrt{6i}}{2}$$

$\therefore z_1 = \dfrac{-(1+i) + \sqrt{6i}}{2}, \quad z_2 = \dfrac{-(1+i) - \sqrt{6i}}{2}$ are invarient points.

To find the image of $|z|<1$ (ie. interior points of the unit circle)

$$w=\frac{i(1-z)}{(1+z)}$$

$$w+wz=i-iz$$

$$wz+iz=i-w$$

$$z(w+i)=i-w$$

$$z=\frac{i-w}{i+w}\text{.............}(2)$$

Now $|z|<1$

$$\left|\frac{i-w}{i+w}\right|<1$$

$$|i-w|<|i+w|$$

$$|i-(u+iv)|<|i+(u+iv)|$$

$$|-u+i(1-v)|<|u+i(i+v)|$$

$$|-[u-i(1-v)]|<|u+i(1+v)|$$

$$|u-i(1-v)|<|u+i(1+v)|$$

$$\sqrt{u^2+(1-v)^2}<\sqrt{u^2+(1+v)^2}$$

$$u^2+v^2-2v+1<u^2+v^2+2v+1$$

$$-4v<0$$

$$4v>0$$

$$\Rightarrow v>0$$

Thus under the given transformation, the circular region $|z| < 1$ (ie. interior of the circle $|z| = 1$) in the z-plane is mapped onto the upper - half of the w-plane.

2) Find the bilinear transformation that the points $z = -1, i, 1$ onto the points $w = 1, i, -1$ respectively.

Solution: Let $z_1 = -1, \quad z_2 = i \quad z_3 = 1$

$$w_1 = -1, \quad w_2 = i \quad w_3 = 1$$

The required bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z+1)(i-1)}{(z-1)(i+1)}$$

$$\frac{(w-1)}{(w+1)} = \frac{(z+1)}{(z-1)} \cdot \frac{(i-1)^2}{(i+1)^2}$$

$$= \frac{(z+1)}{(z-1)} \times \frac{(-2i)}{(2i)}$$

$$\frac{(w-1)}{(w+1)} = -\frac{(1+z)}{(z-1)}$$

$$\frac{(w-1)}{(w+1)} = \frac{(1+z)}{(1-z)}$$

$$(w-1)(1-z) = (w+1)(1+z)$$

$$w - wz - 1 + z = w + wz + 1 + z$$

$$-2wz - 2 = 0$$

$$-2wz = 2$$

$w = -\dfrac{1}{z}$ this is the requird transformation